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Richardson Extrapolation Method for Singularly Perturbed Coupled System of Convection-Diffusion Boundary-Value Problems

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Abstract: This paper presents an almost second–order uniformly convergent Richardson extrapolation method for convection- dominated coupled system of boundary value problems. First, we solve the system by using the classical finite difference scheme on the layer resolving Shishkin mesh, and then we construct the Richardson approximation solution using the solutions obtained on N and $2N$ mesh intervals. Second-order parameter–uniform error estimate is derived. The proposed method is applied to a test example for verification of the theoretical results for the case $\varepsilon \leq N^{-1}$.

Keywords: Singularly perturbed coupled system of equations, Boundary layer, Piecewise-uniform Shishkin mesh, Richardson extrapolation, Uniform convergence.

1 Introduction

Singular perturbation problems (SPPs) are of common occurrence in many applied areas which include fluid dynamics, chemical reaction theory, gas porous electrodes theory, control theory and so on. The well-known examples of SPPs are the linearized Navier-Stokes equations at high Reynolds number, heat transport problems with large Pećlet numbers, magneto-hydrodynamic duct problems at Hartman numbers, etc. The principal characteristic of SPPs is that its solution has a multi-scale character. That is, there are thin layers, known as boundary layers, where the solutions have steep gradients. SPPs can be solved analytically through asymptotic expansion, for details one can see the books of Eckhaus (1979), Lagerstrom (1988) and O'Malley (1991).

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Because of the presence of boundary layers, the classical numerical methods fail to yield satisfactory numerical approximate solution on uniform grids. To overcome these numerical difficulties, one has to seek some special schemes on uniform or nonuniform meshes. One can refer the following books for more details about the numerical treatments of SPPs: Doolan, Miller, and Schildres (1980), Farrell, Hegarty, Miller, O'Riordan, and Shishkin (2000) and Roos, Stynes, and Tobiska (1996). The numerical methods for SPPs are classified into two categories: one is the fitted operator method, *i.e.,* exponentially fitted finite difference schemes on uniform mesh (Doolan, Miller, and Schildres (1980)); and the second approach is fitted mesh method, *i.e.,* classical finite difference schemes will be used on nonuniform grids (Farrell, Hegarty, Miller, O'Riordan, and Shishkin (2000), Roos, Stynes, and Tobiska (1996)).

There are several articles deal with the solution technique for singularly perturbed two-point boundary-value problems (SPTPBVP). For example, Liu (2006a) proposed shooting methods for second-order ordinary differential equations, and also he developed Lie-group shooting technique for SPPs in (Liu (2006b)). Natesan et. al. developed higher-order numerical methods by incorporating the asymptotic expansion for self-adjoint SPPs in (Natesan and Ramanujam (1999a)), and for nonself-adjoint SPPs in (Natesan and Ramanujam (1999b)). Bawa and Natesan (2005) proposed a domain decomposition method using quintic spline for singularly perturbed reaction-diffusion problems.

In this article, we consider the following coupled system of convection-dominated boundary-value problem (BVP):

$$\begin{cases} \mathbf{L}_1\vec{u} \equiv -\varepsilon u_1''(x) - a_1(x)u_1'(x) + b_{11}(x)u_1(x) + b_{12}(x)u_2(x) = f_1(x), x \in \Omega = (0,1), \\ \mathbf{L}_2\vec{u} \equiv -\varepsilon u_2''(x) - a_2(x)u_2'(x) + b_{21}(x)u_1(x) + b_{22}(x)u_2(x) = f_2(x), \\ u_1(0) = u_2(0) = u_1(1) = u_2(1) = 0, \end{cases} \tag{1}$$

where $0 < \varepsilon \ll 1$, and $\vec{u} = (u_1, u_2)^T$. We assume that $B = [b_{ij}]_{i,j=1}^2$ is an L_0-matrix (*i.e.*, whose diagonal entries are positive and off-diagonal entries are nonnegative) with

$$\min_{x\in[0,1]} \, [\, b_{11}(x) + b_{12}(x), \; b_{21}(x) + b_{22}(x) \,] > \beta > 0.$$

Also, we assume that

$$a_1(x) > 2\alpha > 0, \quad a_2(x) > 2\alpha > 0, \forall x \in \overline{\Omega} = [0,1].$$

Under these assumptions, the system of BVPs (1) admits a unique solution, which exhibits a boundary layer of width $O(\varepsilon \ln \varepsilon)$ at $x = 0$.

Richardson extrapolation is a well-known classical postprocessing procedure where two computed solutions approximating a particular quantity are averaged to provide higher-order approximation. Keller (1969) used this technique to improve the accuracy of computed solution to non-singularly perturbed BVP. Vulanović, Herceg, and Petrović (1986) applied this technique to singularly perturbed reaction-diffusion BVPs on the Bakhvalov mesh. Natividad and Stynes (2003) used the Richardson extrapolation technique on a layer resolving Shishkin mesh for solving singularly perturbed convection-diffusion equation.

In this article, to obtain second-order ε-uniform convergence results of the BVP (1), we apply the Richardson extrapolation on the piecewise-uniform Shishkin mesh. First, we solve the system (1) on Shishkin mesh using the classical finite difference scheme and then we construct the Richardson extrapolation solution by using the numerical solutions obtained on N and $2N$ mesh intervals. To estimate the error, we determine the error of the smooth and singular components separately, and obtain the ε-uniform error estimate.

Recently, Natesan and Deb (2007) developed a numerical method consists of cubic spline and classical finite difference scheme for reaction-diffusion systems and obtained ε-uniform second-order error estimates. The classical finite difference scheme is applied to system of convection-diffusion equations on Shishkin mesh by Cen (2005).

The paper is organized in the following way: Section 2 presents bounds on the solution and its derivatives of the continuous problem (1). The difference scheme, Shishkin mesh, the Richardson extrapolation method, and error estimates are obtained in Section 3. A test problem is experimented numerically in Section 4. The paper ends with conclusions.

Throughout the paper, C will denote a generic positive constant that is independent of ε and of the mesh. Note that C does not necessarily assume the same value everywhere. We define the maximum norm by

$$\|\psi(x)\| = \max_{\overline{\Omega}} |\psi(x)|, \quad \|\overrightarrow{\psi}\| = \max_{1\le i\le 2}(\|\psi_i(x)\|), \quad \overrightarrow{\psi}(x) = (\psi_1(x), \psi_2(x)).$$

Further, we assume that $\varepsilon \le N^{-1}$, as is generally the case.

2 The Continuous Problem

In this section, we obtain bounds for the solution and its smooth and singular components of the continuous problem (1), which will be used in the subsequent sections to obtain ε–uniform error estimate.

