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The Artificial Boundary Method for a Nonlinear Interface Problem on Unbounded Domain

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Abstract: In this paper, we apply the artificial boundary method to solve a three-dimensional nonlinear interface problem on an unbounded domain. A spherical or ellipsoidal surface as the artificial boundary is introduced. The exact artificial boundary conditions are derived explicitly in terms of an infinite series and then the well-posedness of the coupled weak formulation in a bounded domain, which is equivalent to the original problem in the unbounded domain, is obtained. The error estimate depends on the mesh size, the term after truncating the infinite series and the location of the artificial boundary. Some numerical examples are presented to demonstrate the effectiveness and accuracy of this method.

Keyword: Artificial boundary method, Dirichlet to Neumann mapping, Finite element method, Natural integral operator, Nonlinear interface problem.

1 Introduction

In many fields of scientific and engineering computing, the boundary value problems of the partial differential equations on some unbounded domains are often met. The boundlessness of the domains brings the essential difficulties for solving these problems numerically. There are several methods for solving these problems. The artificial boundary method is one of them. It is particularly attractive for exterior problems or problems in domains extending to infinity.

The artificial boundary method reduces the original problem in an unbounded domain to an equivalent problem in a bounded domain with some suitable boundary conditions on the artificial boundary. The standard procedure of the method may be simply described as follows. First, one divides the domain into two subregions, a bounded inner region and an unbounded outer region by introducing an auxil-

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iary common boundary. Next, the problem is reduced to an equivalent one in the bounded inner region. This reduction will be accomplished by deriving either a local natural boundary condition or a nonlocal boundary condition, which relates the Cauchy data of the solution on the common boundary. Because of the necessity of deriving this boundary condition on the common boundary, one needs generally to apply boundary integral methods to the unbounded outer region. This reduction to an equivalent problem is by no means a unique process. The first significant result concerning the theoretical justification of a coupling procedure of this type seems due to Brezzi and Johnson (1979) and Johnson and Nedelec (1980). It is based on the classical direct boundary integral method. Further theoretical developments with respect to various coupling procedures may be found in Wendland (1986), Costabel (1987), Feng (1983), Han (1990), Hsiao and Proter (1986), MacCamy and Marin (1980). Most of these coupling methods are not direct and natural. A direct and natural coupling of the finite and boundary elements was suggested first by Feng and Yu (1983), where the imposed boundary condition on the artificial boundary is exact, non-reflective and nonlocal. The method leads to a symmetric and coercive bilinear form and then this method is called the natural coupling method of FEM and BEM, or the exact artificial boundary method. Because the exact artificial boundary condition is just the Dirichlet to Neumann mapping on the artificial boundary, this method is also called the DtN method [Keller and Givoli (1989), Grote and Keller (1995)].

Yu (2002) has given many kinds of equivalent forms of the natural boundary integral operator, i.e., the exact artificial boundary condition for the two-dimensional elliptic problems, Stokes equations and linear elastic equations on the circle artificial boundary. Wu and Yu (2000) obtained the exact artificial boundary condition based on the elliptic boundary while Wu (1999), Grote and Keller (1995) attained the exact artificial boundary one on the spherical boundary. More recently, Huang and Yu (2007) derive the exact artificial boundary one on the spheroidal boundary and general ellipsoidal boundary. The exact artificial boundary condition is generally expressed explicitly in terms of an infinite series. However, in the numerical computation, the infinite series need to be truncated and then this will result in the truncation error. The error estimate of the numerical approximation solution which has first been given by Yu (1985) showed how the error depend on the mesh size, the position of the artificial boundary and the term after truncating the infinite series. The artificial boundary method, originally designed for treating linear problems [Yu (2002)], works equally well for the case of the evolution equation by Du and Yu (2000, 2001), the electromagnetic problems by Liu (2007), an adequate combination of linear and nonlinear partial differential equations by Hu and Yu (2001).

For the three-dimensional exterior problems, a spherical surface [Grote and Keller (1995), Wu (1999)] is usually selected as the artificial boundary. However, for a cigar-shaped or flying saucer-shaped obstacles, a prolate or oblate spheroidal surface [Huang (2007), Huang and Yu (2006)] is used as the artificial boundary very efficiently, since it leads to the smaller computational domain, as shown in Fig. 1, and doesn't result in the increase in the computational complexity of the stiff matrix from the boundary reduction with the spheroidal artificial boundary. On the other hand, an anisotropic exterior problem with constant coefficients with a spherical artificial boundary can be reduced to an isotropic problem with an ellipsoid artificial boundary. For the general ellipsoidal artificial boundary, the exact artificial boundary pertains to the Lamé functions and the ellipsoidal harmonic functions which are too complicated, but the computation complexity of the ellipsoidal harmonic functions is not greater than ones of the spherical harmonic functions when the term of the series amounts to a certain value [Huang (2007)].

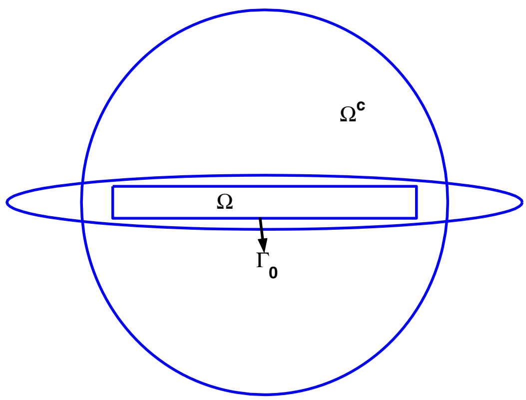

Figure 1: Cross-section of Cigar-shaped, ellipsoid and sphere.

Nonlinear interface problems widely occur in fluid mechanics [Feistauer (1985, 1987)] and elasticity [Carstensen and Gwinner (1997), Costabel and Stephan (1990)]. In this paper, we apply the artificial boundary method to solve a three-dimensional nonlinear interface problem in $\mathbb{R}^3$. Specifically, the boundary value problem consists of a nonlinear second order elliptic equation in divergence form in a bounded inner region, and the Laplace equation in the corresponding unbounded exterior region, in addition to appropriate boundary and transmission conditions. In Sect. 3, a spherical, spheroidal or general ellipsoidal surface as the artificial boundary is introduced. The exact artificial boundary conditions are derived explicitly in terms of an infinite series and then the well-posedness of the coupled variational problem is obtained. In Sect. 4, we give existence and uniqueness of the solution of the discrete problem and derive asymptotic error estimate. The error estimate shows the error depends on the mesh size, the term after truncating the infinite series and the location of the artificial boundary. Some numerical examples are presented in Sect. 5 to demonstrate the effectiveness and accuracy of this method. Finally, we state the conclusions in Sect. 6.

2 The problem described

We consider a nonlinear elliptic differential equation in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ and a linear elliptic differential equation in $\Omega^c := \mathbb{R}^3 \backslash (\Omega \cup \partial\Omega)$ and their solutions are connected by conditions on the interface boundary $\Gamma_0 = \partial\Omega$. For given $f \in L^2(\Omega), u_0 \in H^{1/2}(\Gamma_0), t_0 \in H^{-1/2}(\Gamma_0)$, the interface problem reads: find $u_1 \in H^1(\Omega)$, $u_2 \in H^1_{loc}(\Omega^c)$ such that

$$-div(p(|\nabla u_1|) \cdot \nabla u_1) + u_1 = f, \text{in } \Omega, \tag{1}$$

$$-\Delta u_2 = 0, \text{in } \Omega^c, \tag{2}$$

with the transmission conditions

$$u_1 = u_2 + u_0, p(|\nabla u_1|)\frac{\partial u_1}{\partial \mathbf{n}} = \frac{\partial u_2}{\partial \mathbf{n}} + t_0 \text{ on } \Gamma_0, \tag{3}$$

and the radiation condition at infinity

$$u_2(\mathbf{x}) = O(\frac{1}{|\mathbf{x}|}) \text{ for } |\mathbf{x}| \to \infty, \tag{4}$$

where $p(t) \in C^1(\{0\} \cup \mathbb{R}^+)$ satisfies the condition $p_0 \leq p(t) \leq p_1 < \infty$ and $\alpha \leq p(t) + tp'(t) \leq \beta$ for constants $p_0, p_1, \alpha, \beta > 0$ [see Stephan (1992)] and $\mathbf{n}$ denotes the unit normal vector on Γ_0 defined almost everywhere pointing from Ω into Ω^c.

