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Parameter identification of beam-column structures on two-parameter elastic foundation

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Abstract: In this paper, a finite element model has been developed for analysing the flexural vibrations of a uniform Timoshenko beam-column on a two-parameter elastic foundation. The beam was discretized into a number of finite elements having four degrees of freedom each. The effect of end springs was incorporated in order to identify the end constraints.
The procedure for identifying geometric and mechanical parameters as well as the end restraints of a beam on two-parameter elastic foundation is based on experimentally measured natural frequencies from dynamic tests on the structure itself.
An iterative statistical identification method, based on the Bayesian approach, was used to identify a set of geometric, physical and mechanical parameters of the above mentioned structure.
Simulated measured natural frequencies of the structure were used throughout the identification method. The engineer's confidence in the modelling of the various parameters was also quantified and incorporated into the revision procedure.

Keywords: Bayesian analysis, parameter estimation, finite elements, two-parameter elastic foundation.

List of main symbols

P	Axial force
N	Normal force
V	Shearing stress
M	Bending moment
p	External axial load per unit length
q	External shear load per unit length
m	External moment per unit length

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u	Axial displacement
v	Transverse displacement
ϕ	Rotation
ε	Longitudinal strain
γ	Shearing strain
χ	Curvature
ρ	Mass density
A	Cross-sectional area
I	Moment of inertia
E	Young's modulus
G	Shear modulus
κ	Shear coefficient
k	Winkler foundation modulus
k_p	Pasternak foundation modulus
K_1, K_2, K	Linear end spring moduli
$\gamma_1, \gamma_2, \gamma$	Rotational end spring moduli
δW_e	External virtual work
δW_i	Internal virtual work
W_{Pe}	Work of compressive axial force
Φ_e	Strain energy
EC_e	Kinetic energy
EP_T	Total dynamic potential energy
$\mathbf{N}_v, \mathbf{N}_\phi$	Vectors of interpolation functions
$\mathbf{q}$	Nodal displacement vector
$\mathbf{K}$	Stiffness matrix
$\mathbf{M}$	Mass matrix
$\mathbf{r}$	Vector of unknown parameters
λ	Eigenvalue
ψ	Eigenvector
$\mathbf{S}$	Sensitivity matrix
$\mathbf{C}_{\varepsilon\varepsilon}$	Covariance matrix of experimental error
$\mathbf{C}_{rr}$	Covariance matrix of prior parameters
β	Confidence coefficient
$\mathbf{H}$	Estimator matrix
L	Length of beam
b	Width of beam
h_i	Depth of i-th element
EI_i	Flexural stiffness of i-th element

1 Introduction

In recent years, inverse problems have been extensively treated in engineering for a wide range of applications. The identification of unknown parameters always requires a mathematical model of the structure under consideration. The two main methods used for inverse analysis are the finite element method (FEM) and the boundary element method (BEM), [Mellings and Aliabadi (1995)]. In this paper the FEM, which is a well established procedure for structural analysis [Zienkiewicz (1997); Atluri, Gallagher and Zienkiewicz (1983)], will be used for the parameter identification of beam-column structures on two parameter elastic foundation.

The subject of beam-columns on elastic foundation occupies an important role in the study of soil-structure interaction problems. Several authors studied the free vibrations of Euler-Bernoulli beams on Winkler elastic foundation [Doyle and Pavlovic (1982); Eisenberger, Yankelevsky and Adin (1985); Laura and Cortinez (1987); Pavlovic and Wylie (1983); De Rosa (1989); Lai, Chen and Hsu (2008)], considering partial foundation and non-uniform elastic foundation, too. Particular foundation models were also studied [Kerr (1964); Fletcher and Hermann (1971); Jones and Xenophontos (1977)]. For a more accurate representation of the characteristics of many practical beams, the elastic foundation was idealized by two-parameter model (Winkler-Pasternak), and the effects of shear deformation (Timoshenko beam) and rotatory inertia on the dynamic behaviour were evaluated [Wang and Stephens (1977); Wang and Gagnon (1978); Filipich and Rosales (1988)]. In the aforementioned studies, the direct problem was solved by determining the natural frequencies and modes of vibration in terms of the system parameters.

In order to ensure high reliability of the structures, their actual behaviour has to be accurately predicted. The attaining of the actual behavioural predictions of structures depends on the correctness of all the parameters affecting the structural response. Generally, systems do not have well-defined properties because they manifest a statistical nature. Therefore, it is necessary to gain as many details about the response as possible in order to treat engineering problems. The treatment of structural systems with statistical properties has been presented in a general form including correlation between variables, too.