Lemma 1 *Let $\vec{u}$ be the solution of the system of BVPs (1). Then, we have the following bound:*

$$||\vec{u}^{(k)}(x)|| \leq C\varepsilon^{-k}, \quad for \quad k = 1,2,3.$$

Proof. This lemma can be proved by following the technique used by Cen in (Cen (2005)). ■

The solution $\vec{u}$ of the problem (1) is decomposed in the following manner:

$$\vec{u}(x) = \vec{v}(x) + \vec{w}(x), \quad x \in \overline{\Omega}.$$

The functions $\vec{v}(x)$ and $\vec{w}(x)$ are respectively called the smooth and singular components of $\vec{u}$. Here $\vec{v}(x)$ is the solution of the following problem:

$$\begin{cases} \mathbf{L}_1\vec{v}(x) = f_1(x), \quad x \in \Omega, \\ \mathbf{L}_2\vec{v}(x) = f_2(x), \\ v_1(0) = v_{1,0}(0) + \varepsilon v_{1,1}(0) + \varepsilon^2 v_{1,2}(0), \\ v_2(0) = v_{2,0}(0) + \varepsilon v_{2,1}(0) + \varepsilon^2 v_{2,2}(0), \quad \vec{v}(1) = \vec{u}(1), \end{cases}$$

where, $v_{1,i}, v_{2,i}$ for $i = 0,1,2$ satisfies

$$\begin{cases} -a_1(x)v'_{1,0}(x) + b_{11}(x)v_{1,0}(x) + b_{12}(x)v_{2,0}(x) = f_1(x), \\ -a_2(x)v'_{2,0}(x) + b_{21}(x)v_{1,0}(x) + b_{22}(x)v_{2,0}(x) = f_2(x), \\ v_{1,0}(1) = u_1(1), \quad v_{2,0}(1) = u_2(1), \end{cases}$$

$$\begin{cases} -a_1(x)v'_{1,1}(x) + b_{11}(x)v_{1,1}(x) + b_{12}(x)v_{2,1}(x) = v''_{1,0}(x), \\ -a_2(x)v'_{2,1}(x) + b_{21}(x)v_{1,1}(x) + b_{22}(x)v_{2,1}(x) = v''_{2,0}(x), \\ v_{1,1}(1) = v_{2,1}(1) = 0, \end{cases}$$

and

$$\begin{cases} -\varepsilon v''_{1,0} - a_1(x)(v_{1,0} + \varepsilon v_{1,1})' + b_{11}(x)(v_{1,0} + \varepsilon v_{1,1}) + b_{12}(x)(v_{2,0} + \varepsilon v_{2,1}) = f_1(x), \\ -\varepsilon v''_{2,0} - a_2(x)(v_{2,0} + \varepsilon v_{2,1})' + b_{21}(x)(v_{1,0} + \varepsilon v_{1,1}) + b_{22}(x)(v_{2,0} + \varepsilon v_{2,1}) = f_2(x), \\ v_{1,1}(1) = v_{2,1}(1) = 0. \end{cases}$$

Also $\vec{w}(x)$ satisfies the BVP:

$$\begin{cases} \mathbf{L}_1\vec{w}(x) = 0, \quad x \in \Omega, \\ \mathbf{L}_2\vec{w}(x) = 0, \\ \vec{w}(0) = \vec{u}(0) - \vec{v}(0), \quad \vec{w}(1) = \vec{0}. \end{cases}$$

Theorem 2 *The following bounds hold for the derivatives of the smooth component $\vec{v}$,*

$$\|\vec{v}^{(k)}(x)\| \le C, \quad k = 1,2,3. \tag{2}$$

And the bounds for the singular component $\vec{w}$ is given by

$$\|\vec{w}^{(k)}(x)\| \le C\varepsilon^{-k}\exp\left(-\frac{\alpha x}{\varepsilon}\right), \qquad k = 0,1,2,3. \tag{3}$$

Proof. Using the asymptotic expansion $\vec{v} = \vec{v}_0 + \varepsilon\vec{v}_1 + \varepsilon^2\vec{v}_2$ the theorem easily follows from Cen (2005). ∎

3 The Discrete Problem

In this section, we present the finite difference scheme, Shishkin mesh for singularly perturbed convection-diffusion system of BVPs (1). We also provide the necessary error estimates for the smooth and singular components of the Richardson extrapolated solution.

Let $\overline{\Omega} = [0,1] = \{x_i\}_{i=0}^{N}$ be an arbitrary mesh with $0 = x_0 < x_1 < x_2 < \ldots < x_N = 1$. Set $h_i = x_i - x_{i-1}$ for each i. We use an upwind difference scheme for the problem (1) on the mesh $\overline{\Omega}$ which is given by

$$\begin{cases} \mathbf{L}_1^N \vec{U}_i^N = -\varepsilon\delta^2 U_{1,i}^N - a_{1,i}D^+U_{1,i}^N + b_{11}U_{1,i}^N + b_{12}U_{2,i}^N = f_{1,i}, \quad 1 \le i \le N-1 \\ \mathbf{L}_2^N \vec{U}_i^N = -\varepsilon\delta^2 U_{2,i}^N - a_{2,i}D^+U_{2,i}^N + b_{21}U_{1,i}^N + b_{22}U_{2,i}^N = f_{2,i}, \\ U_{1,0}^N = U_{1,N}^N = U_{2,0}^N = U_{2,N}^N = 0, \end{cases} \tag{4}$$

where

$$D^+z_i = \frac{z_{i+1}-z_i}{h_{i+1}}, \quad D^-z_i = \frac{z_i - z_{i-1}}{h_i} \quad \text{and} \quad \delta^2 z_i = \frac{(D^+ - D^-)z_i}{\hbar_i}, \quad \hbar_i = \frac{h_i + h_{i+1}}{2}.$$

The transition parameter τ is defined as

$$\tau = \min\left\{\frac{1}{2}, \frac{2\varepsilon}{\alpha}\ln N\right\}. \tag{5}$$

A piecewise-uniform mesh $S_{N,\tau} = \{x_i\}_0^N$ is constructed by dividing $\overline{\Omega}$ into two subdomains $[0,\tau]$ and $[\tau,1]$. Then we subdivide each of these two subdomains into $N/2$ mesh intervals. Also the step sizes of the mesh $S_{N,\tau}$ satisfy

$$h_i := \begin{cases} h = \dfrac{2\tau}{N} = \dfrac{4\varepsilon\ln N}{\alpha N}, & \text{for} \quad 1 \le i \le \dfrac{N}{2}, \\ H = \dfrac{2(1-\tau)}{N}, & \text{for} \quad \dfrac{N}{2} < i \le N. \end{cases} \tag{6}$$

Lemma 3 *(Discrete Maximum Principle) Let* $\{\vec{z}_i\}_{i=0}^N$ *be any discrete vector function such that* $\vec{z}_0 \geq \vec{0}$ *and* $\vec{z}_N \geq \vec{0}$. *Suppose that* $\mathbf{L}_1^N \vec{z}_i \geq 0$ *and* $\mathbf{L}_2^N \vec{z}_i \geq 0$ *for* $1 \leq i \leq N-1$. *Then* $\vec{z}_i \geq \vec{0}$ *for all* $0 \leq i \leq N$.