Let $H^s(\Omega), H^s(\Gamma_0)$ and $H^{-1/2}(\Gamma_0)$ denote the usual Sobolev spaces [Lions and Magenes (1972)] and $H^1_{loc}(\Omega^c) = \{v : v|_O \in H^1(O)$ for any $O = \Omega^c \cap B$ with $\Omega \subset\subset B \subset\subset \mathbb{R}^3\}$, where B is any ball. According to the radiation condition Eq. 4 and the essential boundary conditions Eq. 3, define the set of admissible functions $\hat{C} := \{(v_1, v_2) \in H^1(\Omega) \times H^1_{loc}(\Omega^c) : v_1|_{\Gamma_0} = v_2|_{\Gamma_0} + u_0, \text{and } v_2 \text{ satisfies } Eq.4\}$ and the set of trial functions $\hat{C}^* := \{(v_1, v_2) \in H^1(\Omega) \times H^1_{loc}(\Omega^c) : v_1|_{\Gamma_0} = v_2|_{\Gamma_0}, \text{ and } v_2 \text{ satisfies } Eq.4\}$. Obviously, $\hat{C}$ is a nonempty convex set.

The weak form of problem Eq. 1-Eq. 4 is to find $u = (u_1, u_2) \in \hat{C}$ such that for any $v = (v_1, v_2) \in \hat{C}^*$

$$\begin{aligned} &\int_\Omega (p(|\nabla u_1|)\nabla u_1 \cdot \nabla v_1 + u_1 v_1)\, \mathrm{d}\mathbf{x} \\ &+ \int_{\Omega^c} \nabla u_2 \cdot \nabla v_2 \, \mathrm{d}\mathbf{x} = L(v), \end{aligned} \tag{5}$$

where $\mathrm{d}\mathbf{x} = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$,

$$L(v) := \int_\Omega f v_1\,\mathrm{d}\mathbf{x} + \int_{\Gamma_0} t_0 v_2\,\mathrm{d}S. \tag{6}$$

Obviously, $L : H^1(\Omega)\times H^1_{loc}(\Omega^c) \to \mathbb{R}$ is a bounded linear functional.

3 The coupled problem and its well-posedness

In terms of the special shape of the domain Ω, we choose the different boundary Γ (e.g., the spherical, prolate spheroidal, oblate spheroidal or general ellipsoidal surface) as the artificial boundary and $\Gamma \subset \Omega^c$. Then Γ divides Ω^c into two subregions: a bounded inner region Ω_1 and an unbounded outer region Ω_2 such that $\Omega_1 \cap \Omega_2 = \emptyset$. Let $\Omega_0 = \Omega \cup \Omega_1 \cup \Gamma_0$, $u_{21} = u_2|_{\Omega_1}$ and $u_{22} = u_2|_{\Omega_2}$. According to the natural boundary reduction principle [Yu (2002)], if $w \in D_1 := \{\, v \in H^1_{loc}(\Omega_2) :$ v such that Eq.4$\}$, then for any $v \in D_1$

$$<\mathscr{K}(w), v>_\Gamma = -\int_\Gamma \frac{\partial w}{\partial \mathbf{n}} v\,\mathrm{d}S = \int_{\Omega_2} \nabla w\cdot\nabla v\,\mathrm{d}\mathbf{x}.$$

Here, the operator $\mathscr{K} : H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$(i.e., Dirichlet to Neumann map or Steklov-Poincare operator) is the natural integral operator [Yu (2002)]. $\mathscr{K}(u_2)$ is also the exact boundary condition on the artificial boundary Γ. In the following, we will give the explicit expression of the operator $\mathscr{K}$.

When Γ is a spherical surface, let (R,θ,φ) denote its spherical coordinate. From Wu (1999), the solution of the Laplace equation over the unbounded outer domain Ω_2 is for any $u_2 \in H^{1/2}(\Gamma)$

$$u_2(r,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{R^{n+1}}{r^{n+1}} U_{nm} Y_{nm}(\theta,\varphi) \tag{7}$$

and the normal derivative of u_2 on Γ satisfies

$$\frac{\partial u_2}{\partial \mathbf{n}}|_\Gamma = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{n+1}{R} U_{nm} Y_{nm}(\theta,\varphi),$$

where $U_{nm} = \int_0^\pi\int_0^{2\pi} u_2|_\Gamma Y^*_{nm}(\theta,\varphi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$, $Y_{nm}(\theta,\varphi)$ are the spherical harmonic functions and Y^*_{nm} is the conjugate complex number of Y_{nm}. Let $V^*_{nm} = \int_0^\pi\int_0^{2\pi} v|_\Gamma Y_{nm}(\theta,\varphi)\sin\theta d\theta d\varphi$, then

$$<\mathscr{K}(u_2), v>_\Gamma = R\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(n+1)V^*_{nm}U_{nm}. \tag{8}$$

When Γ is a spheroidal surface$\{(x_1,x_2,x_3):(x_1^2+x_2^2)/a^2+x_3^2/b^2=1\}$(If $a>b$, Γ is oblate; If $a<b$, Γ is prolate.) and (μ_1,θ,φ) denotes its oblate or prolate spheroidal coordinate, let

$$T_{nm}(\mu)=\begin{cases}-\dfrac{\mathrm{d}\,Q_n^m(\cosh\mu)/\mathrm{d}\,\mu}{Q_n^m(\cosh\mu)}\sinh\mu, & \text{for } \Gamma \text{ is prolate spheroidal surface}\\ -\dfrac{\mathrm{d}\,T_n^m(\sinh\mu)/\mathrm{d}\,\mu}{T_n^m(\sinh\mu)}\cosh\mu, & \text{for } \Gamma \text{ is oblate spheroidal surface}\end{cases}$$

where $Q_n^m(x)$ denote the associated Legendre functions of the second kind and

$$T_n^m(x)=i\exp(\frac{i\pi n}{2})Q_n^m(ix), \qquad i^2=-1.$$

From Huang (2007), for the prolate spheroidal surface, the solution of the Laplace equation over the unbounded outer domain Ω_2 is

$$u(\mu,\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{Q_n^m(\cosh\mu)}{Q_n^m(\cosh\mu_1)}U_{nm}Y_{nm} \tag{9}$$

and the normal derivative of u_2 on Γ satisfies

$$\frac{\partial u_2}{\partial \mathbf{n}}|_\Gamma=\frac{-\sum\limits_{n=0}^{\infty}\sum\limits_{m=-n}^{n}T_{nm}(\mu_1)U_{nm}Y_{nm}}{f_0\sqrt{\cosh^2\mu_1-\cos^2\theta}}$$

and for the oblate spheroidal surface, the solution of the Laplace equation over the unbounded outer domain Ω_2 is

$$u(\mu,\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{T_n^m(\sinh\mu)}{T_n^m(\sinh\mu_1)}U_{nm}Y_{nm} \tag{10}$$

and the normal derivative of u_2 on Γ satisfies

$$\frac{\partial u_2}{\partial \mathbf{n}}|_\Gamma=\frac{-\sum\limits_{n=0}^{\infty}\sum\limits_{m=-n}^{n}T_{nm}(\mu_1)U_{nm}Y_{nm}}{f_0\sqrt{\cosh^2\mu_1-\sin^2\theta}}.$$

Thus, we obtain

$$<\mathscr{K}(u_2),v>_\Gamma=f_0\sum_{n=0}^{\infty}\sum_{m=-n}^{n}T_{nm}(\mu_1)V_{nm}^*U_{nm}, \tag{11}$$

where $f_0 = \sqrt{|a^2-b^2|}$, Y_{nm}, U_{nm} and V^*_{nm} are the same as the above definition for the spherical surface.