Theoretical bases of identification techniques based on the sensitivity analysis can be found in [Eykhoff (1974); Collins and Thomson (1969); Berman and Flannelly (1971); Baruch and Bar Itzhach (1978); Baruch (1984); Adelman and Haftka (1986); Wang, Huang and Zhang (1993)]. The theory developed is applicable to any problem leading to an eigenvalue equation. Referring to the aim of this paper, sensitivity theory is a mathematical field that has its predominant use in investigating the change in the statistical properties of vibrating structural system behaviour due to parameter variations. Sensitivity of the physical property of a dynamic system to variations of different parameters can be determined by estimating the corresponding partial derivatives at some fixed combinations of the parameters themselves. More recently, there has been strong interest in promoting systematic structural optimisation as a useful tool for the practicing structural engineering of large problems [Araùjo, Mota Soares and Moreira de Freitas (1996); Frederiksen (1998); Hongxing, Sol and de Wilde (2000); Araùjo, Mota Soares, Moreira de Freitas, Pedersen and Herskovits (2000)]. Considerable effort has been devoted to the general problem of structural parameters identification. In the past three decades, statistical identification method, which considers the parameters as stochastic variables and provides the assessment from dynamic response, has been extensively used [Hasselman and Hart (1972); Hart (1973); Collins, Hart, Hasselman and Kennedy (1974); Hart and Torkamani (1974); Hart and Yao (1977); Torkamani and Ahmadi (1988) - a, b, c].

The method of Bayesian estimation has been used for system identification in the field of automatic control, too. It should be noted that since the early 1970s, investigations involving statistical properties of vibrating structural systems have been performed. A few earlier papers [Hasselman and Hart (1972); Hart (1973); Collins, Hart, Hasselman and Kennedy (1974); Hart and Torkamani (1974)] illustrate very clearly the principle and the technique of the Bayesian sensitivity analysis through its application to simple systems. More recently, Bayesian identification techniques have been applied to more complex estimation problems [Lai and Ip (2006); Daghia, de Miranda, Ubertini and Viola (2007)]. In general, both systematic and random errors are present in the identified parameters. A measure of the precision of the estimated values can be provided by the variance matrix of the estimated parameters.

In dealing with an identification procedure, a particular mention has to concern the line of research involving the dynamic behaviour of systems and the damage detection in structures by modal vibration characterization. To this end, it is worth noting that identification procedures to improve a finite element model using experimental modal data are presented in [Antonacci, Capecchi, Silvano and Vestroni (1992); Wu and Li (2004)]. Experimental structural vibration data can be used to identify unknown loads applied to a structure [Huang and Shih (2007)] as well as structural damping, stiffness coefficients and restoring forces [Liu (2008) – a, b]. The problem of modelling for parameters identification in distributed structures is worked out in [Baruh and Boka (1992); Baruh and Meirovitch (1985)]. Identification procedures to study steel structures using experimental modal data are reported in [Morassi and Rovere (1997); Kosmatka and Ricles (1999); Bicanic and Chen (1997); Capecchi and Vestroni (1999); Rytter, Krawczuk and Kirkegaard (2000)].

These techniques can be employed to evaluate the real structural behaviour. Moreover, both the location and the extent of structural damage can be correctly determined using only a limited number of natural frequencies.

Although researchers have focused on the identification of structural parameters, to the authors' knowledge no-one has dealt with the identification of geometrical, physical and mechanical parameters of Timoshenko beam structures on two parameter elastic foundation. So, the aim of the present work is to formulate an appropriate FE model for the identification of physical and mechanical parameters of beam-column structures resting on two-parameter elastic foundation (Winkler and Pasternak). The identification procedure incorporates a more accurate model with respect the ones employed in the above mentioned papers. The effects of axial force and rotatory inertia are also included. It should be noted that an Euler-Bernoulli beam element resting on one-parameter foundation (Winkler model) and consisting of two nodes, each having two degrees of freedom of transverse displacement and bending rotation, was studied in [Viola and Hasan (1996); Hasan, Ricci and Viola (1998); Viola, Ricci and Nobile (1999)] to identify restraint conditions. In this paper, the equations of motion are obtained by Hamilton's principle. In the iterative algorithm an estimator matrix depending on the particular identification method adopted is introduced. It depends on the sensitivity matrix, the diagonal covariance matrix of errors on the measured data and the diagonal covariance matrix of initial parameters. The introduction of a coefficient for accelerating the convergence improves the identification technique and can be defined as improved statistical method [Torkamani and Ahmadi (1988) - a, b, c].

A numerical example is presented, where the sensitivity matrix are calculated using the first three natural frequencies of the structure.

As far as the numerical identification procedure is concerned, some further papers should be mentioned. They deal with problems regarding the parameter estimation from measured displacements of crack edges in isotropic or orthotropic materials [Hasan, Piva and Viola (1998); Federici, Piva and Viola (1999)] or structures [Zhang, He, Xiao and Ojalvo (1993); Salane and Baldwin (1990)], which can be investigated by means of the statistical numerical approach under consideration.