Proof. Following the method of proof given by Cen (2005), one can prove this lemma. ■

Lemma 4 *For* $i = 0, \ldots, N$ *define the mesh function*

$$S_{\varepsilon,i} = \prod_{j=1}^{i} \left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1}$$

with the usual convention that for $i = 0$, $S_{\varepsilon,0} = 1$. *Then, there exist a positive constant* C, *such that for* $i = 1, \ldots, N-1$, *we have*

$$\mathbf{L}_1^N \vec{S}_{\varepsilon,i} \geq \frac{CS_{\varepsilon,i}}{\max\{\varepsilon, h_{i+1}\}}, \quad \textit{and} \quad \mathbf{L}_2^N \vec{S}_{\varepsilon,i} \geq \frac{CS_{\varepsilon,i}}{\max\{\varepsilon, h_{i+1}\}},$$

where $\vec{S}_{\varepsilon,i} = (S_{\varepsilon,i},\ S_{\varepsilon,i})^T$ *with the assumption that*

$$\min_{x\in[0,1]} [a_1(x),\ a_2(x)] \geq 2\alpha > 0.$$

Proof. We have

$$S_{\varepsilon,i+1} - S_{\varepsilon,i} = \prod_{j=1}^{i+1} \left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1} - \prod_{j=1}^{i} \left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1} = -\frac{\alpha h_{i+1}}{\varepsilon + \alpha h_{i+1}} S_{\varepsilon,i}.$$

Therefore,

$$D^+ S_{\varepsilon,i} = -\frac{\alpha}{\varepsilon + \alpha h_{i+1}}\, S_{\varepsilon,i}, \quad \text{and} \quad D^- S_{\varepsilon,i} = -\frac{\alpha}{\varepsilon + \alpha h_i}\, S_{\varepsilon,i-1}.$$

Now,

$$\begin{aligned}
\delta^2 S_{\varepsilon,i} &= \frac{2}{h_i + h_{i+1}} (D^+ - D^-) S_{\varepsilon,i} = \frac{2\alpha}{h_i + h_{i+1}} \left(-\frac{S_{\varepsilon,i}}{\varepsilon + \alpha h_{i+1}} + \frac{S_{\varepsilon,i-1}}{\varepsilon + \alpha h_i}\right) \\
&= 2\alpha^2 \frac{h_{i+1}}{h_i + h_{i+1}}\, \frac{S_{\varepsilon,i}}{\varepsilon + \alpha h_{i+1}}.
\end{aligned}$$

Applying the difference operator $\mathbf{L}_1^N$ to $\vec{S}_{\varepsilon,i}$, we obtain

$$\mathbf{L}_1^N \vec{S}_{\varepsilon,i} = -\,2\alpha^2 \frac{h_{i+1}}{h_i + h_{i+1}}\, \frac{S_{\varepsilon,i}}{\varepsilon + \alpha h_{i+1}} + \alpha \frac{a_1(x_i)}{\varepsilon + \alpha h_{i+1}}\, S_{\varepsilon,i} + (\ b_{11}(x_i) + b_{12}(x_i)\ ) S_{\varepsilon,i}.$$

Using the fact that $\min_{x\in[0,1]} [a_1(x),\ a_2(x)] \geq 2\alpha > 0$, we get

$$\begin{aligned}\mathbf{L}_1^N \vec{S}_{\varepsilon,i} &\geq \frac{2\alpha^2}{\varepsilon+\alpha h_{i+1}}\, S_{\varepsilon,i} - 2\alpha^2 \frac{h_{i+1}}{h_i+h_{i+1}}\, \frac{S_{\varepsilon,i}}{\varepsilon+\alpha h_{i+1}} \\ &\geq \left[1-\frac{h_{i+1}}{h_i+h_{i+1}}\right] \frac{2\alpha^2}{\varepsilon+\alpha h_{i+1}}\, S_{\varepsilon,i}.\end{aligned}$$

Thus, we can always find a positive constant C_1 such that

$$\mathbf{L}_1^N \vec{S}_{\varepsilon,i} \geq \frac{C_1\alpha^2}{\varepsilon+\alpha h_{i+1}}\, S_{\varepsilon,i} \geq \frac{CS_{\varepsilon,i}}{\max\{\varepsilon, h_{i+1}\}}.$$

In a similar manner, one can prove the required estimate for $\mathbf{L}_2^N \vec{S}_{\varepsilon,i}$. ■

Lemma 5 *Let $S_{N,\tau} = \{x_i\}_0^N$ with $h_i = x_i - x_{i-1}$, for all i. Then, we have the following inequalities:*

$$N^{-2} \leq \prod_{j=1}^{N/2} \left(1+\frac{\alpha h_j}{\varepsilon}\right)^{-1} \leq CN^{-2}. \tag{7}$$

Proof. Using the inequality $\ln(1+x) \leq x - x^2/4$ for $0 \leq x \leq 1$, one can express

$$\ln \prod_{j=1}^{N/2} \left(1+\frac{\alpha h_j}{\varepsilon}\right)^{-1} = -\sum_{j=1}^{N/2} \ln\left(1+\frac{\alpha h_j}{\varepsilon}\right) \geq -\frac{\alpha N}{2\varepsilon} h \geq -\frac{\alpha}{\varepsilon}\ \tau = \ln N^{-2}.$$

Thus the first part of the inequality is proved.
For the second part, we use the inequality $x - x^2/2 \leq \ln(1+x)$, and obtain

$$\begin{aligned}\ln \prod_{j=1}^{N/2} \left(1+\frac{\alpha h_j}{\varepsilon}\right) &= \sum_{j=1}^{N/2} \ln\left(1+\frac{\alpha h_j}{\varepsilon}\right) \geq \sum_{j=1}^{N/2} \left(\frac{\alpha h_j}{\varepsilon} - \frac{\alpha^2}{2\varepsilon^2}\, h_j^2\right) \\ &= \frac{\alpha N}{2\varepsilon}\, h - \frac{\alpha^2}{2\varepsilon^2} \sum_{j=1}^{N/2} h_j^2.\end{aligned}$$

i.e., we have

$$\ln \prod_{j=1}^{N/2} \left(1+\frac{\alpha h_j}{\varepsilon}\right)^{-1} \leq \frac{\alpha^2}{2\varepsilon^2} \sum_{j=1}^{N/2} h_j^2 - 2\ln N.$$

The second inequality follows from the above result. ■

The following theorem provides the ε–uniform first-order convergence result for the classical finite difference scheme (4) applied on the Shishkin mesh for the system of BVPs (1).

Theorem 6 *Let $\vec{u}$ be the solution of the system of BVPs (1) and $\vec{U}^N$ be the solution of the discrete problem (4) on the piecewise-uniform Shishkin mesh. Then, we have the error estimate*

$$\|\vec{u} - \vec{U}^N\| \le CN^{-1}\ln N.$$

Proof. Following the similar lines of proof given in Cen (2005), one can prove this theorem. ■

3.1 Extrapolation of the Discrete Solution $\vec{U}^N$

The important goal of this article is to use the Richardson extrapolation technique to increase the order of convergence of the solution $\vec{U}^N$ which is already computed on the mesh $S_{N,\tau}$. For this we solve the BVP again on the mesh $S_{2N,\tau}$, which has $2N$ number of subintervals by keeping the transition parameter τ fixed as in the case of N mesh intervals. The solution in the later case is denoted by $\vec{U}^{2N}$. Since τ is fixed and N is doubled so step sizes h_i in $S_{N,\tau}$ is twice as that of the step size in $S_{2N,\tau}$.
We know that $\ln N = \alpha\tau/2\varepsilon$, and from Theorem 6, we have

$$\vec{U}^N(x_i) - \vec{u}(x_i) = CN^{-1}\ln N + R_N(x_i) = CN^{-1}(\alpha\tau/2\varepsilon) + R_N(x_i), \quad \forall\, x_i \in S_{N,\tau},$$