When Γ is a general ellipsoidal surface $\{(x_1,x_2,x_3): x_1^2/a^2+x_2^2/b^2+x_3^2/c^2=1,\ a>b>c>0\}$ and (a,λ_2,λ_3) denotes its ellipsoidal coordinate. From Huang (2007), the solution of the Laplace equation over the unbounded outer domain Ω_2 is

$$u(\lambda_1,\lambda_2,\lambda_3) = \sum_{n=0}^{\infty}\sum_{p=1}^{2n+1} \frac{F_n^p(\lambda_1)}{F_n^p(a)} U_n^p \overline{E_n^p}(\lambda_2)\overline{E_n^p}(\lambda_3) \tag{12}$$

and the normal derivative of u_2 on Γ satisfies

$$\frac{\partial u_2}{\partial \mathbf{n}}|_\Gamma = \sum_{n=0}^{\infty}\sum_{p=1}^{2n+1} \frac{\mathrm{d}F_n^p(a)/\mathrm{d}\lambda_1}{F_n^p(a)} \frac{bcU_n^p\overline{E_n^p}(\lambda_2)\overline{E_n^p}(\lambda_3)}{\sqrt{(a^2-\lambda_2^2)(a^2-\lambda_3^2)}},$$

and

$$< \mathscr{K}(u_2), v >_\Gamma = -bc\sum_{n=0}^{\infty}\sum_{p=1}^{2n+1} \frac{\mathrm{d}F_n^p(a)/\mathrm{d}\lambda_1}{F_n^p(a)} U_n^p V_n^p, \tag{13}$$

where $E_n^p(\lambda)$ denote the n order Lamé functions of the first kind with the eigenvalue p, $F_n^p(\lambda)$ denote the Lamé functions of the second kind,

$$U_n^p = \int_\Gamma u(a,\lambda_2,\lambda_3)\frac{E_n^p(\lambda_2)E_n^p(\lambda_3)}{\sqrt{(a^2-\lambda_2^2)(a^2-\lambda_3^2)}}\frac{\mathrm{d}S}{\sqrt{\gamma_n^p}},$$

$$\gamma_n^p = \int_\Gamma \frac{\left(E_n^p(\lambda_2)E_n^p(\lambda_3)\right)^2 \mathrm{d}S}{\sqrt{(a^2-\lambda_2^2)(a^2-\lambda_3^2)}}$$

and

$$\mathrm{d}S = \frac{\sqrt{(a^2-\lambda_2^2)(a^2-\lambda_3^2)}(\lambda_2^2-\lambda_3^2)\,\mathrm{d}\lambda_2\,\mathrm{d}\lambda_3}{\sqrt{(k^2-\lambda_2^2)(\lambda_2^2-h^2)}\sqrt{(k^2-\lambda_3^2)(h^2-\lambda_3^2)}},$$

$$\overline{E_n^p}(\lambda_2)\overline{E_n^p}(\lambda_3) = \frac{E_n^p(\lambda_2)E_n^p(\lambda_3)}{\sqrt{\gamma_n^p}}.$$

Here, $k^2 = a^2-c^2, h^2 = a^2-b^2$.

In numerical computing, we replace the series $\sum_{n=0}^{\infty}$ with the Nth partial sum $\sum_{n=0}^{N}$. Let $< \mathscr{K}_N(u_2), v >_\Gamma$ denote the Nth partial sum of the series $< \mathscr{K}(u_2), v >_\Gamma$.

Lemma 1 *[Huang (2007)] The bilinear forms $< \mathscr{K}(u), v >_\Gamma$ and $< \mathscr{K}(u)_N, v >_\Gamma$ on $H^{1/2}(\Gamma) \times H^{1/2}(\Gamma)$ have the following properties:*
1). Two bilinear forms are symmetric about u, v.
2). For any $u, v \in H^{1/2}(\Gamma)$, there exist the positive constant C_1 and C_2 such that

$$| < \mathscr{K}(u), v >_\Gamma | \leq C_1 \|u\|_{H^{1/2}(\Gamma)} \|v\|_{H^{1/2}(\Gamma)},$$

$$| < \mathscr{K}_N(u), v >_\Gamma | \leq C_2 \|u\|_{H^{1/2}(\Gamma)} \|v\|_{H^{1/2}(\Gamma)}.$$

3). For any $v \in H^{1/2}(\Gamma)$, there exists a positive constant α such that

$$< \mathscr{K}(v), v >_\Gamma \geq < \mathscr{K}_N(v), v >_\Gamma > \alpha \left(\int_\Gamma v \, \mathrm{d}\sigma \right)^2.$$

Let

$$V = \{(v_1, v_2) \in H^1(\Omega) \times H^1(\Omega_1) : v_1 = v_2 + u_0 \text{ on } \Gamma_0\},$$

$$V^* = \{(v_1, v_2) \in H^1(\Omega) \times H^1(\Omega_1) : v_1 = v_2 \text{ on } \Gamma_0\}$$

with the norm

$$\|v\|_V = \left(\|v_1\|^2_{H^1(\Omega)} + \|v_2\|^2_{H^1(\Omega_1)} \right)^{\frac{1}{2}}.$$

Denote

$$B(u;v) = \int_\Omega (p(|\nabla u_1|) \nabla u_1 \cdot \nabla v_1 + u_1 v_1) \, \mathrm{d}\mathbf{x} + \int_{\Omega_1} \nabla u_2 \cdot \nabla v_2 \, \mathrm{d}\mathbf{x} + < \mathscr{K}(u_2), v_2 >_\Gamma$$

and

$$B_N(u;v) = \int_\Omega (p(|\nabla u_1|) \nabla u_1 \cdot \nabla v_1 + u_1 v_1) \, \mathrm{d}\mathbf{x} + \int_{\Omega_1} \nabla u_2 \cdot \nabla v_2 \, \mathrm{d}\mathbf{x} + < \mathscr{K}_N(u_2), v_2 >_\Gamma,$$

then the problem Eq. 5 is equivalent to the following weak formulation to find $u = (u_1, u_2) \in V$ such that

$$B(u;v) = L(v), \quad \forall v = (v_1, v_2) \in V^*. \tag{14}$$

By truncating the series $< \mathscr{K}(\cdot), \cdot >$, the weak formulation is to find $u^N = (u_1^N, u_2^N) \in V$ such that

$$B_N(u^N;v) = L(v), \quad \forall v = (v_1, v_2) \in V^*. \tag{15}$$

In order to obtain the existence and uniqueness of solution of the original interface problem, we must give the Euler equations of the problem Eq. 14 and Eq. 15. Define $\Phi : V \mapsto \mathbb{R}$ by

$$\Phi(u) := \int_\Omega (g(|\nabla u_1|) + \frac{1}{2}|u_1|^2)\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_{\Omega_1} |\nabla u_2|^2\,\mathrm{d}\mathbf{x} + \frac{1}{2} < \mathscr{K}(u_2), u_2 >_\Gamma - L(u),$$

where

$$g : [0,\infty) \mapsto [0,\infty), \quad t \longmapsto g(t) = \int_0^t sp(s)\,\mathrm{d}s.$$

Clearly, $\frac{1}{2}p_0 t^2 \le g(t) \le \frac{1}{2}p_1 t^2$. Thus, for any $u \in H^1(\Omega)$,

$$G(u) := \int_\Omega g(|\nabla u|)\,\mathrm{d}\mathbf{x}$$

is bounded. Let

$$\Phi_N(u) = G(u_1) + \int_\Omega \frac{1}{2}|u_1|^2\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_{\Omega_1} |\nabla u_2|^2\,\mathrm{d}\mathbf{x} + \frac{1}{2} < \mathscr{K}_N(u_2), u_2 >_\Gamma - L(u),$$

then the coupled minimization problem is to find $u = (u_1, u_2) \in V$ such that

$$\Phi(u) = \inf_{v \in V} \Phi(v). \tag{16}$$

The approximate minimization problem is to find $u^N = (u_1^N, u_2^N) \in V$ such that

$$\Phi_N(u^N) = \inf_{v \in V} \Phi_N(v). \tag{17}$$

In the following, we refer to Carstensen and Gwinner (1997) and discuss the well-posedness of the problem Eq. 16 and Eq. 17.

Lemma 2 *For any $u \in V$ and $v \in V^*$, the following conclusions hold.*
1). The Gateaux derivative of Φ is

$$D\Phi(u;v) = B(u;v) - L(v). \tag{18}$$

2). $D\Phi$ is strongly monotone and Lipschitz-continuous for the bounded arguments with respect to the norm $\|\cdot\|_V$ [see Ciarlet (1978)].

