

A Novel Fictitious Time Integration Method for Solving the Discretized Inverse Sturm-Liouville Problems, For Specified Eigenvalues

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Abstract: The inverse Sturm-Liouville problem finds its applications in the identification of mechanical properties and/or geometrical configurations of a vibrating continuous medium; however, this problem is hard to solve, either theoretically or numerically. Previously, Liu (2008a) has constructed a Lie-group shooting method to determine the eigenvalues, and the corresponding eigenfunctions, for the direct Sturm-Liouville problem. In this study, we are concerned with solving the inverse Sturm-Liouville problem, by developing a Lie-group of $SL(2, \mathbb{R})$ to construct non-linear algebraic equations (NAEs), when discrete eigenvalues are specified. Our purpose here is to use these NAEs to solve the unknown function in the Sturm-Liouville operator. Then, we use a fictitious time integration method (FTIM) developed by Liu and Atluri (2008), to find the potential function, impedance function or weighting function, in a discretized manner. Numerical examples are presented to show that the Lie-group and FTIM methods have a significantly improved accuracy, along with ease of numerical implementation. The numerical examples also include the inverse problem of determining the material properties and cross-sectional area of a tapered rod undergoing axial vibrations, when the eigen-frequencies are specified.

Keyword: Inverse Sturm-Liouville problem, Eigenvalues, Eigenfunctions, Lie-group method, Lie-group shooting method (LGSM), Fictitious time integration method (FTIM), Inverse problem of a vibrating rod for specified frequencies

1 Introduction

The Sturm-Liouville eigenvalue problem has been of considerable physical interest for a long time. It is important in many engineering and scientific fields, including

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partial differential equations, vibrations of continuum solids, and quantum mechanics as described by the Schrödinger equations.

In most cases, it is not easy to obtain the eigenvalues and eigenfunctions of the Sturm-Liouville problem, analytically. However, there are various numerical methods to approximately determine the eigenvalues and eigenfunctions. Pryce (1993) has provided a comprehensive review of the mathematical background of the Sturm-Liouville problems and their numerical solutions, as well as a detailed discussion of their applications. He summarized the examples of Sturm-Liouville problems that have been considered by numerous authors.

There is a continued interest in the numerical solution of Sturm-Liouville problems with the aim of improving the convergence and ease of implementation of different numerical algorithms. In order to obtain more accurate numerical results efficiently, several numerical methods have been developed, e.g., Andrew (1994, 2000a, 2000b), Andrew and Paine (1985, 1986), Celik (2005a, 2005b), Celik and Gokmen (2005), Condon (1999), Ghelardoni (1997), Ghelardoni, Gheri and Marletta (2001, 2006), Vanden Berghe and De Meyer (1991, 2007), Vanden Berghe, Van Daele and De Meyer (1995), and Yücel (2006). Ghelardoni and Gheri (2001) have discussed a shooting technique for computing the eigenvalues. A very effective modification of the shooting technique is the Lie-group shooting method (LGSM) as suggested recently by Liu (2008a). This technique is applicable for regular as well as singular Sturm-Liouville eigenvalue problems.

In this paper we propose a novel method for computing the potential function $q(x)$, the impedance function $p(x)$, or the weighting function $s(x)$ in the following inverse Sturm-Liouville problem:

$$-\frac{d}{dx} \left[p(x) \frac{dy(x)}{dx} \right] + q(x)y(x) = \lambda s(x)y(x), \quad x_0 < x < x_f, \quad (1)$$

$$y(x_0) = 0, \quad y(x_f) = 0. \quad (2)$$

The inverse problem here is, that for some specified eigenvalues λ , we need to determine the potential function $q(x)$, the impedance function $p(x)$, or the weighting function $s(x)$. Here, we suppose that $p(x)$, $q(x)$, and $s(x)$ are continuous, with $p(x)$ and $s(x)$ being strictly positive. When x_0 and x_f are finite, the Sturm-Liouville problem is regular; otherwise, it is singular.

To motivate the present study, we consider the longitudinal wave motion of a one-dimensional rod with variable Young's modulus $E(x)$ and cross-sectional area $A(x)$:

$$\frac{1}{A(x)} \frac{\partial}{\partial x} \left(E(x)A(x) \frac{\partial u(x,t)}{\partial x} \right) = \rho(x) \frac{\partial^2 u(x,t)}{\partial t^2}, \quad (3)$$

where $\rho(x)$ is the variable mass-density, and $u(x,t)$ is the displacement.

Let $u(x, t) = e^{i\omega t} y(x)$. Eq. (3) can be simplified as

$$-\frac{d}{dx} \left(E(x)A(x) \frac{dy(x)}{dx} \right) = \rho(x)A(x)\omega^2 y(x), \quad (4)$$

where ω is the vibrational frequency. Upon comparing with Eq. (1) we can see that $p(x) = E(x)A(x)$, $s(x) = \rho(x)A(x)$, $q(x) = 0$ and $\lambda = \omega^2$. In the inverse problem, it is technically important to identify the material properties $E(x)$, $\rho(x)$ and the geometric variable $A(x)$ for some specified frequencies ω of the vibrating rod. This problem is known as an inverse problem of a vibrating rod for specified frequencies. Our proposed method can solve this sort of an inverse problem.

The above problem is known as an inverse Sturm-Liouville problem or in short, inverse eigenvalues problem [Hald (1984); McLaughlin (1986); Chu (1998)]. Many algorithms were developed for solving the inverse Sturm-Liouville problem of reconstructing the potential function from the specified eigenvalues [Andrew (2004); Paine (1984)]. Some have been developed analytically. Among them, the best known one is studied by Gel'fand and Levitan (1951), wherein the potential function was determined uniquely by a spectral function. McLaughlin (1986) has given an analytical method to treat this type of an inverse problem.

On the other hand, McLaughlin (1988) first noted that it is possible to obtain the potential function and boundary conditions, using only a set of nodal points. This interesting problem soon became known as the inverse nodal problem [Browne and Sleeman (1996); Cheng, Law and Tsay (2000); Hald and McLaughlin (1989); Yang (1997)].