R_N is of $o(N^{-1}\ln N)$. Since we have computed $\vec{U}^{2N}$ by keeping τ fixed, so we can write

$$\vec{U}^{2N}(x_i) - \vec{u}(x_i) = C(2N)^{-1}(\alpha\tau/2\varepsilon) + R_{2N}(x_i), \quad \forall\, x_i \in S_{N,\tau},$$

where R_{2N} is also of $o(N^{-1}\ln N)$.
We can eliminate the $O(N^{-1})$ terms from the above two expressions to obtain

$$\vec{u}(x_i) - (2\vec{U}^{2N}(x_i) - \vec{U}^N(x_i)) = o(N^{-1}\ln N), \quad \forall\, x_i \in S_{N,\tau}.$$

Therefore, we look for the extrapolation formula in the following way:

$$2\vec{U}^{2N}(x_i) - \vec{U}^N(x_i), \quad \forall x_i \in S_{N,\tau},$$

which will provide an approximation to $\vec{u}$ with better accuracy than $\vec{U}^N$ and $\vec{U}^{2N}$.
As like in the continuous case, here also we decompose the solution $\vec{U}^N$ into smooth and singular components as $\vec{U}^N = \vec{V}^N + \vec{W}^N$, such that

$$\mathbf{L}_1^N\vec{V}^N = f_1, \mathbf{L}_2^N\vec{V}^N = f_2, \vec{V}^N(0) = \vec{v}^N(0), \vec{V}^N(1) = \vec{v}^N(1),$$

$$\mathbf{L}_1^N\vec{W}^N = 0, \mathbf{L}_2^N\vec{W}^N = 0, \vec{W}^N(0) = \vec{w}^N(0), \vec{W}^N(1) = \vec{w}^N(1).$$

A similar decomposition holds true for $\vec{U}^{2N}$.

3.2 Extrapolation of the Discrete Smooth Component $\vec{V}^N$

Lemma 7 *Assume that* $\varepsilon \leq N^{-1}$. *Then for all* $x_i \in (0,1)$, *we have*

$$\mathbf{L}_1^N(\vec{V}^N - \vec{v})(x_i) = O(H^2) + \frac{1}{2}a_1(x_i)(x_{i+1} - x_i)v_1''(x_i),$$
$$\mathbf{L}_2^N(\vec{V}^N - \vec{v})(x_i) = O(H^2) + \frac{1}{2}a_2(x_i)(x_{i+1} - x_i)v_2''(x_i).$$

Proof. By modifying the proof given in page 21 of Miller, O'Riordan, and Shishkin (1996) and using the fact that $\|\vec{v}^{(k)}(x)\|, k = 1,2,3$ is bounded by a constant, one can obtain the required estimate on the truncation error. ∎

Let us assume that $\vec{E} = (E_1, E_2)^T$ be the solution of the following system of BVPs

$$\mathbf{L}_1\vec{E}(x) = \xi_1(x), \quad \mathbf{L}_2\vec{E}(x) = \xi_2(x), \quad \forall x \in (0,1),$$
$$\vec{E}(0) = \vec{E}(1) = 0,$$

where $\xi_1(x)$ and $\xi_2(x)$ are given by

$$\xi_1(x) = \frac{1}{2}a_1(x)v_1''(x), \quad \xi_2(x) = \frac{1}{2}a_2(x)v_2''(x).$$

We decompose $\vec{E}$ into smooth and layer parts by $\vec{E} = \vec{F} + \vec{G}$ such that

$$\begin{aligned} &\mathbf{L}_1\vec{F}(x) = \xi_1(x), && \mathbf{L}_2\vec{F}(x) = \xi_2(x), \\ &\mathbf{L}_1\vec{G}(x) = 0, && \mathbf{L}_2\vec{G}(x) = 0, \\ &\vec{F}(0) = -\vec{G}(0), && \vec{F}(1) = \vec{G}(1) = 0. \end{aligned} \tag{8}$$

One can easily prove that

$$\| \vec{F}^{(k)} \| \leq C, \qquad k = 1,2,3. \tag{9}$$

and

$$\| \vec{G}^{(k)} \| \leq C\varepsilon^{-k} \exp\left(-\frac{\alpha x}{\varepsilon}\right), \qquad k = 0,1,2,3. \tag{10}$$

Lemma 8 *For all* $x_i \in (0,1)$, *we have*

$$(\mathbf{L}_1^N - \mathbf{L}_1)\vec{F}(x_i) = O(H) \quad \textit{and} \quad (\mathbf{L}_2^N - \mathbf{L}_2)\vec{F}(x_i) = O(H).$$

Proof. We know from the truncation error estimate that

$$|(\mathbf{L}_1^N - \mathbf{L}_1)\vec{F}(x_i)| \leq \frac{\varepsilon}{3}(x_{i+1} - x_{i-1})|F_1|_3 + \frac{a_1(x_i)}{2}(x_{i+1} - x_i)|F_1|_2.$$

Now it follows from equation (9) that

$$\begin{aligned}|(\mathbf{L}_1^N-\mathbf{L}_1)\vec{F}(x_i)| &\leq \frac{\varepsilon}{3}C(x_{i+1}-x_{i-1})+C\frac{a_1(x_i)}{2}(x_{i+1}-x_i)\\ &\leq C(x_{i+1}-x_{i-1})+C(x_{i+1}-x_i)\\ &\leq CH.\end{aligned}$$

Hence, we obtain

$$(\mathbf{L}_1^N-\mathbf{L}_1)\vec{F}(x_i)=O(H).$$

Proceeding in the same way, one can prove for $(\mathbf{L}_2^N-\mathbf{L}_2)\vec{F}$. ■

Theorem 9 *For all $x_i\in[\tau,1)$ one has the following*

$$\vec{V}^N(x_i)-\vec{v}(x_i)=H\vec{E}(x_i)+O(N^{-2}).$$

Proof. From Lemma 8, we have

$$\mathbf{L}_1^N\vec{F}(x_i)=\mathbf{L}_1\vec{F}(x_i)+\mathbf{L}_1^N\vec{F}(x_i)-\mathbf{L}_1\vec{F}(x_i)=\mathbf{L}_1\vec{F}(x_i)+O(H).$$

Now, from (8) that

$$H\mathbf{L}_1^N\vec{F}(x_i)=H\xi_1(x_i)+O(H^2).$$

Similarly, for $\mathbf{L}_2^N$, one can obtain

$$H\mathbf{L}_2^N\vec{F}(x_i)=H\xi_2(x_i)+O(H^2).$$

From Lemma 7, it follows that

$$\mathbf{L}_1^N(\vec{V}^N-\vec{v})(x_i)-H\mathbf{L}_1^N\vec{F}(x_i)=O(H^2)+\frac{1}{2}a_1(x_i)(x_{i+1}-x_i)v_1''(x_i)-H\xi_1(x_i)-O(H^2),$$

therefore,

$$\mathbf{L}_1^N\{(\vec{V}^N-\vec{v})(x_i)-H\vec{F}(x_i)\}=O(H^2)+\{(x_{i+1}-x_i)-H\}\xi_1(x_i).$$

And for $x_i\in[\tau,1)$, we have

$$\begin{aligned}\mathbf{L}_1^N\{(\vec{V}^N-\vec{v})(x_i)-H\vec{F}(x_i)\}&=O(H^2)\quad\text{in}\quad[\tau,1),\\ \mathbf{L}_2^N\{(\vec{V}^N-\vec{v})(x_i)-H\vec{F}(x_i)\}&=O(H^2)\quad\text{in}\quad[\tau,1).\end{aligned}$$