Proof. For any $u \in V$ and $v \in V^*$, $p(t) \in C^1(\{0\} \cup \mathbb{R}^+)$ implies

$$\lim_{t\to 0} \frac{\Phi(u+tv) - \Phi(u)}{t} = \int_\Omega p(|\nabla u_1|)\nabla u_1 \cdot \nabla v_1\,\mathrm{d}\mathbf{x} + \int_\Omega u_1 v_1\,\mathrm{d}\mathbf{x} + \int_{\Omega_1} \nabla u_2 \cdot \nabla v_2\,\mathrm{d}\mathbf{x}$$
$$+ < \mathscr{K}(u_2), v >_\Gamma - L(v).$$

Since

$$\left|\int_\Omega p(|\nabla u|)\nabla u\cdot\nabla v\,\mathrm{d}\mathbf{x}\right|\le p_1\int_\Omega |\nabla u||\nabla v|\,\mathrm{d}\mathbf{x}\le p_1\|\nabla u\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)},$$

for a fixed u, $D\Phi(u;v)-L(v)$ is a bounded linear functional on V^*. This solves 1). Similarly, we can obtain

$$\begin{aligned} D^2\Phi(u;v;w) \quad &= \quad \int_\Omega\big((\nabla v_1)^{\mathrm{T}}\mathbf{E}(\nabla u_1)\nabla w_1+w_1v_1\big)\,\mathrm{d}\mathbf{x} \\ &\quad +\int_{\Omega_1}\nabla w_2\cdot\nabla v_2\,\mathrm{d}\mathbf{x}+<\mathscr{K}(w_2),v_2>_\Gamma \end{aligned}$$

with the 3×3-unit matrix $I_{3\times 3}$ and

$$\mathbf{E}(\mathbf{t}):=p(t)I_{3\times3}+tp'(t)(sign\,\mathbf{t})\cdot(sign\,\mathbf{t})^{\mathrm{T}}\in\mathbb{R}^{3\times3},$$

where $\mathbf{t}\in\mathbb{R}^3; t:=|\mathbf{t}|$, $sign\mathbf{t}$ is the unit vector with respect to the direction $\mathbf{t}$. Three eigenvalues of the matrix $\mathbf{E}(\mathbf{t})$ are $tp'(t)+p(t)$ and $p(t)$ (double). Since $tp'(t)+p(t)>\alpha>0$ and $p(t)>p_0>0$, $\mathbf{E}(\mathbf{t})$ is symmetric positive definite. Then $D^2\Phi(u;v;w)$ with respect to w is still a bounded linear functional on V^*. For all $u,w\in V$, there is $t_1\in(0,1)$ such that

$$D\Phi(u;u-w)-D\Phi(w;u-w)=D^2\Phi(w+t_1(u-w);u-w;u-w).$$

Thus, Friedrichs inequality and the properties of the matrix $\mathbf{E}(\mathbf{t})$ satisfy

$$\begin{aligned} D\Phi(u;u-w)-D\Phi(w;u-w)&\ge C_1\|u_1-w_1\|^2_{H^1(\Omega)} \\ &+|u_2-w_2|^2_{H^1(\Omega_1)}+C_2\Big(\int_\Gamma(u_2-w_2)\,\mathrm{d}\sigma\Big)^2 \\ &\ge C\left(\|u_1-w_1\|^2_{H^1(\Omega)}+\|u_2-w_2\|^2_{H^1(\Omega_1)}\right) \end{aligned}$$

and for any bounded ball $B(0;M)=\{w\in V:\|w\|_V\le M\}$, we have

$$D\Phi(u;u-w)-D\Phi(w;u-w)\le C\left(\|u_1-w_1\|^2_{H^1(\Omega)}+\|u_2-w_2\|^2_{H^1(\Omega_1)}\right).$$

Theorem 1 *The functional Φ has a unique minimizer on V and the minimizer is just the unique solution of the weak formulation Eq. 14. The functional Φ_N has a unique minimizer on V and the minimizer is just the unique solution of the weak formulation Eq. 15.*

Proof. Given $u_0 \in H^{1/2}(\Gamma_0)$, there exists $u_1 \in H^1(\Omega)$ with $u_1|_{\Gamma_0} = u_0$. Taking $u_2 = 0$ in $H^1(\Omega_1)$, we obtain $u = (u_1, u_2) \in V$ and $\Phi(u) < \infty$, that is to say $V \neq \emptyset$ and if Φ has a minimizer on V, then the minimizer must be finite. Obviously, V is also closed and convex. In the following, we prove that Φ is weakly lower semicontinuous in V. For all $u \in V$, if the sequence $\{u^n\}_1^\infty$ weakly converge to u , then the monotonicity of $D\Phi$ and the Taylor formula of Φ imply that there exists a $t \in (0,1)$, such that

$$\Phi(u^n) - \Phi(u) = D\Phi(u + t(u^n - u); u^n - u) \geq D\Phi(u; u^n - u).$$

Since $D\Phi(u; w)$ with respect to w is a linear bounded functional on V^*, we have $D\Phi(u; u^n - u) \to 0$ and then Φ is weakly lower semicontinuous in V. On the other hand, for any $v = (v_1, v_2) \in V$, Frierichs inequality and the properties of $p(t)$ satisfy

$$\Phi(v) \geq C_1 \|v_1\|^2_{H^1(\Omega)} + C_2 \|v_2\|^2_{H^1(\Omega_1)} - \|f\|_{L^2(\Omega)} \|v_1\|_{L^2(\Omega)} - \|t_0\|_{H^{-1/2}(\Gamma_0)} \|v_2\|_{H^{1/2}(\Gamma_0)}.$$

Thus,

$$\lim_{\substack{\|v\|_V \to \infty \\ v \in V}} \Phi(v) = +\infty.$$

i.e., for given $w \in V$, there exists a positive constant M such that

$$\|v\|_V > M \Rightarrow \Phi(v) \geq \Phi(w), \quad \forall v \in V.$$

According to the concerned conclusions [Ciarlet (1978)], Φ has at most one minimizer on V. Let $u = (u_1, u_2) \in V$ is the minimizer of Φ on V, for any $\pm v \in V^*$, lemma 2 implies that

$$0 \leq \lim_{t \to 0^+} \frac{\Phi(u \pm tv) - \Phi(u)}{t} = \pm(B(u; v) - L(v)).$$

Thus, $u = (u_1, u_2)$ must be the solution of the weak formulation Eq. 14. Conversely, if $u = (u_1, u_2)$ is the solution of the weak formulation Eq. 14, then for all $w \in V$, Gateaux differential of Φ and the strong monotonicity of $D\Phi(w; v)$ with respect to w imply that there exists $t \in (0,1)$ such that

$$\Phi(w) - \Phi(u) = D\Phi(u + t(w - u); w - u) \geq D\Phi(u; w - u) = 0.$$

Hence, u is the minimizer of Φ on V. Finally, we prove the uniqueness. Suppose that u, w are both the solution of the weak formulation Eq. 14, then we have

$$B(u; u - w) - B(w; u - w) = 0.$$

$u = w$ follows from the strong monotonicity of $D\Phi$. This proves that the functional Φ has a unique minimizer on V and the minimizer is just the unique solution of the variational problem Eq. 14. Similarly, we obtain that the functional Φ_N has a unique minimizer on V and the minimizer is just the unique solution of the variational problem Eq. 15.

Let Γ_1 take the spherical surface (R_2,θ,φ), the spheroidal surface (μ_2,θ,φ) or the general ellipsoidal surface $(a_2,\lambda_2,\lambda_3)$ such that $\Gamma_1 \subset \Omega_1$. Set

$$S_n^m(\Gamma_1,\Gamma) = \begin{cases} \dfrac{R_2^{n+1}}{R^{n+1}} & \text{for } \Gamma_1 \text{ is spherical surface,} \\ \dfrac{Q_n^m(\cosh\mu_1)}{Q_n^m(\cosh\mu_2)} & \text{for } \Gamma_1 \text{ is prolate spheroidal surface,} \\ \dfrac{T_n^m(\cosh\mu_1)}{T_n^m(\cosh\mu_2)} & \text{for } \Gamma_1 \text{ is oblate spheroidal surface,} \\ \dfrac{F_n^p(a)}{F_n^p(a_2)} & \text{for } \Gamma_1 \text{ is general ellipsoidal surface.} \end{cases}$$

$$H_n(\Gamma_1,\Gamma) = \begin{cases} \dfrac{R_2^{n+1}}{R^{n+1}} & \text{for } \Gamma_1 \text{ is spherical surface,} \\ \dfrac{(\cosh\mu_2)^{n+1}}{(\cosh\mu_1)^{n+1}} & \text{for } \Gamma_1 \text{ is spheroidal surface,} \\ \left(\dfrac{a\,(a_2^2+k^2)}{a_2\,(a^2+k^2)}\right)^{n+1} & \text{for } \Gamma_1 \text{ general ellipsoidal surface.} \end{cases}$$

$$T_n^m(\Gamma) = \begin{cases} \dfrac{n+1}{R} & \text{for } \Gamma \text{ is spherical surface,} \\ f_0 T_{nm}(\mu_1) & \text{for } \Gamma \text{ is spheroidal surface,} \\ -bc\dfrac{\mathrm{d}F_n^p(a)/\mathrm{d}\lambda}{F_n^p(a)} & \text{for } \Gamma \text{ is general ellipsoidal surface.} \end{cases}$$