Discretization methods often transform the inverse Sturm-Liouville problem into an inverse eigenvalue problem of a certain matrix [Boley and Golub (1987)]. However, many of these discretizations resulting in a matrix form lead to higher eigenvalues which are significantly different from the true eigenvalues of the continuous system. As a consequence, the inverse algorithms based on those discretizations require careful implementation [Andrew (2004); Paine (1984)]. Previously, Liu (2008b) has used the specified data on eigenfunctions in order to identify the potential function. In this study, the following inverse Sturm-Liouville problem is considered. Some specified data on eigenvalues is to be used to reconstruct the potential function $q(x)$, the impedance function $p(x)$, or the weighting function $s(x)$. The present inverse problem is more difficult than that studied in Liu (2008b).

It would be obvious that the reconstruction algorithm in this paper appears to be more efficient than the algorithms for approximate solutions given by Rundell and Sacks (1992), Knobel and Lowe (1993), and Neher (1994, 1998).

This paper is organized as follows. In Section 2 we give a brief description of the one-step group preserving scheme, which includes a shooting method for a

second-order boundary value problem, the group preserving scheme (GPS), a generalized mid-point rule to construct a one-step GPS, and a one-step mapping of GPS between two points on the cone. In Section 3 we introduce our main tool of the Lie-group shooting method (LGSM), and its use in deriving the unknown slope is given in Section 4. For later use, we demonstrate a technique of calculating the eigenvalues in Section 5. The main results are given in Section 6, where we develop the Lie-group and fictitious time integration methods to determine the potential function or other function in the Sturm-Liouville operator for specified eigenvalues, and the numerical examples are given in Section 7. The numerical examples also include the physical problem of determining the material properties and/or cross-sectional area of a tapered rod undergoing axial vibrations, for a prescribed set of eigen-frequencies. Finally, we draw some conclusions in Section 8, wherein a flow-chart of the present algorithm for the inverse Sturm-Liouville problem is also presented.

2 One-step group preserving scheme

2.1 Shooting method

By letting

$$x = x_0 + (x_f - x_0)t, \quad (5)$$

$$u(t) = y(x) + t(1-t) + c, \quad (6)$$

where c is a given positive constant, we can transform Eqs. (1) and (2) into an equivalent system:

$$\ddot{u}(t) = -\frac{\dot{p}(t)}{p(t)}[\dot{u}(t) + 2t - 1] + \frac{(x_f - x_0)^2}{p(t)}[q(t) - \lambda s(t)][u(t) - t(1-t) - c] - 2, \quad (7)$$

$$u(0) = c, \quad u(1) = c. \quad (8)$$

In above $p(t)$, $q(t)$ and $s(t)$ should be understood as $p(x(t))$, $q(x(t))$ and $s(x(t))$.

The time stepping techniques developed for initial value problems (IVPs) require both the initial conditions of $u_1 = u$ and $u_2 = \dot{u}$ for the second-order ODEs. If the initial value of u_2 is also available, then we can numerically integrate the following IVP step-by-step in a forward direction from $t = 0$ to $t = 1$:

$$\dot{u}_1 = u_2, \quad (9)$$

$$\dot{u}_2 = f(t, u_1, u_2), \quad (10)$$

$$u_1(0) = c, \quad (11)$$

$$u_2(0) = A, \quad (12)$$

where

$$f(t, u_1, u_2) := -\frac{\dot{p}(t)}{p(t)}[u_2(t) + 2t - 1] + \frac{(x_f - x_0)^2}{p(t)}[q(t) - \lambda s(t)][u_1(t) - t(1 - t) - c] - 2. \quad (13)$$

The shooting technique involves in simply finding a suitable A , such that the solution of $u(t) = u_1(t)$ can also match the right-end boundary condition of $u(1) = u_1(1) = c$.

We will use the group preserving scheme (GPS) developed by Liu (2001) for the integration of IVPs. The GPS method is very effective in dealing with ordinary differential equations (ODEs), endowing them with special structures as shown by Liu (2005) for stiff equations, and by Liu (2006a) for ODEs with constraints. Liu (2006b, 2006c, 2006d) has extended the GPS for ODEs to solve the boundary value problems (BVPs), and the numerical results reveal that the Lie-group shooting method is a rather promising method to effectively solve the two-point BVPs. In the construction of the Lie-group method for the solutions of BVPs, Liu (2006b) has introduced the idea of one-step GPS by utilizing the closure property of the Lie group, and hence, the new shooting method has been named the Lie-group shooting method. Then, Liu (2008a) has modified the Lie-group shooting method to solve the Sturm-Liouville problems.

2.2 The GPS

We rewrite Eqs. (9) and (10) in a vector form:

$$\dot{\mathbf{u}} = \mathbf{f}(t, \mathbf{u}), \quad (14)$$

where

$$\mathbf{u} := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \mathbf{f} := \begin{bmatrix} u_2 \\ f(t, u_1, u_2) \end{bmatrix}. \quad (15)$$

Liu (2001) has embedded Eq. (14) into an augmented differential equations system:

$$\dot{\mathbf{X}} := \frac{d}{dt} \begin{bmatrix} \mathbf{u} \\ \|\mathbf{u}\| \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{2 \times 2} & \frac{\mathbf{f}(t, \mathbf{u})}{\|\mathbf{u}\|} \\ \frac{\mathbf{f}^T(t, \mathbf{u})}{\|\mathbf{u}\|} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \|\mathbf{u}\| \end{bmatrix} := \mathbf{A}\mathbf{X}, \quad (16)$$

where \mathbf{A} is an element of the Lie algebra $so(2, 1)$ satisfying

$$\mathbf{A}^T \mathbf{g} + \mathbf{g}\mathbf{A} = \mathbf{0} \quad (17)$$

with

$$\mathbf{g} = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_{2 \times 1} \\ \mathbf{0}_{1 \times 2} & -1 \end{bmatrix} \tag{18}$$

being a Minkowski metric. Here, \mathbf{I}_2 is the identity matrix, and the superscript τ stands for the transpose.

The augmented variable \mathbf{X} satisfies the cone condition:

$$\mathbf{X}^T \mathbf{g} \mathbf{X} = \mathbf{u} \cdot \mathbf{u} - \|\mathbf{u}\|^2 = 0. \tag{19}$$

Accordingly, Liu (2001) has developed a group-preserving scheme (GPS) as follows:

$$\mathbf{X}_{k+1} = \mathbf{G}(k) \mathbf{X}_k, \tag{20}$$

where \mathbf{X}_k denotes the numerical value of \mathbf{X} at the discrete t_k , and $\mathbf{G}(k) \in SO_o(2, 1)$ satisfies

$$\mathbf{G}^T \mathbf{g} \mathbf{G} = \mathbf{g}, \tag{21}$$

$$\det \mathbf{G} = 1, \tag{22}$$

$$G_0^0 > 0, \tag{23}$$

where G_0^0 is the 00th component of \mathbf{G} .