Proceeding in the same way for $x_i\in(0,\tau)$, we obtain

$$\mathbf{L}_1^N(\vec{V}^N-\vec{v})(x_i)-H\mathbf{L}_1^N\vec{F}(x_i)=O(H^2)+\{h-H\}\xi_1(x_i).$$

Therefore, we have

$$\mathbf{L}_1^N(\vec{V}^N - \vec{v})(x_i) - H\mathbf{L}_1^N\vec{F}(x_i) = O(H) \quad \text{in} \quad (0,\tau),$$

and

$$\mathbf{L}_2^N(\vec{V}^N - \vec{v})(x_i) - H\mathbf{L}_2^N\vec{F}(x_i) = O(H) \quad \text{in} \quad (0,\tau).$$

Now consider the discrete function

$$\vec{z}_i = CN^{-2}(1,1)^T + \vec{S}_{\varepsilon,i}, \quad \text{for} \quad i = 0,\ldots,N,$$

where $\vec{S}_{\varepsilon,i} = (S_{\varepsilon,i}, S_{\varepsilon,i})^T$. Then applying the difference operators on $\vec{z}_i$, we obtain that

$$\mathbf{L}_1^N\vec{z}_i \geq CN^{-2} + C\frac{S_{\varepsilon,i}}{\max(\varepsilon, h_{i+1})}, \quad \text{for } i = 1,\ldots,N-1$$
$$\mathbf{L}_2^N\vec{z}_i \geq CN^{-2} + C\frac{S_{\varepsilon,i}}{\max(\varepsilon, h_{i+1})}, \quad \text{for } i = 1,\ldots,N-1.$$

Hence, for $N/2 < i < N-1$,

$$\mathbf{L}_1^N\vec{z}_i \geq CN^{-2},$$
$$\mathbf{L}_2^N\vec{z}_i \geq CN^{-2}.$$

Using the fact that $N^{-1} \leq H \leq 2N^{-1}$, we obtain

$$\mathbf{L}_1^N\vec{z}_i \geq CN^{-2} \geq CH^2 \quad \text{in} \quad [\tau,1)$$
$$\mathbf{L}_2^N\vec{z}_i \geq CN^{-2} \geq CH^2 \quad \text{in} \quad [\tau,1).$$

From Lemma 5, for $i = 1,\ldots,(N/2-1)$, we have

$$\begin{aligned}\mathbf{L}_1^N\vec{z}_i &\geq C\frac{S_{\varepsilon,i}}{\max(\varepsilon, h_{i+1})} \geq C\varepsilon^{-1}S_{\varepsilon,i} \\ &\geq C\varepsilon^{-1}S_{\varepsilon,N/2} \geq C\varepsilon^{-1}N^{-2} \geq CH.\end{aligned}$$

In a similar fashion one can show that $\mathbf{L}_2^N\vec{z}_i \geq CH$.
At $x_i = 0$, we have

$$\| \vec{V}^N(0) - \vec{v}(0) - H\vec{F}(0) \| \leq CH \leq \vec{z}_0,$$

and at $x_i = 1$,

$$\| \vec{V}^N(1) - \vec{v}(1) - H\vec{F}(1) \| = 0 \leq \vec{z}_N.$$

Now using the discrete maximum principle (Lemma 3), we can show that $\vec{z}_i$ is a barrier function for $\pm(\vec{V}^N - \vec{v} - H\vec{F})(x_i)$. Thus, we have for all i

$$\| (\vec{V}^N - \vec{v} - H\vec{F})(x_i) \| \leq \vec{z}_i.$$

But, for all $i = N/2, \dots, N$

$$\vec{z}_i \ \leq \ CN^{-2} + \prod_{j=1}^{N/2} \left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1} \ \leq \ CN^{-2}.$$

Hence, for $N/2 \leq i \leq N$, we have

$$\begin{aligned} \| (\vec{V}^N - \vec{v} - H\vec{E})(x_i) \| \ &\leq \ \| (\vec{V}^N - \vec{v} - H\vec{F})(x_i) \| + H \| \vec{G}(x_i) \| \\ &\leq \ CN^{-2} + CH \exp\left(-\frac{\alpha x_i}{\varepsilon}\right) \\ &\leq \ CN^{-2}. \end{aligned}$$

Here we have used the fact that $\vec{E} = \vec{F} + \vec{G}$. Thus,

$$\vec{V}^N(x_i) - \vec{v}(x_i) = H\vec{E}(x_i) + O(N^{-2}). \qquad \blacksquare$$

Theorem 10 *For all $x_i \in [\tau, 1)$, we have the second-order convergent result for the smooth component*

$$\| \vec{v}(x_i) - [2\vec{V}^{2N}(x_i) - \vec{V}^N(x_i)] \| \leq CN^{-2}.$$

Proof. Let $x_i \in [\tau, 1)$. Then from Theorem 9 on the mesh $S_{N,\tau}$ we have

$$\vec{V}^N(x_i) - \vec{v}(x_i) = H\vec{E}(x_i) + O(N^{-2}).$$

Similarly, by keeping τ fixed on the mesh $S_{2N,\tau}$, it follows that

$$\vec{V}^{2N}(x_i) - \vec{v}(x_i) = \frac{H}{2}\vec{E}(x_i) + O(N^{-2}).$$

From the above two expressions, we can obtain that

$$\vec{v}(x_i) - (2\vec{V}^{2N}(x_i) - \vec{V}^N(x_i)) = O(N^{-2}). \qquad \blacksquare$$

Theorem 11 *For all $x_i \in (0, \tau)$, we have the following estimate*

$$\| \vec{v}(x_i) - [2\vec{V}^{2N}(x_i) - \vec{V}^N(x_i)] \| \leq CN^{-2} \ln N.$$

Proof. Define the function $\vec{Y}$ on $[0,\tau]$ such that $\vec{Y}(0)=0$, $\vec{Y}(\tau)=1$ and

$$\mathbf{L}_1\vec{Y}(x)=0, \qquad \mathbf{L}_2\vec{Y}(x)=0 \qquad \text{in} \quad 0<x<\tau.$$

Define the discrete approximation $\vec{Y}^N$ of $\vec{Y}$ by $\vec{Y}^N(0)=0$, $\vec{Y}^N(\tau)=1$, and

$$\mathbf{L}_1^N\vec{Y}^N(x_i)=0, \quad \mathbf{L}_2^N\vec{Y}^N(x_i)=0, \quad 1<i<N/2.$$

Then it can be easy to prove that

$$\| \vec{Y}^N(x_i)-\vec{Y}(x_i) \| \leq CN^{-1}\ln N, \quad 0\leq i\leq N/2.$$

Again we set for $0\leq i\leq N/2$

$$\vec{\Phi}_i=\vec{V}^N(x_i)-\vec{v}(x_i)-H\vec{E}(\tau)\,\vec{Y}^N(x_i).$$

Then, $\vec{\Phi}_0=0$, and

$$\vec{\Phi}_\tau=\vec{V}^N(\tau)-\vec{v}(\tau)-H\vec{E}(\tau)\,\vec{Y}^N(\tau)=O(N^{-2}).$$

Now from Theorem 9, we have

$$\mathbf{L}_1^N\vec{\Phi}(x_i)=\mathbf{L}_1^N(\vec{V}^N-\vec{v})(x_i)=O(H^2)+\frac{1}{2}a_1(x_i)(x_{i+1}-x_i)v_1''(x_i),$$
$$\mathbf{L}_2^N\vec{\Phi}(x_i)=\mathbf{L}_2^N(\vec{V}^N-\vec{v})(x_i)=O(H^2)+\frac{1}{2}a_2(x_i)(x_{i+1}-x_i)v_2''(x_i).$$