Theorem 2 *Assume that $u = (u_1,u_2)$ and $u^N = (u_1^N,u_2^N)$ are the solutions of the weak formulation Eq. 14 and Eq. 15 respectively. If $u \in H^{3/2}(\Gamma_1)$, then there exists a positive constant C independent of the term N after truncating the series such that*

$$\|u-u^N\|_{H^1(\Omega_1)} \le \frac{C}{N+2} H_{N+1}(\Gamma_1,\Gamma)\|u\|_{H^{3/2}(\Gamma_1)}. \tag{19}$$

Proof. Because $u = (u_1, u_2)$ and $u^N = (u_1^N, u_2^N)$ are the solutions of Eq. 14 and Eq. 15 respectively, for any $v \in V^*$, we have

$$B(u;v) - B_N(u^N;v) = 0.$$

From lemma 2 and lemma 1, there exists a positive constant C_1 independent of N such that

$$\begin{aligned} C_1\|u-u^N\|_V^2 &\leq B_N(u;u-u^N) - B_N(u^N;u-u^N) \\ &= B_N(u;u-u^N) - B(u;u-u^N) \\ &= <\mathscr{K}_N(u) - \mathscr{K}(u), u-u^N>_\Gamma . \end{aligned}$$

Let F_{nm}, U_{nm} and P_{nm} are the generalized Fourier coefficient of $(u-u^N)|_\Gamma$, $u|_\Gamma$ and $u|_{\Gamma_1}$, respectively, then we have

$$\begin{aligned} &<\mathscr{K}_N(u), u-u^N>_\Gamma - <\mathscr{K}(u), u-u^N>_\Gamma \\ &\quad \leq \sum_{n=N+1}^{\infty} \sum_{m=-n}^{n} T_n^m(\Gamma)|U_{nm}||F_{nm}| \\ &\quad \leq C\|u-u^N\|_{H^{1/2}(\Gamma)} \left(\sum_{n=N+1}^{\infty} \sum_{m=-n}^{n} (n+1)|U_{nm}|^2 \right)^{\frac{1}{2}} \end{aligned}$$

The formula Eq. 7, Eq. 9, Eq. 10,

$$U_{nm} = S_n^m(\Gamma_1, \Gamma)P_{nm}.$$

Therefore,

$$\begin{aligned} \|u-u^N\|_V &\leq C \left(\sum_{n=N+1}^{\infty} \sum_{m=-n}^{n} (n+1)|U_{nm}|^2 \right)^{\frac{1}{2}} \\ &\leq C \left(\sum_{n=N+1}^{\infty} \sum_{m=-n}^{n} \frac{(n+1)^3}{(N+2)^2} (S_n^m\Gamma_1, \Gamma)^2 |P_{nm}|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{C}{N+2} H_{N+1}(\Gamma_1, \Gamma)\|u\|_{H^{3/2}(\Gamma_1)}. \end{aligned}$$

Here, C is independent of N.

4 Discrete problem and error estimate

To describe a discrete formulation of Eq. 16 and Eq. 17, we divide Ω and Ω_1 into quasi-uniform regular tetrahedral meshes with mesh size h, such that the nodes

on Γ_0 are matching (i.e., coincident) and these tetrahedrons nearby Γ are curved. The conforming linear finite element spaces associated with Ω and Ω_1 are denoted by $V_h(\Omega)$ and $V_h(\Omega_1)$ respectively. Usually, the curved tetrahedrons are approximated by the straight edge tetrahedrons which have the same nodes as the curved tetrahedrons. This approximate only generates small error. Letting $\mathscr{N}_h$ denote the set of nodes in the domain $\Omega \cup \Omega_1 \cup \partial\Omega \cup \partial\Omega_1$, we let $U_h = \{v^h = (v_1^h, v_2^h) \in V_h(\Omega) \times V_h(\Omega_1) : \forall \mathbf{b} \in \mathscr{N}_h \cap \Gamma_0, v_1^h(\mathbf{b}) = v_2^h(\mathbf{b}) + u_0(\mathbf{b})\}$, $U_h^* = \{v^h = (v_1^h, v_2^h) \in V_h(\Omega) \times V_h(\Omega_1) : \forall \mathbf{b} \in \mathscr{N}_h \cap \Gamma_0, v_1^h(\mathbf{b}) = v_2^h(\mathbf{b})\}$. Then, we have $U_h^* \subset V^*$.
The discrete formulation of the problem Eq. 14 is to find $u^h = (u_1^h, u_2^h) \in U_h$ such that

$$B(u^h; v^h) = L(v^h), \quad \forall v^h \in U_h^*. \tag{20}$$

The discrete formulation of the problem Eq. 15 is to find $u^{Nh} = (u_1^{Nh}, u_2^{Nh}) \in U_h$ such that

$$B_N(u^{Nh}; v^h) = L(v^h), \quad \forall v^h \in U_h^*. \tag{21}$$

Theorem 3 *The discrete problem Eq. 20 and Eq. 21 both exist a unique solution.*

The proof is similar with the proof of theorem 1.
Suppose that the interpolation operator $\Pi_h : H^2(\Omega \cup \Omega_1 \cup \Gamma_0) \mapsto U_h$ such that interpolation error

$$|v - \Pi_h v|_{H^1(\Omega)} \le Ch|v|_{H^2(\Omega)}, \quad \forall v \in H^2(\Omega)$$

and

$$|v - \Pi_h v|_{H^1(\Omega_1)} \le Ch|v|_{H^2(\Omega_1)}, \quad \forall v \in H^2(\Omega_1).$$

Theorem 4 *Assume that u and u^{Nh} are the solution of the problem Eq. 14 and Eq. 21 respectively, and $u \in H^2(\Omega) \times H^2(\Omega_1)$ and $u|_{\Gamma_1} \in H^{3/2}(\Gamma_1)$, then there exists a positive constant C independent of the mesh size h and the term N after truncating the series such that*

$$\|u - u^{Nh}\|_V \le C\left(h\|u\|_{H^2(\Omega)\times H^2(\Omega_1)} + \left(\frac{1}{N+2} H_{N+1}(\Gamma_1, \Gamma)\|u\|_{H^{3/2}(\Gamma_1)}\right)\right).$$

Proof. Let $u^N = (u_1^N, u_2^N)$ is the solution of the problem Eq. 15. Owing to $U_h^* \in V^*$, we have

$$B_N(u^N; v^h) - B_N(u^{Nh}; v^h) = 0, \quad \forall v^h \in U_h^*.$$

For any $w^h \in U_h$, $w^h - u^{Nh} \in U_h^*$. The strong monotonicity and Lipschitz continuity about the bounded variable of $D\Phi$ satisfy

$$\begin{aligned} C_1\|u^N - u^{Nh}\|_V^2 &\leq B_N(u^N - u^{Nh}; u^N - u^{Nh}) \\ &= B_N(u^N - u^{Nh}; u^N - w^h) \\ &\leq C_2\|u^N - u^{Nh}\|_V \|u^N - w^h\|_V \end{aligned}$$

Namely,

$$\|u^N - u^{Nh}\|_V \leq C \inf_{w^h \in U_h} \|u^N - w^h\|_V.$$

Thus, theorem 2 implies that

$$\begin{aligned} \|u - u^{Nh}\|_V &\leq \|u - u^N\|_V + \|u^N - u^{Nh}\|_V \\ &\quad \leq \|u - u^N\|_V + C\|u^N - \Pi_h u\|_V \\ &\leq (1+C)\|u - u^N\|_V + C\|u - \Pi_h u\|_V \\ &\leq C\left(h\|u\|_{H^2(\Omega)\times H^2(\Omega_1)} + \frac{1}{N+2} H_{N+1}(\Gamma_1, \Gamma)\|u\|_{H^{3/2}(\Gamma_1)}\right). \end{aligned}$$

Remark. The above error estimate indicates that $N_{opt} = \ln(h)/\ln(\cosh\mu_2/\cosh\mu_1) - 2$ may be chosen as the optimal truncation term with respect to the norm in H^1, when the mesh size is fixed.