2.3 Generalized mid-point rule

Applying scheme (20) to Eq. (16) with a specified initial condition $\mathbf{X}(0) = \mathbf{X}_0$ we can compute the solution $\mathbf{X}(t)$ by GPS. Assuming that the stepsize used in GPS is $\Delta t = 1/K$, and starting from an initial augmented condition $\mathbf{X}_0 = \mathbf{X}(0) = (\mathbf{u}_0^T, \|\mathbf{u}_0\|)^T$ we can calculate the value $\mathbf{X}(1) = (\mathbf{u}^T(1), \|\mathbf{u}(1)\|)^T$ at $t = 1$ by

$$\mathbf{X}_f = \mathbf{G}_K(\Delta t) \cdots \mathbf{G}_1(\Delta t) \mathbf{X}_0. \tag{24}$$

However, we recall that each $\mathbf{G}_i, i = 1, \dots, K$, is an element of the Lie group $SO_o(2, 1)$, and by the closure property of the Lie group, $\mathbf{G}_K \cdots \mathbf{G}_1$ is also a Lie group element of $SO_o(2, 1)$ denoted by \mathbf{G} . Hence, we have

$$\mathbf{X}_f = \mathbf{G} \mathbf{X}_0. \tag{25}$$

This is a one-step transformation from \mathbf{X}_0 to \mathbf{X}_f .

Usually it is very hard to obtain an exact solution of \mathbf{G} . To be an approximation, we can calculate \mathbf{G} by a generalized mid-point rule, which is obtained from an

exponential mapping of \mathbf{A} by taking the values of the argument variables of \mathbf{A} at a generalized mid-point. The Lie group generated from this constant $\mathbf{A} \in so(2, 1)$ is known as a proper orthochronous Lorentz group, which admits a closed-form representation:

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_2 + \frac{(a-1)\hat{\mathbf{f}}\hat{\mathbf{f}}^T}{\|\hat{\mathbf{f}}\|^2} & \frac{b\hat{\mathbf{f}}}{\|\hat{\mathbf{f}}\|} \\ \frac{b\hat{\mathbf{f}}^T}{\|\hat{\mathbf{f}}\|} & a \end{bmatrix}, \quad (26)$$

where

$$\hat{\mathbf{u}} = r\mathbf{u}_0 + (1-r)\mathbf{u}_f, \quad (27)$$

$$\hat{\mathbf{f}} = \mathbf{f}(\hat{t}, \hat{\mathbf{u}}), \quad (28)$$

$$a = \cosh\left(\frac{\|\hat{\mathbf{f}}\|}{\|\hat{\mathbf{u}}\|}\right), \quad (29)$$

$$b = \sinh\left(\frac{\|\hat{\mathbf{f}}\|}{\|\hat{\mathbf{u}}\|}\right). \quad (30)$$

Here, we use the initial \mathbf{u}_0 and the final \mathbf{u}_f through a suitable weighting factor r to calculate \mathbf{G} , where $0 < r < 1$ is a parameter and $\hat{t} = r$. The above method employed a generalized mid-point rule to calculate \mathbf{G} , and the result is a single-parameter Lie group element $\mathbf{G}(r)$.

2.4 A Lie group mapping between two points on the cone

Let us define a new vector

$$\mathbf{F} := \frac{\hat{\mathbf{f}}}{\|\hat{\mathbf{u}}\|}, \quad (31)$$

so that, Eqs. (26), (29) and (30) can also be expressed as

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_2 + \frac{a-1}{\|\mathbf{F}\|^2}\mathbf{F}\mathbf{F}^T & \frac{b\mathbf{F}}{\|\mathbf{F}\|} \\ \frac{b\mathbf{F}^T}{\|\mathbf{F}\|} & a \end{bmatrix}, \quad (32)$$

$$a = \cosh(\|\mathbf{F}\|), \quad (33)$$

$$b = \sinh(\|\mathbf{F}\|). \quad (34)$$

From Eqs. (25) and (32) it follows that

$$\mathbf{u}_f = \mathbf{u}_0 + \eta\mathbf{F}, \quad (35)$$

$$\|\mathbf{u}_f\| = a\|\mathbf{u}_0\| + b\frac{\mathbf{F} \cdot \mathbf{u}_0}{\|\mathbf{F}\|}, \quad (36)$$

where

$$\eta := \frac{(a-1)\mathbf{F} \cdot \mathbf{u}_0 + b\|\mathbf{u}_0\|\|\mathbf{F}\|}{\|\mathbf{F}\|^2}. \tag{37}$$

Eq. (35) is rewritten as:

$$\mathbf{F} = \frac{1}{\eta}(\mathbf{u}_f - \mathbf{u}_0). \tag{38}$$

Substituting Eq. (38) into Eq. (36) and dividing both sides by $\|\mathbf{u}_0\|$ we can obtain

$$\frac{\|\mathbf{u}_f\|}{\|\mathbf{u}_0\|} = a + b \frac{(\mathbf{u}_f - \mathbf{u}_0) \cdot \mathbf{u}_0}{\|\mathbf{u}_f - \mathbf{u}_0\|\|\mathbf{u}_0\|}, \tag{39}$$

where

$$a = \cosh\left(\frac{\|\mathbf{u}_f - \mathbf{u}_0\|}{\eta}\right), \tag{40}$$

$$b = \sinh\left(\frac{\|\mathbf{u}_f - \mathbf{u}_0\|}{\eta}\right) \tag{41}$$

are obtained by inserting Eq. (38) for \mathbf{F} into Eqs. (33) and (34).