This paper is arranged into six sections and three appendices. Section 1 covers the state of the art, that is the introduction to the problem. Section 2 reports the Timoshenko beam equations, namely the equilibrium, congruence, constitutive and fundamental equations for the static case. Section 3 deals with the finite element formulation where various matrices of the system under consideration are assessed. The identification method is illustrated in Section 4. In Section 5, an illustrative example is worked out and numerical results are graphically shown. Finally, in section 6 some conclusions are drawn.

The Appendices of the paper report, in extended notation, the relationships involved in the matrices considered in the eigenvalue problem.

2 Timoshenko beam equations for static case

Consider a plane prismatic beam of length L. The structure is made of linear-elastic, homogeneous and isotropic material. Denote E and G the Young's modulus of elasticity and the shear modulus, respectively, and by ρ the mass density. Let $Oxyz$ be a cartesian coordinate system in which the origin O is located at the centroid of the left end cross-section of the beam, the x-axis coincides with the geometric axis, the y and z axes coincide with the principal axes of the cross-section. The cross sectional area and the moment of inertia with respect to the neutral axis are functions of x and can be denoted by $A(x)$ and $I(x)$, respectively. Consequently, the flexural rigidity $EI(x)$ and mass per unit length $\rho A(x)$ are variable along the x-axis.

The plane and straight beam in Fig. 1, restrained in an arbitrary way at its ends, is supposed to be in equilibrium under a general load system. The y and z components of external forces per unit length are denoted by $q = q(x)$ and $p = p(x)$, respectively, and the external moments per unit length by $m = m(x)$.

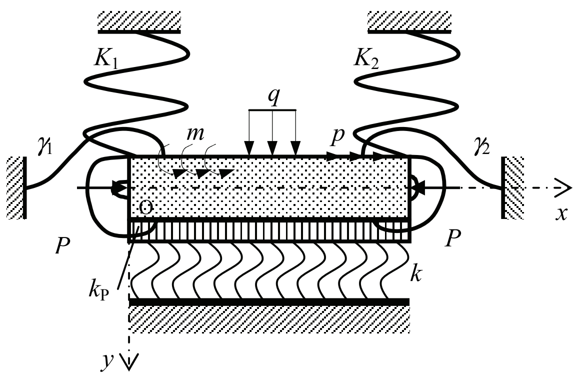

Figure 1: Typical column finite element

At some point on the beam's axis, a normal force $N = N(x)$, a shearing stress $V = V(x)$ and bending moment $M = M(x)$ are considered to act on the left side of an element dx. On the opposite side where the location is x+dx from the origin the stress resultants acquire incremental changes of dN, dV and dM in the interval dx. The equilibrium conditions on stress resultants are

$$\frac{dN}{dx} + p = 0, \frac{dV}{dx} + q = 0, \frac{dM}{dx} + m = V \tag{1}$$

Axial displacement $u = u(x)$, transverse displacement $v = v(x)$ and rotation of the cross-section $\phi = \phi(x)$ are independent variables that allow us to determine the components of strain through the strain-displacement relations. The deformation of the beam is completely defined by the strain components which are the longitudinal strain $\varepsilon(x)$, the shearing strain $\gamma(x)$ and the curvature $\chi(x)$.

The strain-displacement relations can be obtained by the application of the principle of virtual forces. The external virtual work δW_e is

$$\delta W_e = \int_0^l (pu + qv + m\phi)dx, \tag{2}$$

the internal virtual work is

$$\delta W_i = \int_0^l (N\varepsilon + V\gamma + M\chi)dx \tag{3}$$

Combining eqs. (1) and (2) and integrating by parts, the condition for compatibility $\delta W_e = \delta W_i$ gives

$$\int_0^l \left[N\left(\frac{du}{dx} - \varepsilon\right) + V\left(\frac{dv}{dx} + \phi - \gamma\right) + M\left(\frac{d\phi}{dx} - \chi\right)\right] dx = 0 \tag{4}$$

from which it follows that

$$\varepsilon = \frac{du}{dx},\ \gamma = \frac{dv}{dx} + \phi,\ \chi = \frac{d\phi}{dx} \tag{5}$$

for any kinematic boundary conditions.

When the beam is made of linear-elastic, homogeneous and isotropic material, the constitutive equations are:

$$N = EA\varepsilon, \quad V = G\Lambda\gamma, \quad M = EI\chi \tag{6}$$

where $\Lambda = A/\kappa$, with κ being the shear coefficient depending on the geometry of the cross-section.

Combining eqs. (1), (5) and (6), the system of three ordinary differential equations of second order can be obtained

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + p(x) = 0 \tag{7}$$

$$\frac{d}{dx}\left[G\Lambda\left(\frac{dv}{dx}+\phi\right)\right]+q(x)=0 \tag{8}$$

$$\frac{d}{dx}\left(EI\frac{d\phi}{dx}\right)+m(x)=G\Lambda\left(\frac{dv}{dx}+\phi\right) \tag{9}$$

It should be noted that eqs. (8), (9) are coupled through the variables v and ϕ.