5 Numerical examples

In numerical computation, we first do harmonic extension of u_0 to Ω_1. Let $v_0^h \in V_h(\Omega_1)$ be the weak solution of the following problem

$$\begin{cases} \Delta v_0^h = 0, & \text{in } \Omega_1, \\ v_0^h = u_0, & \text{on } \Gamma_0, \\ v_0^h = 0, & \text{on } \Gamma. \end{cases}$$

Set

$$\tilde{L}(v^h) = L(v^h) + \int_{\Omega_1} \nabla v_2^h \cdot \nabla v_0^h \,\mathrm{d}\mathbf{x}.$$

Clearly, $\tilde{L}(v^h)$ is still the bounded linear functional on U_h^*. Thus, solve numerically the following problem: find $w^{Nh} = (u_1^{Nh}, w_2^{Nh}) \in U^*$ such that

$$B_N(w^{Nh}; v^h) = \tilde{L}(v^h), \quad \forall v^h \in U^*. \tag{22}$$

If $w^{Nh} = (u_1^{Nh}, w_2^{Nh})$ is the unique solution of the problem Eq. 22, then $u^{Nh} = (u_1^{Nh}, w_2^{Nh} - v_0^h)$ must be the unique solution of the problem Eq. 21. Conversely, the conclusion is also hold.

To solve the nonlinear problem Eq. 22, we apply Newton iteration. Let $e_1(h,N) = \|u^{Nh} - u\|_{H^1(\Omega_s)}$, $e_0(h,N) = \|u^{Nh} - u\|_{L^2(\Omega_s)}$, $e_\infty(h,N) = \|u^{Nh} - u\|_{L^\infty(\Omega_s)}$. Here, if $s = 0$, the domain Ω_s denotes Ω; if $s = 1$, the domain Ω_s denotes Ω_1. $iters$, $nodes$ and $tetra$ denote the times of Newton iteration, the total number of the nodes in Ω_s and the total number of the tetrahedrons elements, respectively. $Error$ in Fig. 8 and Fig. 14 denotes the norm of $u^{Nh} - u^{50h}$ in three spaces as the mesh size is fixed while $Error$ in Fig. 19 denotes the norm of $u^{Nh} - u^{20h}$ in three spaces. In other figures, $Error$ denotes the norm of $u - u^{Nh}$. According to the special shape of the inner domain Ω, we apply the different artificial boundary, e.g., a spherical, prolate spheroidal, oblate spheroidal and general ellipsoidal surface.

Example 1 *Let* $\Omega = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : r = \sqrt{x_1^2 + x_2^2 + x_3^2} < 1\}$*, the artificial boundary* $\Gamma = \partial\Omega$*,* $u_0 = 0$ *and* $t_0 = 4r + 2r/(1+2r) + 1/r^2$*. The exact solution of the problem Eq. 1-Eq. 4 is* $u = (r^2, 1/r)$*. The corresponding results are the case as Tab. 1.*

Table 1: Convergence results on quasi-uniform mesh, $N = 5$.

h	$e_1(h,N)$	$e_0(h,N)$	$e_\infty(h,N)$	iters
0.5077	6.2884e-1	3.3593e-1	2.7286e-1	3
0.2239	2.0490e-1	1.0390e-1	7.1050e-2	5
0.1060	5.6080e-2	2.8562e-2	1.8400e-2	7
0.0516	1.4437e-2	7.3060e-3	5.1237e-3	7

Example 2 *Let* $\Omega = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : |x_i| < 1, i = 1,2,3\}$*, the artificial boundary* $\Gamma = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : r = \sqrt{x_1^2 + x_2^2 + x_3^2} = 2\}$*,* $p(t) = 1 + e^{-t^2}$*,* $f = e^{-r^2}(2r^2 - 3) - 3 + r^2/2$*,* $u_0 = r^2/2 - 1/r$ *and*

$$t_0 = \begin{cases} |x_1|(1 + e^{-r^2} + 1/r^3), & |x_1| = 1, \\ |x_2|(1 + e^{-r^2} + 1/r^3), & |x_2| = 1, \\ |x_3|(1 + e^{-r^2} + 1/r^3), & |x_3| = 1. \end{cases}$$

The exact solution of the problem Eq. 1-Eq. 4 is $u = (r^2/2, 1/r)$*. The concerned results are the case as Tab. 2 and Fig. 2-3.*

Table 2: Convergence results on quasi-uniform mesh, $N = 50$

h	domain	$e_1(h,N)$	$e_0(h,N)$	$e_\infty(h,N)$	iters
0.5000	Ω	2.3313e-1	4.4980e-2	5.6569e-2	4
0.2500	Ω	1.1038e-1	1.5017e-2	2.9795e-2	5
0.1250	Ω	4.8815e-2	4.6182e-3	1.7115e-2	5
0.4050	Ω_1	1.9565e-1	4.9411e-2	4.6950e-2	4
0.2025	Ω_1	9.7283e-2	1.5609e-2	2.9795e-2	5
0.1013	Ω_1	4.7756e-2	5.6206e-3	1.7115e-2	5

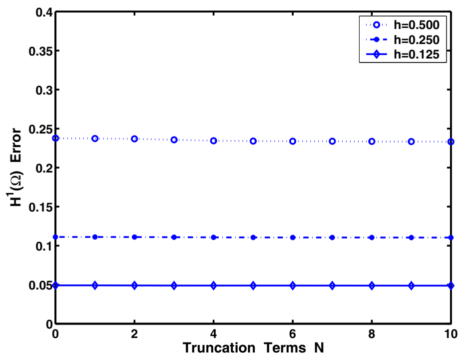

Figure 2: the relation between mesh size, error in $H^1(\Omega)$ and N.

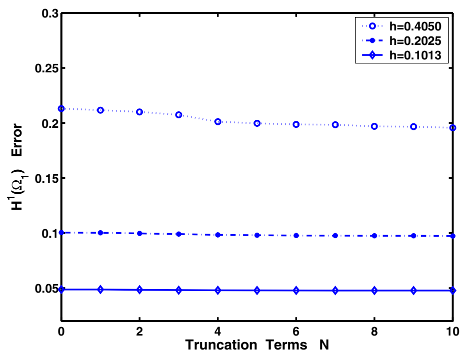

Figure 3: the relation between mesh size, error in $H^1(\Omega_1)$ and N.

Example 3 *Let* $\Omega = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : |x_1| < 1, |x_2| < 1, |x_3| < 3\}$, $p(t) = 1 + e^{-t^2}$, $f = e^{-r^2}(2r^2 - 3) - 3 + r^2/2$, $u_0 = r^2/2 - 1/r$ *and*

$$t_0 = \begin{cases} |x_1|(1+e^{-r^2}+1/r^3), & |x_1| = 1, \\ |x_2|(1+e^{-r^2}+1/r^3), & |x_2| = 1, \\ |x_3|(1+e^{-r^2}+1/r^3), & |x_3| = 3. \end{cases}$$

The exact solution of the problem Eq. 1-Eq. 4 is $u = (r^2/2, 1/r)$. *Owing to the shape of* Ω,

$$\Gamma = \left\{ (x_1,x_2,x_3) : x_1^2 + x_2^2 + \frac{x_3^2}{9} = a^2, a \geq \sqrt{3} \right\}$$

is chosen as the artificial boundary. The concerned results are the case as Tab. 3 and Fig. 4-9.

Table 3: Convergence results on quasi-uniform mesh, $N = 30$, $a = 2$.

h	domain	$e_1(h,N)$	$e_0(h,N)$	$e_\infty(h,N)$	iters
0.5	Ω	3.3794e-1	5.9625e-2	7.1325e-2	5
0.25	Ω	1.2295e-1	1.7618e-2	3.2277e-2	5
0.1667	Ω	6.7098e-2	8.3863e-3	1.8281e-2	5
0.7368	Ω_1	2.7427e-1	4.8479e-2	5.0519e-2	5
0.3889	Ω_1	1.0689e-1	1.6062e-2	2.2889e-2	5
0.2623	Ω_1	6.3718e-2	8.6362e-3	1.5641e-2	5

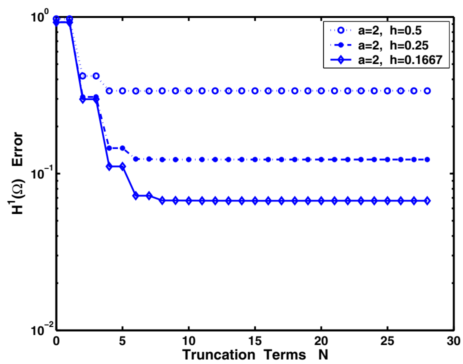

Figure 4: the relation between mesh size, error in $H^1(\Omega)$ and N.