Let

$$\cos \theta := \frac{[\mathbf{u}_f - \mathbf{u}_0] \cdot \mathbf{u}_0}{\|\mathbf{u}_f - \mathbf{u}_0\|\|\mathbf{u}_0\|}, \tag{42}$$

$$S := \|\mathbf{u}_f - \mathbf{u}_0\|. \tag{43}$$

From Eqs. (39)-(41) it follows that

$$\frac{\|\mathbf{u}_f\|}{\|\mathbf{u}_0\|} = \cosh\left(\frac{S}{\eta}\right) + \cos \theta \sinh\left(\frac{S}{\eta}\right). \tag{44}$$

By defining

$$Z := \exp\left(\frac{S}{\eta}\right), \tag{45}$$

we obtain from Eq. (44), a quadratic equation for Z :

$$(1 + \cos \theta)Z^2 - \frac{2\|\mathbf{u}_f\|}{\|\mathbf{u}_0\|}Z + 1 - \cos \theta = 0. \tag{46}$$

The solution for Z in Eq. (46) is found to be

$$Z = \frac{\frac{\|\mathbf{u}_f\|}{\|\mathbf{u}_0\|} + \sqrt{\left(\frac{\|\mathbf{u}_f\|}{\|\mathbf{u}_0\|}\right)^2 - 1 + \cos^2 \theta}}{1 + \cos \theta}. \tag{47}$$

From Eqs. (45) and (43) we obtain

$$\eta = \frac{\|\mathbf{u}_f - \mathbf{u}_0\|}{\ln Z}. \tag{48}$$

Therefore, between any two points $(\mathbf{u}_0, \|\mathbf{u}_0\|)$ and $(\mathbf{u}_f, \|\mathbf{u}_f\|)$ on the cone, there exists a Lie group element $\mathbf{G} \in SO_o(2, 1)$ mapping $(\mathbf{u}_0, \|\mathbf{u}_0\|)$ onto $(\mathbf{u}_f, \|\mathbf{u}_f\|)$, which is given by

$$\begin{bmatrix} \mathbf{u}_f \\ \|\mathbf{u}_f\| \end{bmatrix} = \mathbf{G} \begin{bmatrix} \mathbf{u}_0 \\ \|\mathbf{u}_0\| \end{bmatrix}, \tag{49}$$

where \mathbf{G} is uniquely determined by \mathbf{u}_0 and \mathbf{u}_f through Eqs. (32)-(34), (38) and (48).

3 The Lie-group shooting method

The Sturm-Liouville problem considered in Section 1 requires both the information at the initial point $t = 0$ and at the terminal point $t = 1$. However, the usual stepping scheme requires a complete information at the starting point $t = 0$. Some effort is then required to reconcile the stepping scheme for the integration of the Sturm-Liouville problem presented there.

From Eqs. (9)-(12) it follows that

$$\dot{u}_1 = u_2, \tag{50}$$

$$\dot{u}_2 = f(t, u_1, u_2), \tag{51}$$

$$u_1(0) = c, \quad u_1(1) = c, \tag{52}$$

$$u_2(0) = A, \quad u_2(1) = B, \tag{53}$$

where A and B are two supplemental unknown constants, and c is a given positive constant.

From Eqs. (38), (52) and (53) it follows that

$$\mathbf{F} := \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{1}{\eta} \begin{bmatrix} 0 \\ B - A \end{bmatrix}. \tag{54}$$

By inserting Eq. (15) for \mathbf{u} into Eqs. (48), (47) and (42) we can obtain

$$\eta = \frac{\sqrt{(A - B)^2}}{\ln Z}, \tag{55}$$

$$Z = \frac{\frac{\sqrt{c^2+B^2}}{\sqrt{c^2+A^2}} + \sqrt{\frac{c^2+B^2}{c^2+A^2} - 1 + \cos^2 \theta}}{1 + \cos \theta}, \tag{56}$$

$$\cos \theta = \frac{A(B - A)}{\sqrt{(A - B)^2 \sqrt{c^2 + A^2}}}. \tag{57}$$

Comparing Eq. (54) with Eq. (31), and with the aid of Eqs. (27), (28) and (50)-(53), we obtain

$$rA + (1 - r)B = 0, \tag{58}$$

$$A - B + \frac{\eta}{\xi} \hat{f} = 0, \tag{59}$$

where

$$\hat{f} := f(r, c, rA + (1 - r)B) = f(r, c, 0), \tag{60}$$

$$\xi := \sqrt{c^2 + [rA + (1 - r)B]^2} = c, \tag{61}$$

because of $\hat{u}_1 = c$ and $\hat{u}_2 = rA + (1 - r)B = 0$.

Eq. (58) is a crucial result for the further development of a closed-form formula about A . This equation is obtained by using the two identical boundary values of $u_1 = u$ in Eq. (8). From the above equations we can see that c must be a positive value, and the advantage of adjusting the two boundary values in Eq. (8) to be equal is that we can derive Eq. (58), and that a closed-form solution of A will be available as follows.

The above derivation of the governing equations (55)-(61) is based on equating the two \mathbf{F} 's in Eqs. (31) and (38). It also means that the two Lie groups defined by Eqs. (26) and (32) are equal. In this sense, we have called our shooting technique a Lie-group shooting method (LGSM).

4 The solution of A

From Eqs. (58)-(60), (13), and (61) we obtain an algebraic equation for A :

$$Ac + \eta_0 f_1 = 0, \tag{62}$$

where

$$f_1(r) = -\frac{\dot{p}(r)}{p(r)}(2r - 1) - \frac{(x_f - x_0)^2 r(1 - r)}{p(r)}[q(r) - \lambda s(r)] - 2, \tag{63}$$

$$Z = \frac{\sqrt{c^2 + B^2} + \sqrt{B^2}}{\sqrt{c^2 + A^2} - \sqrt{A^2}}, \tag{64}$$

$$\eta_0 = \frac{\sqrt{A^2}}{\ln Z}. \tag{65}$$

Here, $B = rA/(r - 1)$ has a different sign from A because of the fact that $0 < r < 1$. Eq. (62) can be used to solve A for a given r , analytically. If A is solved for, we can return to integrating Eqs. (9)-(12) by a suitable forward IVP solver. Here we consider only the case of $A > 0$. For this case, inserting Eq. (65) for η_0 into Eq. (62) we obtain

$$\ln Z = \frac{-f_1}{c}. \tag{66}$$

Defining

$$f_2(r) := \exp\left(-\frac{f_1(r)}{c}\right), \tag{67}$$

and substituting Eq. (64) for Z into Eq. (92) we obtain

$$\frac{\sqrt{c^2 + B^2} + \sqrt{B^2}}{\sqrt{c^2 + A^2} - \sqrt{A^2}} = f_2. \tag{68}$$