3 Finite element formulation

The beam column is partitioned into a finite number of elements. For the typical element shown in Fig. 1, the strain energy including the effects of the two-parameter foundation, shear deformation and end springs is

$$\begin{aligned}\Phi_e=&\frac{1}{2}\int_0^l EI\left(\frac{\partial\phi}{\partial x}\right)^2dx+\frac{1}{2}\int_0^l G\Lambda\left(\frac{\partial v}{\partial x}+\phi\right)^2dx\\&+\frac{1}{2}\int_0^l kv^2dx+\frac{1}{2}\int_0^l k_P\left(\frac{\partial v}{\partial x}\right)^2dx+\frac{1}{2}K_1v\left(x,t\right)^2_{x=0}\\&+\frac{1}{2}\gamma_1\,\phi\left(x,t\right)^2_{x=0}+\frac{1}{2}K_2v\left(x,t\right)^2_{x=l}+\frac{1}{2}\gamma_2\,\phi\left(x,t\right)^2_{x=l}\end{aligned} \tag{10}$$

where x is the local coordinate along the geometrical axis, k is the Winkler foundation modulus, k_P is the shear foundation modulus, K_i (i=1,2) is the linear end spring modulus and γ_i (i=1,2) is the rotational end spring modulus. Note that the effect of end springs is incorporated in order to identify end constraints.

The work done by the compressive axial force is

$$W_{Pe}=-\frac{1}{2}\int_0^l P\left(\frac{\partial v}{\partial x}\right)^2dx \tag{11}$$

The Kinetic energy of the beam element including the rotatory inertia effect is given by

$$EC_e=\frac{1}{2}\int_0^l\rho A\left(\frac{\partial v}{\partial t}\right)^2dx+\frac{1}{2}\int_0^l\rho I\left(\frac{\partial\phi}{\partial t}\right)^2dx=\frac{1}{2}\int_0^l\rho A\dot{v}^2dx+\frac{1}{2}\int_0^l\rho I\dot{\phi}^2dx \tag{12}$$

where ρ is the mass density of the beam material and t is the time.

The representation of displacement v and rotation ϕ is performed by means of algebraic shape functions that exactly satisfy the homogeneous form of the static equations (8) and (9).

Thus, displacement v and rotation ϕ can be expressed as

$$v(x) = \mathbf{N}_v^T \mathbf{q}_e \tag{13}$$

and

$$\phi(x) = \mathbf{N}_\phi^T \mathbf{q}_e \tag{14}$$

where $\mathbf{N}_v$ and $\mathbf{N}_\phi$ are vectors of interpolation functions of displacement $v(x)$ and rotation $\phi(x)$, respectively, and $\mathbf{q}_e$ is the nodal displacement vector.
The shape functions for displacement are given by the following:

$$\begin{aligned}
N_{v1} &= \left[1 - 3\frac{x^2}{l^2} + 2\frac{x^3}{l^3} + \left(1 - \frac{x}{l}\right)\Omega\right]\frac{1}{1+\Omega} \\
N_{v2} &= \left[-x + 2\frac{x^2}{l} - \frac{x^3}{l^2} - \left(x - \frac{x^2}{l}\right)\frac{\Omega}{2}\right]\frac{1}{1+\Omega} \\
N_{v3} &= \left[3\frac{x^2}{l^2} - 2\frac{x^3}{l^3} + \frac{x}{l}\Omega\right]\frac{1}{1+\Omega} \\
N_{v4} &= \left[\frac{x^2}{l} - \frac{x^3}{l^2} + \left(x - \frac{x^2}{l}\right)\frac{\Omega}{2}\right]\frac{1}{1+\Omega}
\end{aligned} \tag{15}$$

while the shape functions for rotation are:

$$\begin{aligned}
N_{\phi 1} &= \frac{6}{l^2}\left(-x + \frac{x^2}{l}\right)\frac{1}{1+\Omega} \\
N_{\phi 2} &= \left[-1 + 4\frac{x}{l} - 3\frac{x^2}{l^2} - \left(1 - \frac{x}{l}\right)\Omega\right]\frac{1}{1+\Omega} \\
N_{\phi 3} &= \frac{6}{l^2}\left(+x - \frac{x^2}{l}\right)\frac{1}{1+\Omega} \\
N_{\phi 4} &= \left[2\frac{x}{l} - 3\frac{x^2}{l^2} - \frac{x}{l}\Omega\right]\frac{1}{1+\Omega}
\end{aligned} \tag{16}$$

where

$$\Omega = \frac{12EI}{G\Lambda l^2} \tag{17}$$

is the ratio of the beam flexural stiffness to the shear stiffness.
The bending curvature χ and shear strain γ are expressed as

$$\chi = \frac{\partial \phi}{\partial x} = \mathbf{B}_b \mathbf{q}_e \tag{18}$$

$$\gamma = \frac{\partial v}{\partial x} + \phi = \mathbf{B}_s \mathbf{q}_e \tag{19}$$

where

$$\mathbf{B}_b = \frac{\partial}{\partial x} \mathbf{N}_\phi \tag{20}$$

$$\mathbf{B}_s = \frac{\partial}{\partial x} \mathbf{N}_v + \mathbf{N}_\phi = \mathbf{B}_v + \mathbf{N}_\phi \tag{21}$$