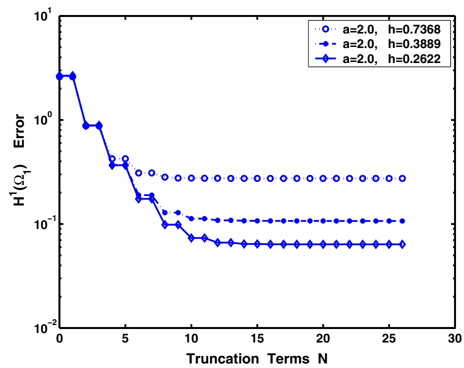

Figure 5: the relation between mesh size, error in $H^1(\Omega_1)$ and N.

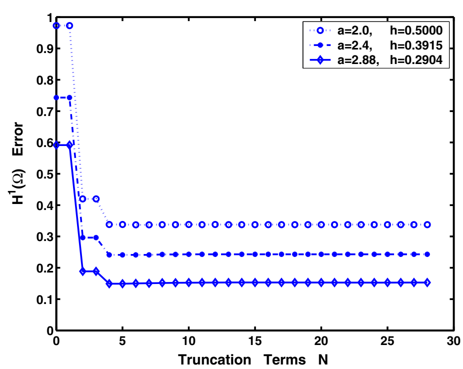

Figure 6: the relation between the location of Γ, error in $H^1(\Omega)$ and N.

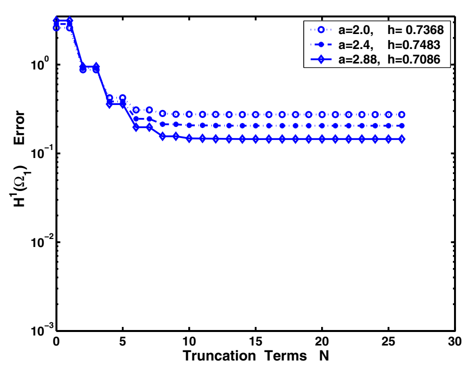

Figure 7: the relation between the location of Γ, error in $H^1(\Omega_1)$ and N.

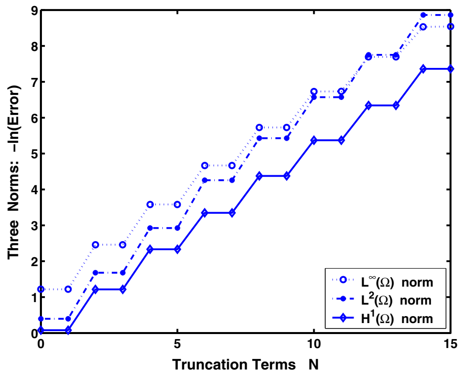

Figure 8: the relation between N and error in Ω, $h = 0.1667$, $a = 2$.

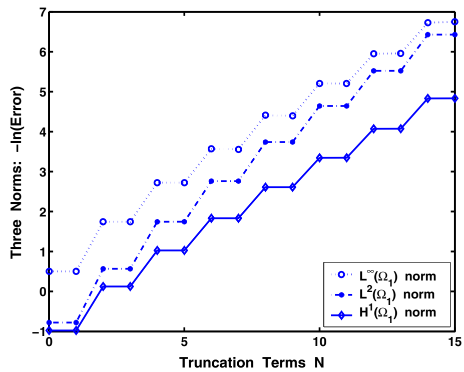

Figure 9: the relation between N and error in Ω_1, $h = 0.2622$, $a = 2$.

Example 4 *Let* $\Omega = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : |x_1| < 3, |x_2| < 3, |x_3| < 1\}$, $p(t) = 1 + e^{-t^2}$, $f = e^{-r^2}(2r^2 - 3) - 3 + r^2/2$, $u_0 = r^2/2 - 1/r$ *and*

$$t_0 = \begin{cases} |x_1|(1+e^{-r^2}+1/r^3), & |x_1| = 3, \\ |x_2|(1+e^{-r^2}+1/r^3), & |x_2| = 3, \\ |x_3|(1+e^{-r^2}+1/r^3), & |x_3| = 1. \end{cases}$$

The exact solution of the original problem is $u = (r^2/2, 1/r)$. *Owing to the shape of* Ω,

$$\Gamma = \left\{ (x_1,x_2,x_3) : \frac{x_1^2}{9} + \frac{x_2^2}{9} + x_3^2 = a^2, a \geq \sqrt{3} \right\}$$

is chosen as the artificial boundary. The corresponding results are the case as Tab. 4 and Fig. 10-15.

Example 5 *Let* $\Omega = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : |x_1| < 2.5, |x_2| < 2.0, |x_3| < 1.5\}$, $p(t) = 1 + e^{-t^2}$, $f = e^{-r^2}(2r^2 - 3) - 3 + r^2/2$, $u_0 = r^2/2 - 1/r$ *and*

$$t_0 = \begin{cases} |x_1|(1+e^{-r^2}+1/r^3), & |x_1| = 2.5, \\ |x_2|(1+e^{-r^2}+1/r^3), & |x_2| = 2.0, \\ |x_3|(1+e^{-r^2}+1/r^3), & |x_3| = 1.5. \end{cases}$$

The exact solution of the original problem is $u = (r^2/2, 1/r)$. *A general ellipsoidal surface*

$$\Gamma = \left\{ (x_1,x_2,x_3) : \frac{x_1^2}{6.25} + \frac{x_2^2}{4} + \frac{x_3^2}{2.25} = a^2, a \geq \sqrt{3} \right\}$$

Table 4: Convergence results on quasi-uniform mesh, $N = 30$, $a = 2$.

h	domain	$e_1(h,N)$	$e_0(h,N)$	$e_\infty(h,N)$	iters
1.0	Ω	1.0896	3.1240e-1	1.7754e-1	5
0.5	Ω	7.7651e-1	1.4491e-1	1.4368e-1	5
0.25	Ω	4.0698e-1	5.5045e-2	7.6554e-2	5
1.3896	Ω_1	1.32701	4.5018e-1	1.7754e-1	5
0.7612	Ω_1	7.4871e-1	1.8982e-1	1.4368e-1	5
0.3924	Ω_1	3.93731e-1	7.8092e-2	7.6554e-2	5

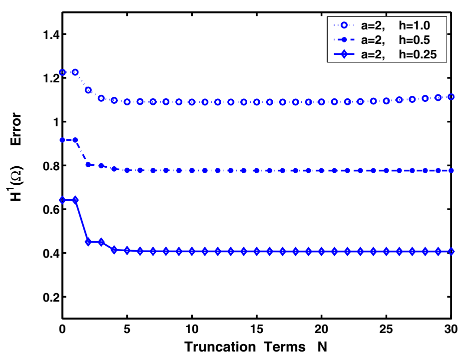

Figure 10: the relation between mesh size, error in $H^1(\Omega)$ and N.

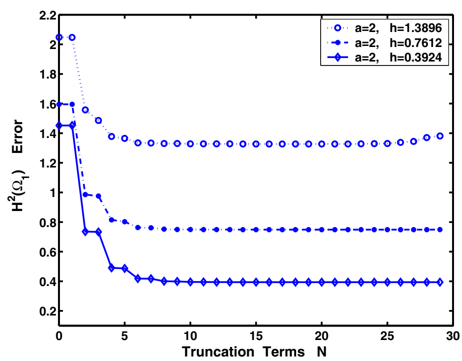

Figure 11: the relation between mesh size, error in $H^1(\Omega_1)$ and N.

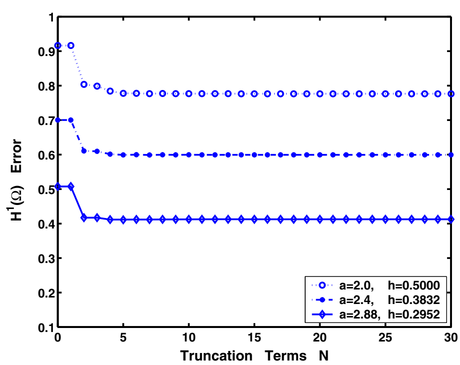

Figure 12: the relation between the location of Γ, error in $H^1(\Omega)$ and N.