Using these truncation errors in $(0,\tau)$ and the relation $\varepsilon\leq N^{-1}$, we have

$$| \mathbf{L}_1^N\vec{\Phi}(x_i) | \leq Ch+CH^2\leq C\varepsilon(N^{-1}\ln N)+CN^{-2}\leq CN^{-2}\ln N.$$

Similarly, one can show that $| \mathbf{L}_2^N\vec{\Phi}(x_i) | \leq CN^{-2}\ln N$.
Consequently, we can use the barrier function $\vec{\Psi}_i=CN^{-2}\ln N(1,1)^T$ to obtain

$$\| \vec{\Phi}(x_i) \| \leq CN^{-2}\ln N, \quad \text{for} \quad 0\leq i\leq N/2.$$

Now bounding $\vec{E}(\tau)$ by a constant and using the fact that $\vec{Y}^N(x_i)-\vec{Y}(x_i)=O(N^{-1}\ln N)$. We have, for $0\leq i\leq N/2$

$$\vec{V}^N(x_i)-\vec{v}(x_i)\;=\;H\vec{E}(\tau)\vec{Y}(x_i)+O(N^{-2}\ln N).$$

Now by making N double and keeping τ fixed, we obtain

$$\vec{V}^{2N}(x_i)-\vec{v}(x_i)\;=\;\frac{H}{2}\vec{E}(\tau)\vec{Y}(x_i)+O(N^{-2}\ln N).$$

Hence, we have the following result

$$\vec{v}(x_i)-[2\vec{V}^{2N}(x_i)-\vec{V}^N(x_i)]=O(N^{-2}\ln N).$$ ■

3.3 Extrapolation of the Discrete Singular Component $\vec{W}^N$

Theorem 12 *For all $x_i \in [\tau, 1]$, we have*

$$\| \vec{w}(x_i) - [2\vec{W}^{2N}(x_i) - \vec{W}^N(x_i)] \| \le CN^{-2}.$$

Proof. From [Theorem 1 of Cen (2005)], we have

$$\| \vec{w}_i - \vec{W}_i^N \| \le \| \vec{w}_i \| + \| \vec{W}_i^N \| \le C\exp\left(-\frac{\alpha x_i}{\varepsilon}\right) + CS_{\varepsilon,i}.$$

Now using Lemma 5, one can easily prove that for $N/2 \le i \le N$,

$$\| \vec{w}_i - \vec{W}_i^N \| \le CN^{-2}.$$

The statement of this theorem is an immediate consequence of the above result. ■

Let $\vec{M} = (M_1, M_2)^T$ be the solution of the following BVP:

$$\begin{cases} \mathbf{L}_1\vec{M}(x) = \dfrac{2\varepsilon}{\alpha} a_1(x) w_1''(x), \quad (0,\tau) \\ \mathbf{L}_2\vec{M}(x) = \dfrac{2\varepsilon}{\alpha} a_2(x) w_2''(x), \\ \vec{M}(0) = \vec{M}(\tau) = 0. \end{cases} \tag{11}$$

Then, $\vec{M}$ depends upon τ and is independent of N. Now using the fact that $\| \vec{M}'(0) \| \le C\varepsilon^{-1}$, we have for $0 < x < \tau$

$$\| \vec{M}^{(k)}(x) \| \le C\varepsilon^{-k} \exp\left(-\frac{\alpha x}{\varepsilon}\right), \quad \text{for } k = 1,2,3. \tag{12}$$

Theorem 13 *For all $x_i \in [0, \tau]$, we have the estimate*

$$\| \vec{w}(x_i) - [2\vec{W}^{2N}(x_i) - \vec{W}^N(x_i)] \| \le CN^{-2}\ln^2 N.$$

Proof. It is easy to prove from Taylor's series expansion in $(0, \tau)$ that

$$\mathbf{L}_1^N(\vec{W}^N - \vec{w})(x_i) = \frac{a_1(x_i)}{2} h w_1''(x_i) + \frac{a_1(x_i)}{3!} h^2 w_1'''(x_i) + \{2\varepsilon + a_1(x_i)h\} \frac{h^2}{4!} w_1^{iv}(\xi_i),$$

$$\mathbf{L}_2^N(\vec{W}^N - \vec{w})(x_i) = \frac{a_2(x_i)}{2} h w_2''(x_i) + \frac{a_2(x_i)}{3!} h^2 w_2'''(x_i) + \{2\varepsilon + a_2(x_i)h\} \frac{h^2}{4!} w_2^{iv}(\xi_i),$$

where $\xi_i \in [x_{i-1}, x_{i+1}]$. Now using equation (3) and on simple calculation, we obtain that

$$|\,\mathbf{L}_1^N(\vec{W}^N - \vec{w})(x_i) - \frac{a_1(x_i)}{2} h w_1''(x_i)\,| \leq C\varepsilon^{-1}\exp\left(-\frac{\alpha x_i}{\varepsilon}\right)(N^{-2}\ln^2 N).$$

Thus, we have

$$\mathbf{L}_1^N(\vec{W}^N - \vec{w})(x_i) = \frac{2\varepsilon}{\alpha}\, a_1(x_i)\, w_1''(x_i)(N^{-1}\ln N) + O\left(\varepsilon^{-1}\exp\left(-\frac{\alpha x_i}{\varepsilon}\right)(N^{-2}\ln^2 N)\right).$$

Proceeding in the same way, we can have

$$\mathbf{L}_2^N(\vec{W}^N - \vec{w})(x_i) = \frac{2\varepsilon}{\alpha}\, a_2(x_i)\, w_2''(x_i)(N^{-1}\ln N) + O\left(\varepsilon^{-1}\exp\left(-\frac{\alpha x_i}{\varepsilon}\right)(N^{-2}\ln^2 N)\right).$$

Again, for all $x_i \in (0, \tau)$

$$|\,(\mathbf{L}_1^N - \mathbf{L}_1)\vec{M}(x_i)\,| \leq C\varepsilon^{-1}\exp\left(-\frac{\alpha x_i}{\varepsilon}\right)(N^{-1}\ln N).$$

So, we have

$$(N^{-1}\ln N)\mathbf{L}_1^N\vec{M}(x_i) = \frac{2\varepsilon}{\alpha} a_1(x_i) w_1''(x_i)(N^{-1}\ln N) + O\left(\varepsilon^{-1}\exp\left(-\frac{\alpha x_i}{\varepsilon}\right)(N^{-2}\ln^2 N)\right).$$

Similarly, for $\mathbf{L}_2^N$

$$(N^{-1}\ln N)\mathbf{L}_2^N\vec{M}(x_i) = \frac{2\varepsilon}{\alpha} a_2(x_i) w_2''(x_i)(N^{-1}\ln N) + O\left(\varepsilon^{-1}\exp\left(-\frac{\alpha x_i}{\varepsilon}\right)(N^{-2}\ln^2 N)\right).$$

Therefore,

$$|\mathbf{L}_1^N\{\vec{W}^N - \vec{w} - (N^{-1}\ln N)\vec{M}\}(x_i)| \leq C\varepsilon^{-1}\exp\left(-\frac{\alpha x_i}{\varepsilon}\right)(N^{-2}\ln^2 N),$$

$$|\mathbf{L}_2^N\{\vec{W}^N - \vec{w} - (N^{-1}\ln N)\vec{M}\}(x_i)| \leq C\varepsilon^{-1}\exp\left(-\frac{\alpha x_i}{\varepsilon}\right)(N^{-2}\ln^2 N).$$