Substituting eqs. (18)-(21) into eqs. (10)-(12) gives

$$\Phi_e = \frac{1}{2}\mathbf{q}_e^T \mathbf{k}_{eb} \mathbf{q}_e + \frac{1}{2}\mathbf{q}_e^T \mathbf{k}_{es} \mathbf{q}_e + \frac{1}{2}\mathbf{q}_e^T \mathbf{k}_{eW} \mathbf{q}_e + \frac{1}{2}\mathbf{q}_e^T \mathbf{k}_{eP} \mathbf{q}_e + \frac{1}{2}\mathbf{q}_e^T \mathbf{k}_{e1} \mathbf{q}_e + \frac{1}{2}\mathbf{q}_e^T \mathbf{k}_{e2} \mathbf{q}_e \tag{22}$$

$$EC_e = \frac{1}{2}\dot{\mathbf{q}}_e^T \mathbf{m}_{ev} \dot{\mathbf{q}}_e + \frac{1}{2}\dot{\mathbf{q}}_e^T \mathbf{m}_{e\phi} \dot{\mathbf{q}}_e \tag{23}$$

$$W_{Pe} = -\frac{1}{2}\mathbf{q}_e^T \mathbf{k}_{eg} \mathbf{q}_e \tag{24}$$

where

$$\mathbf{k}_{eb} = \int_0^l \mathbf{B}_b^T EI \mathbf{B}_b dx \tag{25}$$

is the flexural stiffness matrix,

$$\mathbf{k}_{es} = \int_0^l \mathbf{B}_s^T G\Lambda \mathbf{B}_s dx \tag{26}$$

is the shear stiffness matrix.

$$\mathbf{k}_{eW} = \int_0^l \mathbf{N}_v^T k \mathbf{N}_v dx \tag{27}$$

is the stiffness matrix due to Winkler foundation,

$$\mathbf{k}_{eP} = \int_0^l \mathbf{B}_v^T k_p \mathbf{B}_v dx \tag{28}$$

is the stiffness matrix due to shear foundation,

$$\mathbf{k}_{e1} = K_1 \left(\mathbf{N}_v^T \mathbf{N}_v\right)_{x=0} + \gamma_1 \left(\mathbf{N}_\phi^T \mathbf{N}_\phi\right)_{x=0} \tag{29}$$

is the stiffness matrix due to left end springs,

$$\mathbf{k}_{e2} = K_2 \left(\mathbf{N}_v^T \mathbf{N}_v\right)_{x=l} + \gamma_2 \left(\mathbf{N}_\phi^T \mathbf{N}_\phi\right)_{x=l} \tag{30}$$

is the stiffness matrix due to right end springs,

$$\mathbf{k}_{eg} = \int_0^l \mathbf{B}_v^T P \mathbf{B}_v dx \tag{31}$$

is the geometric stiffness matrix,

$$\mathbf{m}_{ev} = \int_0^l \rho A \mathbf{N}_v \mathbf{N}_v^T dx \tag{32}$$

is the consistent mass matrix for translational inertia,

$$\mathbf{m}_{e\phi} = \int_0^l \rho I \mathbf{N}_\phi \mathbf{N}_\phi^T dx \tag{33}$$

is the consistent mass matrix for rotatory inertia, and the superposed dot denotes differentiation with respect to time t.

The stiffness matrix $\mathbf{k}_e$ and the consistent mass matrix $\mathbf{m}_e$ for the beam element can be obtained as

$$\mathbf{k}_e = \mathbf{k}_{eb} + \mathbf{k}_{es} + \mathbf{k}_{eW} + \mathbf{k}_{eP} + \mathbf{k}_{e1} + \mathbf{k}_{e2} - \mathbf{k}_{eg} \tag{34}$$

$$\mathbf{m}_e = \mathbf{m}_{ev} + \mathbf{m}_{e\phi} \tag{35}$$

Matrices (34)-(35) are listed in extensive notation in Appendix A.

Inserting the total dynamic potential energy

$$EP_T = \sum_e \left(\Phi_e + EC_e - W_{Pe}\right) \tag{36}$$

into Hamilton's principle leads to the governing matrix equation for free vibrations of the Timoshenko beam-column on the two-parameter elastic foundation as

$$\mathbf{Kq} + \mathbf{M\ddot{q}} = \mathbf{0} \tag{37}$$

where $\mathbf{q}$ is the global displacement vector and

$$\mathbf{K} = \sum_{i=1}^{n} \left(\mathbf{K}_{eb} + \mathbf{K}_{es} + \mathbf{K}_{eW} + \mathbf{K}_{eP} + \mathbf{K}_{e1} + \mathbf{K}_{e2} - \mathbf{K}_{eg}\right) \tag{38}$$

is the global stiffness matrix,

$$\mathbf{M} = \sum_{i=1}^{n} \left(\mathbf{M}_{ev} + \mathbf{M}_{e\phi}\right) \tag{39}$$

is the global consistent mass matrix.