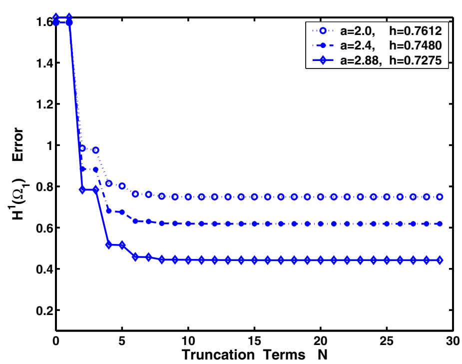

Figure 13: the relation between the location of Γ, error in $H^1(\Omega_1)$ and N.

is chosen as the artificial boundary. The concerned results are the case as Tab. 5 and Fig. 16-19. $a = 2.0$.

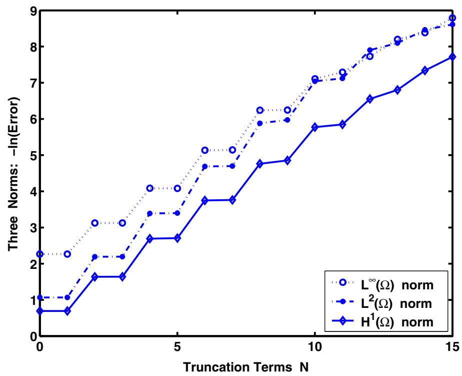

Figure 14: the relation between N and error in Ω, $h = 0.2500$,$a = 2$.

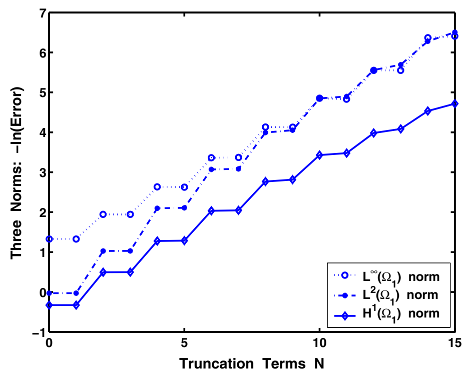

Figure 15: the relation between N and error in Ω_1, $h = 0.3924$,$a = 2$.

Table 5: Convergence results on quasi-uniform mesh, $N = 20$, $a = 2$.

h	domain	$e_1(h,N)$	$e_0(h,N)$	$e_\infty(h,N)$	iters
1.00	Ω	1.1571	3.2559e-1	1.6243e-1	5
0.50	Ω	5.9577e-1	1.2818e-1	1.0490e-1	5
0.25	Ω	2.9484e-1	4.5164e-2	6.6466e-2	5
1.3449	Ω_1	9.8758e-1	3.3065e-1	1.6243e-01	5
0.7265	Ω_1	5.3383e-1	1.3750e-1	1.0490e-1	5
0.3708	Ω_1	2.8247e-1	5.8797e-2	6.6466e-2	5

6 Conclusions

In this paper we apply the artificial boundary method to solve a three-dimensional nonlinear interface problem on an unbounded domain. According to the shape of the inner boundary Γ_0 of the domain, we can use some different artificial boundaries, e.g., a spherical, spheroidal or ellipsoidal surface, in order to reduce the unbounded domain into a small computational bounded region Ω_1. The DtN mapping on the artificial boundary are presented, it is just the exact condition on the outer boundary of the reduced domain Ω_1. The well-posedness of the coupled weak formulation is proved, and the error estimate is given. The error not only depends on the mesh size, but also on the number of term after truncating the series and the location of the artificial boundary. Finally, all of the numerical examples given in Sect. 5 show us that the method is effective, and we have following conclusions:

1. From Fig. 2-5, Fig. 10, Fig. 11, Fig. 16 and Fig. 17 we can see that for

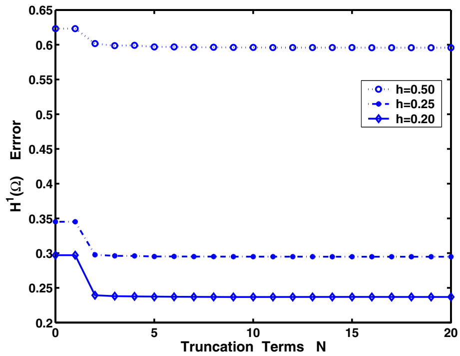

Figure 16: the relation between mesh size, error in $H^1(\Omega)$ and N.

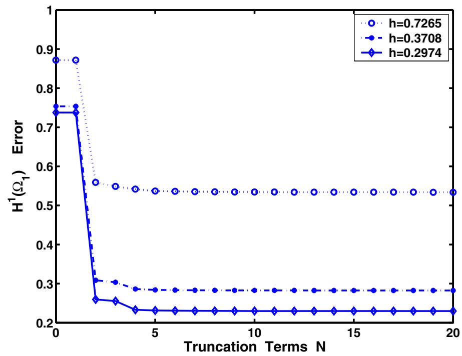

Figure 17: the relation between mesh size, error in $H^1(\Omega_1)$ and N.

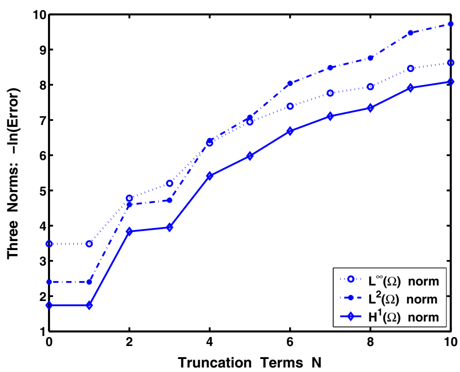

Figure 18: the relation between N and error in Ω, $h = 0.2500$.

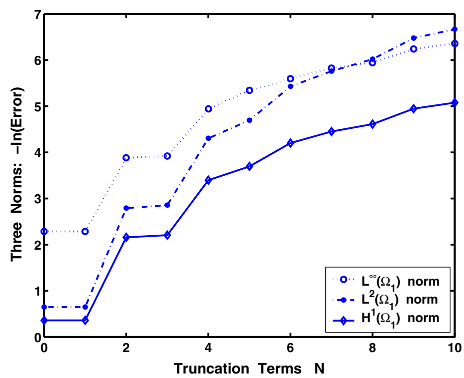

Figure 19: the relation between N and error in Ω_1, $h = 0.3708$.

different mesh size h, after the truncation term N increases to a certain value (pertaining to the mesh size h), the error of approximate solution with respect to the norms in $H^1(\Omega)$ and $H^1(\Omega_1)$ changes very small. It has shown that the error mainly comes from the mesh size when the truncation term has already amounted to this value.

2. Tab. 1-5 all have indicated that when the truncation term arrive at a certain value, e.g., $N = 30$ or 20, the convergent order of $||u - u^{Nh}||_{H^1(\Omega_s)}$ with respect to h is approximate to 1, while the convergent order of $||u - u^{Nh}||_{L^2(\Omega_1)}$ with respect to h is greater than 1.5.

3. From Fig. 6, Fig. 7, Fig. 12 and Fig. 13 we see that the error depends on

the location of the artificial boundary. When the mesh size and the truncation term N are fixed, the error become smaller as the distant between the artificial boundary and the inner boundary Γ_0 is farther. However, in order to reduce the error, it is inadvisable to increase the distant between the artificial boundary and the inner boundary Γ_0 because the computational cost resulted from the larger computational domain will become higher.

4. Fig. 8, Fig. 9, Fig. 14, Fig. 15, Fig. 18 and Fig. 19 all have illustrated that when the mesh size is given, the relation between the logarithm of the error of $u^{Nh} - u^{20h}$ with respect to three norms and the truncation term N is approximate to a line. These have shown that the error of approximate solution resulted from the truncation term is approximately an exponential function, whose base is less than 1 and N is the exponential. It is also consistent with the theoretical analysis in Sect. 4.

Of course, there are also some other methods developed in recent years for solving exterior problems. We can see the book by Ying (2006) where the numerical methods for exterior problems, including both traditional and novel methods, are introduced comprehensively. Moreover, we can see many papers, for examples, the papers by Yang, Chen and Liang (1995), Chen (2002), Chen (2007), where a dual boundary integral method and a semi-analytical approach are applied for exterior problems.

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