Now, for $i = 0, \ldots, N/2$ we define the discrete function

$$\vec{\Gamma}_i = CN^{-2}(1,1)^T + CN^{-2}\vec{S}_{\varepsilon,i}\ \ln^2 N.$$

Then, for all $i = 1, \ldots, N/2 - 1$

$$\mathbf{L}_1^N\vec{\Gamma}_i \geq CN^{-2}\ln^2 N(\mathbf{L}_1^N\vec{S}_{\varepsilon,i}) \geq CN^{-2}\varepsilon^{-1}S_{\varepsilon,i}\ln^2 N \geq CN^{-2}\varepsilon^{-1}\exp\left(\frac{\alpha x_i}{\varepsilon}\right)\ln^2 N.$$

Similarly, we can prove that

$$\mathbf{L}_2^N \vec{\Gamma}_i \geq CN^{-2}\varepsilon^{-1} \exp\left(\frac{\alpha x_i}{\varepsilon}\right) \ln^2 N.$$

Again at $x_i = 0$,

$$\| (\vec{W}^N - \vec{w})(x_i) - (N^{-1}\ln N)\vec{M}(x_i) \| = 0 \leq \vec{\Gamma}_0,$$

at $x_i = \tau$,

$$\| (\vec{W}^N - \vec{w})(x_i) - (N^{-1}\ln N)\vec{M}(x_i) \| \leq CN^{-2} \leq \vec{\Gamma}_{N/2}.$$

Therefore, for $i = 0, \ldots, N/2$, we can use $\vec{\Gamma}_i$ as a barrier function for $\| (\vec{W}^N - \vec{w})(x_i) - (N^{-1}\ln N)\vec{M}(x_i) \|$. But, we know that

$$\| \vec{\Gamma}_i \| \leq CN^{-2}\ln^2 N \quad \text{for} \quad i = 0, \ldots, N/2.$$

Hence, we have for $i = 0, \ldots, N/2$

$$\| (\vec{W}^N - \vec{w})(x_i) - (N^{-1}\ln N)\vec{M}(x_i) \| \leq CN^{-2}\ln^2 N.$$

It follows that for all $x_i \in [0, \tau]$

$$(\vec{W}^N - \vec{w})(x_i) = (N^{-1}\ln N)\vec{M}(x_i) + O(N^{-2}\ln^2 N),$$

that is,

$$(\vec{W}^N - \vec{w})(x_i) = N^{-1}(\alpha\tau/2\varepsilon)\vec{M}(x_i) + O(N^{-2}\ln^2 N). \tag{13}$$

Now $\vec{W}^{2N}$ is computed on the mesh $S_{2N,\tau}$ keeping τ fixed. Then, we have

$$(\vec{W}^{2N} - \vec{w})(x_i) = (2N)^{-1}(\alpha\tau/2\varepsilon)\vec{M}(x_i) + O(N^{-2}\ln^2 N). \tag{14}$$

From equations (13) and (14), we obtain the following:

$$\| \vec{w}(x_i) - [2\vec{W}^{2N}(x_i) - \vec{W}^N(x_i)] \| \leq CN^{-2}\ln^2 N. \qquad \blacksquare$$

The following theorem is the main result of this paper, which proves the ε–uniform second-order error estimate for the Richardson extrapolation solution.

Theorem 14 *For all $x_i \in S_{N,\tau}$, we have the following second-order ε-uniform error estimate*

$$\| \vec{u}(x_i) - [2\vec{U}^{2N}(x_i) - \vec{U}^N(x_i)] \| \leq CN^{-2}\ln^2 N.$$

Proof. For all $x_i \in S_{N,\tau}$, we have

$$\| \vec{u}(x_i) - [2\vec{U}^{2N}(x_i) - \vec{U}^N(x_i)] \| \leq \Big[\| \vec{v}(x_i) - [2\vec{V}^{2N}(x_i) - \vec{V}^N(x_i)] \| + \| \vec{w}(x_i) - [2\vec{W}^{2N}(x_i) - \vec{W}^N(x_i)] \|\Big]$$

Using Theorems 10, 11 and 12, 13 respectively for the first and second terms on the RHS of the above inequality, one can obtain the required ε–uniform second-order error estimate. ■

4 Numerical Example

In this section, to validate the theoretical results, we apply the Richardson extrapolation method to a test problem which was studied by many researchers. The results are presented in the form of tables with maximum point-wise errors and rates of convergence before and after extrapolation. Also, the maximum errors are plotted in loglog scale.

Table 1: Maximum point-wise errors and rates of convergence for u_1 before and after extrapolation.

ε	Number of intervals N						
		64	128	256	512	1024	2048
10^{-3}	before	3.1679e-1	2.0338e-1	1.3183e-1	8.0379e-2	4.6604e-2	2.6318e-2
	rate	0.63937	0.62546	0.71379	0.78638	0.82437	0.85143
	after	5.3699e-2	2.8312e-2	1.1183e-2	4.1536e-3	1.4178e-3	4.7184e-4
	rate	0.92350	1.3401	1.4288	1.5507	1.5873	1.4983
10^{-4}	before	3.1728e-1	2.0367e-1	1.3199e-1	8.0453e-2	4.6631e-2	2.6318e-2
	rate	0.63956	0.62580	0.71418	0.78687	0.82525	0.85299
	after	5.3711e-2	2.8314e-2	1.1165e-2	4.1300e-3	1.3922e-3	4.4527e-4
	rate	0.92372	1.3426	1.4347	1.5688	1.6446	1.6844
10^{-5}	before	3.1734e-1	2.0370e-1	1.3201e-1	8.0462e-2	4.6634e-2	2.6318e-2
	rate	0.63957	0.62583	0.71423	0.78692	0.82533	0.85315
	after	5.3712e-2	2.8314e-2	1.1163e-2	4.1277e-3	1.3897e-3	4.4262e-4
	rate	0.92374	1.3428	1.4353	1.5706	1.6506	1.7047
10^{-6}	before	3.1734e-1	2.0370e-1	1.3201e-1	8.0463e-2	4.6635e-2	2.6318e-2
	rate	0.63958	0.62584	0.71423	0.78692	0.82534	0.85317
	after	5.3713e-2	2.8314e-2	1.1163e-2	4.1275e-3	1.3894e-3	4.4235e-4
	rate	0.92374	1.3428	1.4353	1.5708	1.6512	1.7068
10^{-7}	before	3.1734e-1	2.0370e-1	1.3201e-1	8.0463e-2	4.6635e-2	2.6318e-2
to	rate	0.63958	0.62584	0.71423	0.78692	0.82534	0.85317
10^{-14}	after	5.3713e-2	2.8314e-2	1.1163e-2	4.1274e-3	1.3894e-3	4.4233e-4
	rate	0.92375	1.3428	1.4354	1.5708	1.6513	1.7070