The global stiffness matrix can be expressed as

$$\mathbf{K} = diag(\boldsymbol{\alpha}, \boldsymbol{\beta}, \ldots, \boldsymbol{\beta}, \boldsymbol{\xi}) \tag{40}$$

where matrices $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\xi}$ are expressed as

$$\boldsymbol{\alpha} = \begin{bmatrix} a' & b & c & d \\ b & e' & f & g \\ c & f & 2a & 0 \\ d & g & 0 & 2e \end{bmatrix} \tag{41}$$

$$\boldsymbol{\beta} = \begin{bmatrix} 2a & 0 & c & d \\ 0 & 2e & f & g \\ c & f & 2a & 0 \\ d & g & 0 & 2e \end{bmatrix} \tag{42}$$

$$\boldsymbol{\xi} = \begin{bmatrix} 2a & 0 & c & d \\ 0 & 2e & f & g \\ c & f & a'' & h \\ d & g & h & e'' \end{bmatrix} \tag{43}$$

The explicit expressions for the respective element matrices are listed in appendix B.

The global consistent mass matrix can be expressed as

$$\mathbf{M} = diag(\boldsymbol{\alpha}_m, \boldsymbol{\beta}_m, \ldots, \boldsymbol{\beta}_m, \boldsymbol{\xi}_m) \tag{44}$$

where

$$\boldsymbol{\alpha}_m = \begin{bmatrix} A & B & C & D \\ B & E & F & G \\ C & F & 2A & 0 \\ D & G & 0 & 2E \end{bmatrix} \tag{45}$$

$$\boldsymbol{\beta}_m = \begin{bmatrix} 2A & 0 & C & D \\ 0 & 2E & F & G \\ C & F & 2A & 0 \\ D & G & 0 & 2E \end{bmatrix} \tag{46}$$

$$\boldsymbol{\xi}_m = \begin{bmatrix} 2A & 0 & C & D \\ 0 & 2E & F & G \\ C & F & A & H \\ D & G & H & E \end{bmatrix} \tag{47}$$

The explicit expressions for the respective element matrices are listed in appendix C.

4 The identification method

Denote by

$$\mathbf{r} = (r_1\ r_2\ \dots\ r_m)^T \tag{48}$$

the vector of unknown parameters r_i (i=1, 2, ... m) to be identified, e.g. geometric or structural parameters. Mass and stiffness matrices in eq. (49) are functions of these parameters of the system and therefore the eigenvalues and eigenvectors are implicit functions of these same parameters. Eq. (37) can be written as

$$[\mathbf{K} - \lambda_i(\mathbf{r})\mathbf{M}]\,\boldsymbol{\psi}_i(\mathbf{r}) = \mathbf{0} \tag{49}$$

where $\lambda_i(r)$, $\boldsymbol{\psi}_i(r)$ are the eigenvalues and the eigenvectors, respectively.
The functional relationship between the modal characteristics and the parameters can be expressed in terms of a Taylor's series expansion

$$\left\{\frac{\boldsymbol{\lambda}(\mathbf{r})}{\boldsymbol{\psi}(\mathbf{r})}\right\} = \left\{\frac{\boldsymbol{\lambda}(\mathbf{r}_a)}{\boldsymbol{\psi}(\mathbf{r}_a)}\right\} + \mathbf{S}(\mathbf{r} - \mathbf{r}_a) \tag{50}$$

where

$$\mathbf{r}_a = (r_{1a}\ r_{2a}\ \dots\ r_{ma})^T \tag{51}$$

is the vector of prior estimates of parameters, $\boldsymbol{\lambda}(\mathbf{r}_a)$ and $\boldsymbol{\psi}(\mathbf{r}_a)$ are are the vectors of eigenvalues and eigenvectors when $\mathbf{r} = \mathbf{r}_a$,

$$\mathbf{S} = \left|\frac{\partial\boldsymbol{\lambda}/\partial\mathbf{r}}{\partial\boldsymbol{\psi}/\partial\mathbf{r}}\right| \tag{52}$$

is the sensitivity matrix. The partial derivatives of $\boldsymbol{\lambda}(\mathbf{r}_a)$ e $\boldsymbol{\psi}(\mathbf{r}_a)$ with respect to $\mathbf{r}$ are

$$\frac{\partial\lambda_i}{\partial r_j} = \boldsymbol{\psi}_i^T\left(\frac{\partial\mathbf{K}}{\partial r_j} - \lambda_i\frac{\partial\mathbf{M}}{\partial r_j}\right)\boldsymbol{\psi}_i \tag{53}$$

$$\frac{\partial \boldsymbol{\psi}_i}{\partial r_j} = -\left[\mathbf{K} - \lambda_i \mathbf{M}\right]^{-1} \boldsymbol{\psi}_i^T \left(\frac{\partial \mathbf{K}}{\partial r_j} - \frac{\partial \lambda_i}{\partial r_j}\mathbf{M} - \lambda_i \frac{\partial \mathbf{M}}{\partial r_j}\right) \boldsymbol{\psi}_i \tag{54}$$