Eq. (68) can be written as

$$f_2 A - B = f_2 \sqrt{c^2 + A^2} - \sqrt{c^2 + B^2} \tag{69}$$

by using $A > 0$ and $B < 0$. Squaring the above equation and cancelling the common terms we can rearrange it as:

$$2f_2 \sqrt{c^2 + B^2} \sqrt{c^2 + A^2} = (1 + f_2^2)c^2 + 2f_2 AB. \tag{70}$$

Squaring again and cancelling the common term and factor, we can obtain:

$$4f_2^2(A^2 + B^2) - 4f_2(1 + f_2^2)AB = (1 - f_2^2)^2 c^2. \tag{71}$$

Inserting $B = rA/(r - 1)$ and through some algebraic manipulations we eventually obtain:

$$\frac{4f_2}{(r - 1)^2} [f_2 - (1 - f_2)^2 r^2 + (1 - f_2)^2 r] A^2 = (1 - f_2^2)^2 c^2. \tag{72}$$

If the following condition holds

$$f_3(r) := f_2 - (1 - f_2)^2 r^2 + (1 - f_2)^2 r > 0, \tag{73}$$

then A has a positive solution:

$$A = \sqrt{\frac{(r - 1)^2 (1 - f_2^2)^2 c^2}{4 f_2 f_3}}. \tag{74}$$

5 Calculating the eigenvalues for the direct problem

In the previous section we have derived a closed-form solution to calculate the slope A for each r in its admissible range. If A is available, then we can apply the GPS to integrate the (\mathbf{u}, t) -IVP in Eqs. (9)-(12).

In principle, if there exists one solution y of Eqs. (1) and (2), there are many solutions of the type αy , $\alpha \neq 0 \in \mathbb{R}$. Assume that one of these solutions has a slope $y'(x_0) \neq 0$ at the left-end, then there are many different solutions with slopes $\alpha y'(x_0)$, $\alpha \neq 0 \in \mathbb{R}$. It means that the slope A can be an arbitrary value. So the factor r in Eq. (74) can be any value in the interval of $r \in (0, 1)$. Therefore, we can fix $r = 1/2$, and then we obtain the following equation for A :

$$A = \sqrt{\frac{(1 - f_4^2)^2 c^2}{4(1 + f_4^2)^2 f_4}}, \tag{75}$$

where

$$f_4(\lambda) := \exp\left(\frac{(x_f - x_0)^2}{4cp(1/2)}[q(1/2) - \lambda s(1/2)] + \frac{2}{c}\right). \tag{76}$$

Here A is only dependent on λ .

In order to calculate the eigenvalues we let λ run in a selected interval in which we are interested, and then insert λ into Eq. (75) we can obtain A . When c and A are given, we can calculate $y(x_f)$ by integrating Eqs. (9)-(12) and using Eq. (6). Therefore, we can plot a curve of the variation of $y(x_f)$ with respect to λ , namely the eigenvalue curve, of which the intersecting points with the zero line give the values of the required eigenvalues. In order to obtain more accurate eigenvalues, we can adjust the λ near the marked one until $y(x_f)$ satisfies $|y(x_f)| < \varepsilon_1$, where ε_1 is a given tolerance of error of mismatching the right-end boundary condition $y(x_f) = 0$. Numerical examples will be given below by using the above method.

6 Calculating the potential function, for specified eigenvalues

When n eigenvalues of $\lambda_i, i = 1, \dots, n$ are specified, we can insert them into Eq. (75) to calculate A . Corresponding to each λ_i we denote the corresponding A by A_i . However, q is an unknown function, which we are required to identify, with the aid of n eigenvalues λ_i .

Below we will develop a numerical procedure to determine $q(x)$ at the n discretized points of $x_i = x_0 + i\Delta x$, where $\Delta x = (x_f - x_0)/n$ is a uniform spatial length. First from Eqs. (5) and (6) we have

$$y'_i(x_0) = \frac{A_i - 1}{x_f - x_0}, \tag{77}$$

where y_i is the i -th eigenfunction. The above condition together with $y_i(x_0) = 0$ can be used as the initial conditions of Eq. (1). However, we require a numerical integrator. Second, we let $z_i(x) = p(x)y'_i(x)$, and from Eq. (1) we have

$$\frac{d}{dx} \begin{bmatrix} y_i(x) \\ z_i(x) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{p(x)} \\ q(x) - \lambda_i s(x) & 0 \end{bmatrix} \begin{bmatrix} y_i(x) \\ z_i(x) \end{bmatrix}, \tag{78}$$

where the initial conditions are given by

$$\begin{bmatrix} y_i(x_0) \\ z_i(x_0) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{p(x_0)(A_i - 1)}{x_f - x_0} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{79}$$

As mentioned by Liu (2008a) there are many eigenfunctions, which satisfy Eqs. (1) and (2); in order to give a unique solution we can choose A_i by $p(x_0)(A_i - 1) = x_f - x_0$, such that $p(x_0)(A_i - 1)/(x_f - x_0) = 1$.

Because the state matrix in Eq. (78) is a $sl(2, \mathbb{R})$ we can employ the following numerical integrator:

$$\begin{bmatrix} y_i^{k+1} \\ z_i^{k+1} \end{bmatrix} = \mathbf{G}_k(\lambda_i) \begin{bmatrix} y_i^k \\ z_i^k \end{bmatrix}, \tag{80}$$

where

$$\mathbf{G}_k := \frac{1}{1 - \frac{(\Delta x)^2(\bar{q}_k - \lambda_i \bar{s}_k)}{4\bar{p}_k}} \begin{bmatrix} 1 + \frac{(\Delta x)^2(\bar{q}_k - \lambda_i \bar{s}_k)}{4\bar{p}_k} & \frac{\Delta x}{\bar{p}_k} \\ \Delta x(\bar{q}_k - \lambda_i \bar{s}_k) & 1 + \frac{(\Delta x)^2(\bar{q}_k - \lambda_i \bar{s}_k)}{4\bar{p}_k} \end{bmatrix} \tag{81}$$

is an element of the Lie-group $SL(2, \mathbb{R})$. In the above we have simplified the notations by $s_k = s(x_0 + k\Delta x)$, $p_k = p(x_0 + k\Delta x)$ and $\bar{s}_k = (s_k + s_{k+1})/2$, $\bar{p}_k = (p_k + p_{k+1})/2$ and $\bar{q}_k = (q_k + q_{k+1})/2$.