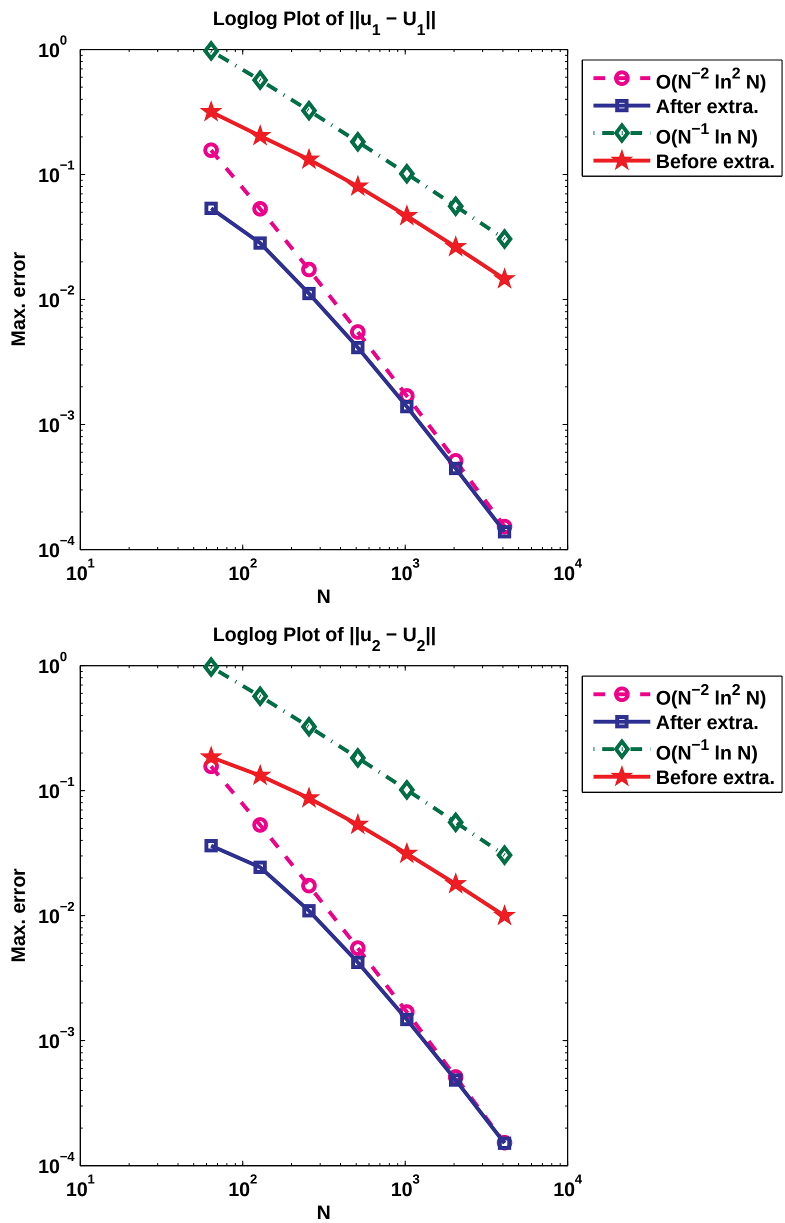

Figure 1: Loglog plot of the error before and after extrapolation for $\varepsilon = 10^{-4}$.

Table 2: Maximum point-wise errors and rates of convergence for u_2 before and after extrapolation.

ε	Number of intervals N						
		64	128	256	512	1024	2048
10^{-3}	before	1.8427e-1	1.3152e-1	8.6982e-2	5.3404e-2	3.1356e-2	1.7935e-2
	rate	0.48655	0.59644	0.70377	0.76820	0.80594	0.84092
	after	3.6322e-2	2.4384e-2	1.0983e-2	4.2806e-3	1.5349e-3	5.4714e-4
	rate	0.57494	1.1507	1.3594	1.4797	1.4882	1.3392
10^{-4}	before	1.8486e-1	1.3181e-1	8.7148e-2	5.3469e-2	3.1361e-2	1.7905e-2
	rate	0.48793	0.59694	0.70477	0.76973	0.80856	0.84571
	after	3.6316e-2	2.4358e-2	1.0948e-2	4.2286e-3	1.4838e-3	4.9068e-4
	rate	0.57618	1.1537	1.3725	1.5109	1.5964	1.6429
10^{-5}	before	1.8492e-1	1.3184e-1	8.7166e-2	5.3476e-2	3.1362e-2	1.7903e-2
	rate	0.48807	0.59698	0.70487	0.76988	0.80882	0.84618
	after	3.6316e-2	2.4356e-2	1.0945e-2	4.2234e-3	1.4787e-3	4.8505e-4
	rate	0.57631	1.1540	1.3738	1.5141	1.6081	1.6765
10^{-6}	before	1.8492e-1	1.3185e-1	8.7168e-2	5.3477e-2	3.1362e-2	1.7903e-2
	rate	0.48808	0.59699	0.70488	0.76990	0.80885	0.84623
	after	3.6316e-2	2.4356e-2	1.0945e-2	4.2229e-3	1.4782e-3	4.8448e-4
	rate	0.57632	1.1541	1.3739	1.5144	1.6093	1.6799
10^{-7}	before	1.8492e-1	1.3185e-1	8.7168e-2	5.3477e-2	3.1362e-2	1.7903e-2
to	rate	0.48808	0.59699	0.70488	0.76990	0.80885	0.84624
10^{-14}	after	3.6316e-2	2.4356e-2	1.0944e-2	4.2228e-3	1.4781e-3	4.8443e-4
	rate	0.57632	1.1541	1.3739	1.5145	1.6094	1.6803

Example 15 *Consider the following system of convection-diffusion BVPs:*

$$\begin{cases} -\varepsilon u_1''(x) - u_1'(x) + 2u_1(x) - u_2(x) = f_1(x), \quad x \in (0,1) \\ -\varepsilon u_2''(x) - 2u_2'(x) - u_1(x) + 4u_2(x) = f_2(x), \\ u_1(0) = u_1(1) = u_2(0) = u_2(1) = 0, \end{cases}$$

where $f_1(x)$, $f_2(x)$ are chosen such that the exact solution is given by

$$\begin{cases} u_1(x) = \dfrac{2-2\exp(-x/\varepsilon)}{1-\exp(-1/\varepsilon)} - 2\sin\left(\dfrac{\pi x}{2}\right), \\ u_2(x) = \dfrac{1-\exp(-x/\varepsilon)}{1-\exp(-1/\varepsilon)} - x\exp(x-1). \end{cases}$$

The maximum point-wise error is calculated as

$$E^N = \| \vec{u}(x_i) - \vec{U}^N(x_i) \|,$$

and the rate of convergence by

$$r = \log_2 \left(\frac{E^N}{E^{2N}} \right).$$

Tables 1 and 2 respectively show the maximum errors and rates of convergence before, after Richardson extrapolation for u_1 and u_2.

The maximum error obtained before and after extrapolation is plotted in Figure 1 for $\varepsilon = 10^{-4}$.

The numerical results shown in the tables and in the figures reveal the theoretical claim of the ε-uniform second-order convergence.

5 Conclusions

In this paper, the Richardson extrapolation method is applied to the coupled system of convection–dominated boundary-value problems of the form (1). First, we discretize the domain by the piecewise-uniform Shishkin mesh and then we apply the classical finite difference scheme to solve the system. Finally, we combine the solutions obtained on N and $2N$-mesh intervals to have the Richardson extrapolation solution, which provides the second-order ε–uniform convergent approximation. ε–uniform error estimate is derived, and the present method is applied to a test problem for verification of the theoretical results.

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