Substituting $\lambda_i = \omega_i^2$ into eqs. (53) and (54) gives

$$\frac{\partial \omega_i}{\partial r_j} = \frac{1}{2\omega_i} \boldsymbol{\psi}_i^T \left(\frac{\partial \mathbf{K}}{\partial r_j} - \omega_i^2 \frac{\partial \mathbf{M}}{\partial r_j}\right) \boldsymbol{\psi}_i \tag{55}$$

$$\frac{\partial \boldsymbol{\psi}_i}{\partial r_j} = -\left[\mathbf{K} - \omega_i^2 \mathbf{M}\right]^{-1} \boldsymbol{\psi}_i^T \left(\frac{\partial \mathbf{K}}{\partial r_j} - 2\omega_i \frac{\partial \omega_i}{\partial r_j}\mathbf{M} - \omega_i^2 \frac{\partial \mathbf{M}}{\partial r_j}\right) \boldsymbol{\psi}_i \tag{56}$$

The iterative algorithm for the identification method can be written as

$$\hat{\mathbf{r}} = \mathbf{r} + \mathbf{H} \left\{ \frac{\Delta \boldsymbol{\lambda}}{\Delta \boldsymbol{\psi}} \right\} = \left(\hat{r}_1 \quad \hat{r}_2 \quad \ldots\ldots\ldots \quad \hat{r}_m\right)^T \tag{57}$$

where $\mathbf{r}$ is the vector of estimated parameters at the i-th iteration, $\hat{\mathbf{r}}$ the same vector at the $(i+1)$-th iteration,

$$\mathbf{H} = \beta^{-1} \mathbf{C}_{rr} \mathbf{S}^T (\beta^{-1} \mathbf{S} \mathbf{C}_{rr} \mathbf{S}^T + \mathbf{C}_{\varepsilon\varepsilon})^{-1} \tag{58}$$

is an estimator matrix, with $\mathbf{C}_{\varepsilon\varepsilon} = diag(b_1 \ldots b_n)$ the diagonal covariance matrix of errors on the measured data, $\mathbf{C}_{rr} = diag(a_1 \ldots a_n)$ the diagonal covariance matrix of the priori parameters and β [Berman and Flannelly (1971); Baruch and Bar (1978)] a confidence coefficient, and

$$\left\{ \frac{\Delta \boldsymbol{\lambda}}{\Delta \boldsymbol{\psi}} \right\} = \left\{ \frac{\boldsymbol{\lambda}_s}{\boldsymbol{\psi}_s} \right\} - \left\{ \frac{\boldsymbol{\lambda}(\mathbf{r})}{\boldsymbol{\psi}(\mathbf{r})} \right\} \tag{59}$$

where $\boldsymbol{\lambda}_s$ and $\boldsymbol{\psi}_s$ are vectors of experimentally measured values, $\boldsymbol{\lambda}(\mathbf{r})$ and $\boldsymbol{\psi}(\mathbf{r})$ are vectors of eigenvalues and eigenvectors, respectively, obtained from solution to eq. (49) with $\mathbf{r} = \mathbf{r}_a$.

Convergence is evaluated at the end of each iteration prescribing confidence bounds, as a rule 95%, to $\boldsymbol{\lambda}(\mathbf{r})$ and $\boldsymbol{\psi}(\mathbf{r})$.

If the number of measured eigenvalues and independent elements of eigenvectors is equal to the number of parameters

$$\mathbf{H} = \mathbf{S}^{-1} \tag{60}$$

Usually, the number of parameters exceeds the number of measured eigenvalues and eigenvectors (under-determined system). The solution of the problem depends on the prior estimates, the covariance matrices $\mathbf{C}_{\varepsilon\varepsilon}$ and $\mathbf{C}_{rr}$, and the coefficient β. The structural analyst establishes the prior parameter values, the covariance matrix of the priori parameters and the confidence coefficient; the structural experimentalist establishes the covariance matrix of errors on the measured data.

It should be noted that, as appears from equation (57), during the iteration process the estimator matrix (58) relates the revised parameters to the prior ones. Moreover, comparison of the vectors for the initial estimates and for the revised parameters will indicate which of the properties are found more accurately. According to the Bayesian point of view, different tests must be repeated on series of tests in order to obtain statistically relevant results.