Applying the above integration method to Eq. (78) n times, and by noting the initial conditions given in Eq. (79) we have

$$\begin{bmatrix} 0 \\ z_i(x_f) \end{bmatrix} = \mathbf{G}_n(\lambda_i) \cdots \mathbf{G}_1(\lambda_i) \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{82}$$

Inserting each λ_i into the above equation, replacing the term $q(1/2)$ in A_i by $q_{n/2}$ where n is taken to be an even number, similarly $p(1/2)$ by $p_{n/2}$ and $s(1/2)$ by $s_{n/2}$, and taking its first row we get a nonlinear algebraic equation denoted by $F_i = 0$, such that when i runs from $i = 1$ to $i = n$ we have

$$\begin{aligned} F_1(q_1, \dots, q_n) &= 0, \\ &\vdots \\ F_n(q_1, \dots, q_n) &= 0. \end{aligned} \tag{83}$$

Therefore, we have n equations to solve n unknowns q_1, \dots, q_n . However, in order to solve these equations we may require a feasible numerical method.

Now we propose a new method to solve the above algebraic equations. Recently, Liu and Atluri (2008) have developed a very powerful fictitious time integration method (FTIM) to solve nonlinear algebraic equations, which is based on the following idea.

In order to apply our new method to solve the system of nonlinear algebraic equations, we demonstrate it through a single nonlinear algebraic equation:

$$F(x) = 0, \tag{84}$$

where we only have one independent variable x . We transform it into a first-order ODE by introducing a fictitious time variable τ into the following transformation of variables from x to y :

$$y(\tau) = (1 + \tau)x. \tag{85}$$

Here, τ is a variable which is independent of x ; hence, $y' = dy/d\tau = x$. If $v \neq 0$, Eq. (84) is equivalent to

$$0 = -vF(x). \tag{86}$$

Adding the equation $y' = x$ to Eq. (86) we obtain:

$$y' = x - vF(x). \tag{87}$$

By using Eq. (85) we can derive

$$y' = \frac{y}{1 + \tau} - vF\left(\frac{y}{1 + \tau}\right). \tag{88}$$

This is a first-order ODE for $y(\tau)$. The initial condition for the above equation is $y(0) = x$, which is however an unknown and requires a guess.

Multiplying Eq. (88) by an integrating factor of $1/(1 + \tau)$ we can obtain

$$\frac{d}{d\tau} \left(\frac{y}{1 + \tau} \right) = -\frac{v}{1 + \tau} F\left(\frac{y}{1 + \tau}\right). \tag{89}$$

Further using $y/(1 + \tau) = x$, leads to

$$x' = -\frac{v}{1 + \tau} F(x). \tag{90}$$

Therefore, we have transformed the algebraic Eq. (84) into a first-order nonautonomous ODE. Under certain conditions, we expect that the solution of Eq. (90) starting from an initial guess of $x(0)$ can approximate the true solution x of Eq. (84).

The above idea was first employed by Liu (2008b) to treat an inverse Sturm-Liouville problem by transforming an ODE into a PDE. Then, Liu (2008c, 2008d) and Liu, Chang, Chang and Chen (2008) extended this idea to develop new methods for estimating parameters in the inverse vibration problems.

Now, by applying Eq. (90) to Eq. (83) we can obtain

$$\begin{aligned} q_1' &= -\frac{v}{1 + \tau} F_1(q_1, \dots, q_n), \\ &\vdots \\ q_n' &= -\frac{v}{1 + \tau} F_n(q_1, \dots, q_n). \end{aligned} \tag{91}$$

Starting from an initial value of $q_i(0)$, we may employ the GPS in Section 2.1 to integrate Eq. (91) from $\tau = 0$ to a selected final time τ_f . In the numerical integration process we can check the residual norm by

$$\left(\sum_{i=1}^n F_i^2 \right)^{1/2} \leq \epsilon_2, \tag{92}$$

where ϵ_2 is a given convergent criterion. If at a time $\tau_0 \leq \tau_f$ the above criterion is satisfied, then the solution of q_i is obtained.

Similarly, we can determine the unknown function $s(x)$ when $p(x)$ and $q(x)$ are given. For this problem we need to replace Eq. (83) by a new set of algebraic equations in terms of the discretized s_j . Then, a similar FTIM is used to solve s_j . By the same token, we can determine the unknown function $p(x)$ when $q(x)$ and $s(x)$ are given.

7 Numerical examples

7.1 Example 1

For this example we consider the Sturm-Liouville eigenvalues problem with [Ghe-lardoni, Gheri and Marletta (2001); Yücel (2006)]:

$$\begin{aligned}
 -y''(x) + e^x y(x) &= \lambda y(x), \\
 y(0) = y(\pi) &= 0.
 \end{aligned}
 \tag{93}$$

The eigenvalue in this case does not have a closed-form solution, and we first employ the Lie-group shooting method in Section 5 to search the first thirty eigenvalues as shown in Table 1.

Table 1: The first thirty eigenvalues for Example 1

k	λ_k	k	λ_k	k	λ_k
1	4.897	2	10.045	3	16.0192
4	23.267	5	32.264	6	43.221
7	56.181	8	71.161	9	88.132
10	107.117	11	128.105	12	151.096
13	176.089	14	203.083	15	232.079
16	263.075	17	296.072	18	331.069
19	368.067	20	407.065	21	448.064
22	491.062	23	536.061	24	583.060
25	632.059	26	683.058	27	736.057
28	791.057	29	848.056	30	907.055

For this example we use $n = 30$, $\Delta\tau = 1$ used in the GPS, $\varepsilon_2 = 0.01$ and $\nu = 0.001$. The eigenvalues listed in Table 1 are then used in the inverse problem, for the estimation of $q(x)$ by utilizing the Lie-group and FTIM methods in Section 6. In Fig. 1 we compare the calculated potential function and exact one plotted respectively by the dashed line and solid line. It can be seen that the numerical solution is very close to the exact one.

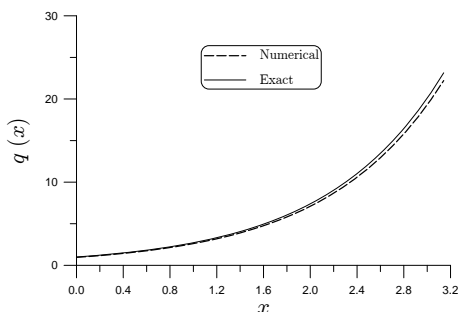


Figure 1: Comparing the calculated potential function from the proposed numerical method, with the exact one for Example 1.

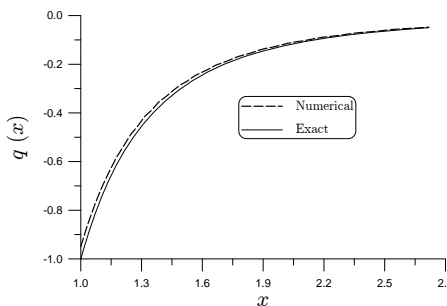


Figure 2: Comparing the computed potential function with the exact one for Example 2.

7.2 Example 2

For a second test example we consider an inverse Sturm-Liouville problem with

$$[x^{-1}y'(x)]' + (\lambda + 1)x^{-3}y(x) = 0, \tag{94}$$

$$y(1) = y(e) = 0.$$

The direct problem has the solution: eigenvalues are $\lambda_k = k^2\pi^2$, $k \in \mathbb{N}$, and eigenfunctions are $y_k = x \sin(k\pi \ln x)$. In the inverse problem, we specify the eigenvalues to be $\lambda_k = k^2\pi^2$, and seek to determine the potential function numerically.

For this example we use $n = 50$, $\Delta\tau = 0.5$ used in the GPS, $\varepsilon_2 = 0.01$ and $\nu = 0.005$. In Fig. 2 we compare the calculated potential function and the exact one. It can be seen that the numerical solution is very close to the exact one.

7.3 Example 3

As another example, we consider a singular inverse Sturm-Liouville eigenvalues problem to calculate the potential function $q(x)$ in the Schrödinger equation

$$\frac{d^2y(x)}{dx^2} + [\lambda - q(x)]y(x) = 0, \tag{95}$$

$$y(-\infty) = y(\infty) = 0,$$

where $q(x) = x^2$ is used as a test example. Here $\lambda_k = 2k + 1$, $k \in \mathbb{Z}^+$ is the eigenvalue, and

$$y_k(x) = H_k(x) \exp\left(-\frac{x^2}{2}\right) \tag{96}$$

is the eigenfunction, where the Hermite polynomials are given by

$$H_k(x) = (-1)^k e^{x^2} d^k e^{-x^2} / dx^k, \quad k = 0, 1, \dots$$

In Fig. 3 we compare the calculated potential function in the inverse problem with specified λ_k , and the exact one in the range of $-5 \leq x \leq 5$. Here we use $n = 100$, $\Delta\tau = 10^{-5}$, $\varepsilon_2 = 0.01$ and $\nu = 10^{-6}$. The numerical error is about in the order of 10^{-2} .

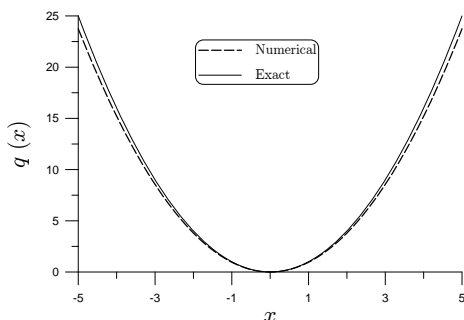


Figure 3: Comparing the computed potential function with the exact one for Example 3.

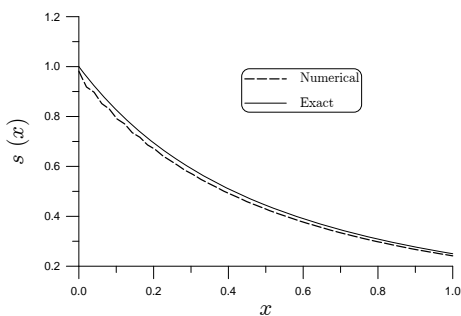


Figure 4: Comparing the computed weighting function with the exact one for Example 4.

7.4 Example 4

Alternatively, we may identify the weighting function $s(x)$ in the following inverse Sturm-Liouville problem:

$$\begin{aligned} \frac{d^2 y(x)}{dx^2} + \lambda s(x) y(x) &= 0, \\ y(0) = y(1) &= 0, \end{aligned} \tag{97}$$

where $s(x) = 1/(1+x)^2$ is used as a test example. Here,

$$\lambda_k = \frac{1}{4} + \left(\frac{k\pi}{\ln 2} \right)^2, \quad k \in \mathbb{Z}^+, \tag{98}$$

$$y_k(x) = \sqrt{1+x} \sin \left(\frac{k\pi \ln(1+x)}{\ln 2} \right) \tag{99}$$

are respectively the eigenvalues and eigenfunctions.

In Fig. 4 we compare the calculated $s(x)$ in the inverse problem with specified λ_k , and the exact $s(x)$ under the following parameters: $n = 50$, $\Delta\tau = 1$, $\varepsilon_2 = 0.01$ and $\nu = -0.1$. The numerical error is about in the order of 10^{-2} .

7.5 Example 5

In this example we consider the direct and inverse problems of an axial vibrating tapered rod:

$$-\frac{d}{dx} \left(E(x)A(x) \frac{dy(x)}{dx} \right) = \rho(x)A(x)\omega^2 y(x), \quad 0 < x < L,$$

$$y(0) = y(L) = 0. \tag{100}$$

Let $y = Lw$, $x = L\xi$, and

$$\hat{E}(\xi) = \frac{E(x)}{E_0}, \quad \hat{A}(\xi) = \frac{A(x)}{A_0}, \quad \hat{\rho}(\xi) = \frac{\rho(x)}{\rho_0}, \tag{101}$$

where $E_0 = E(0)$, $A_0 = A(0)$, and $\rho_0 = \rho(0)$, and then we have a non-dimensionalized equation:

$$-\frac{d}{d\xi} \left(\hat{E}(\xi)\hat{A}(\xi) \frac{dw(\xi)}{d\xi} \right) = \hat{\rho}(\xi)\hat{A}(\xi)\lambda^2 w(\xi), \quad 0 < \xi < 1,$$

$$w(0) = w(1) = 0, \tag{102}$$

where $\lambda^2 = \rho_0 L^2 \omega^2 / E_0$.

In order to give some specified data of λ^2 for the use in the inverse problem, as an example we take

$$\hat{E}(\xi) = \hat{A}(\xi) = \hat{\rho}(\xi) = 1 - 0.5\xi. \tag{103}$$

The direct problem is finding λ^2 ; however, the eigenvalues do not have a closed-form solution, and we have employed the Lie-group shooting method in Section 5 to search the eigenvalues in the range of $0 < \lambda < 5000$ as shown in Fig. 5. There are 24 intersecting points of the eigenvalue curve with the zero line, and Table 2 lists the first twenty eigenvalues.

Next, we turn to the inverse problem by identifying one of $(\hat{E}(\xi), \hat{A}(\xi), \hat{\rho}(\xi))$, for the specified eigenvalues λ_k^2 in Table 2. For the first case we identify $\hat{E}(\xi)$, and this problem leads to p_i being unknowns. Under the following parameters: $n = 20$, $\Delta\tau = 0.1$, and $\nu = -0.0001$, the computed $\hat{E}(\xi)$ is compared with the exact one in Fig. 6(a). For the second case we identify $\hat{A}(\xi)$, which leads to both p_i and s_i being

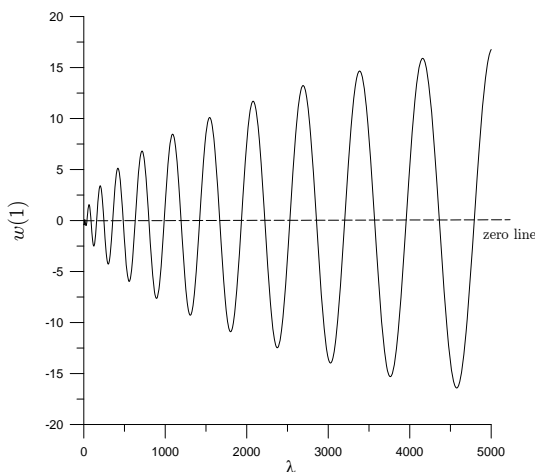


Figure 5: Eigenvalue curve for the problem of a vibrating rod.

Table 2: The first twenty eigenvalues for Example 5

k	λ_k^2	k	λ_k^2	k	λ_k^2	k	λ_k^2
1	0.32	2	9.87	3	15.67	4	39.48
5	88.82	6	157.92	7	246.74	8	355.31
9	483.63	10	631.69	11	799.52	12	987.11
13	1194.48	14	1421.66	15	1668.66	16	1935.52
17	2222.28	18	2528.98	19	2855.67	20	3202.42

unknowns. The computed $\hat{A}(\xi)$ is compared with the exact one in Fig. 6(b). Then, the last case is identifying $\hat{\rho}(\xi)$. This problem leads to s_i being unknowns, and the computed $\hat{\rho}(\xi)$ is compared with the exact one in Fig. 6(c). All the numerical errors are in the order of about 10^{-2} .

8 Conclusions

In this paper we have proposed a novel $SL(2, \mathbb{R})$ Lie-group shooting method (LGSM), combined with the use of FTIM, for solving the inverse Sturm-Liouville problems. We solved the resulting algebraic equations by the Fictitious Time Integration Method (FTIM), to find the unknown potential function, the unknown impedance function, or the unknown weighting function in the Sturm-Liouville operator, when the discrete eigenvalues are specified. Several numerical examples were given to confirm the efficiency and accuracy of the present LGSM and FTIM approaches, which are rather easy to implement with a low computational cost than the previous

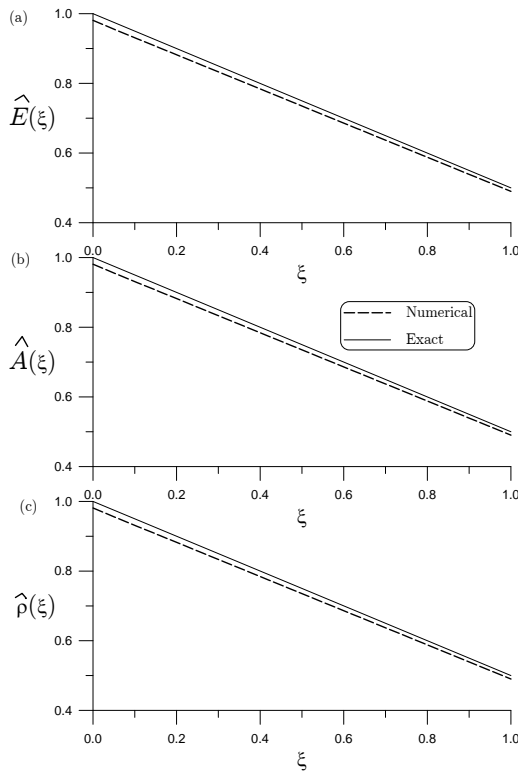


Figure 6: Comparing the computed (a) Young’s modulus, (b) cross-sectional area, and (c) mass-density, with the exact ones for a rod undergoing axial vibrations.

numerical methods. Moreover, the accuracy is improved significantly. The numerical examples also include the mechanics problem of determining the mechanical properties and/or geometric configuration of a tapered bar undergoing axial vibrations, when the discrete eigen-frequencies are specified.

In summary, we present here the following algorithmic flow chart in Fig. 7 for determining one of $(q(x), p(x), s(x))$ when the eigenvalues λ_i are given. We demonstrate it by finding the unknown potential $q(x)$, while the other two unknown functions $p(x)$ or $s(x)$ can be found by a similar numerical algorithm.

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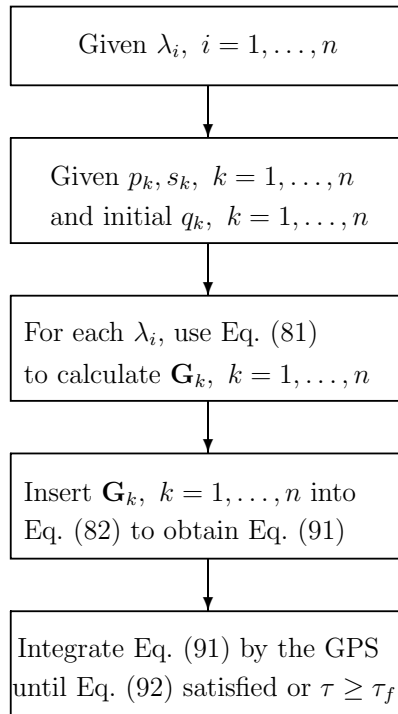


Figure 7: An algorithmic flow chart to determine, for example, the unknown function $q(x)$, when λ_i are specified.

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