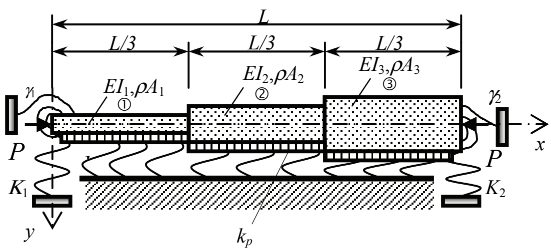

Figure 2: Timoshenko beam-column

Table 1: Geometric and mechanical characteristics

lenght	L	7.5 m
width	b	0.2 m
Depth of element 1	h_1	0.30 m
Depth of element 2	h_2	0.40 m
Depth of element 3	h_3	0.50 m
Young's modulus	E	31000000 KN·m^{-2}
flexural stiffness of element 1	EI_1	13950 KN·m^2
flexural stiffness of element 2	EI_2	33067 KN·m^2
flexural stiffness of element 3	EI_3	64583 KN·m^2
Mass density	ρ	25 KN·m^{-3}
Winkler foundation modulus	k	21.7 KN·m^{-2}
Pasternak foundation modulus	k_p	25 KN·m^{-2}
Linear end spring modulus	K_1=K_2=K	300 KN·m^{-1}
Rotational end spring modulus	$\gamma_1 = \gamma_2 = \gamma$	250 KN·m
Shear modulus	G	3/8E
Shear coefficient	κ	1.5
Axial force	P	50 KN

5 Illustrative example

Consider a Timoshenko beam-column with end springs and discontinuity in thickness, supported on an elastic foundation as depicted in Fig. 2. The elastic foundation is idealized as constant two-parameter model characterized by two moduli k and k_P.

The geometric and mechanical properties of the beam-column are illustrated in Tab. 1.

The previously illustrated iterative method of identification is now applied to identify the nine parameters collected in the vector

$$\mathbf{r} = \begin{pmatrix} K & \gamma_1 & k & k_p & EI_1 & EI_2 & EI_3 & G & P \end{pmatrix} = \begin{pmatrix} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 & r_8 & r_9 \end{pmatrix} \quad (61)$$

In the present identification procedure, only the first three natural frequencies and modal shapes are required to obtain an estimate.

To simulate the experimental data, the measured natural frequencies are obtained from the eigenvalues obtained by solving the characteristic equation with the "actual" parameters as in Tab. 1.

$$\mathbf{r} = \begin{pmatrix} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 & r_8 & r_9 \end{pmatrix} = \begin{pmatrix} 300 & 250 & 21.7 & 25 & 13950 & 33067 & 64583 & 11625000 & 50 \end{pmatrix} \quad (62)$$

The analysis is performed by using ten finite elements for each beam element of constant depth.

The coefficients of variation in the diagonal covariance matrix of errors in measured data are chosen as: $a_1=a_2=a_3=0.10$.

The priori coefficients of variation are chosen as: $b_1=0.0037$, $b_2=0.0155$, $b_3=0.0093$, $b_4=0.0245$, $b_5=0.0065$, $b_6=0.0042$, $b_7=0.0145$,$b_8=0.04$, $b_9=0.019$.

Note that the choice of these coefficients is based on the experience of the analyst and convergence is based on the prescribed confidence bound equal to 99% of the experimentally measured frequencies.

Assuming the initial parameters 15% greater than the exact values and β=0.001, the iterative procedure allows the determination of the final estimates as listed in Tab. 2. Note that all the parameters are well estimated, with errors varying from 0.05% to 2.84%. The convergence for this case has been obtained after 11 iterations.

Figg. 3-11 show the influence of the coefficient β on the convergence velocity of the identified parameters.

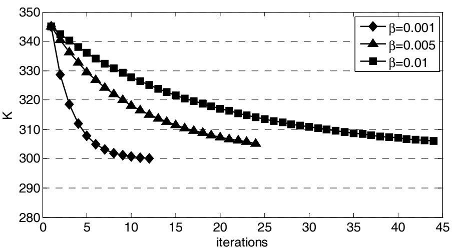

Figure 3: Convergence of linear end spring modulus

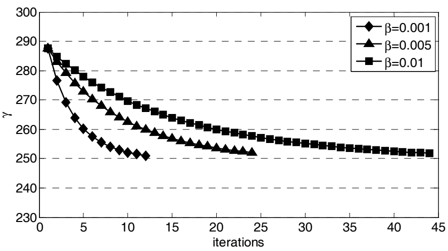

Figure 4: Convergence of rotational end spring modulus

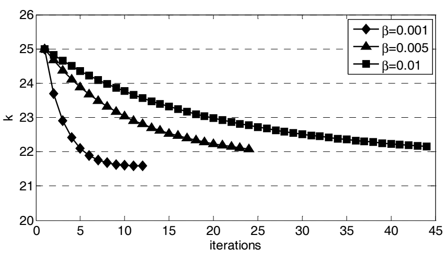

Figure 5: Convergence of Winkler foundation modulus

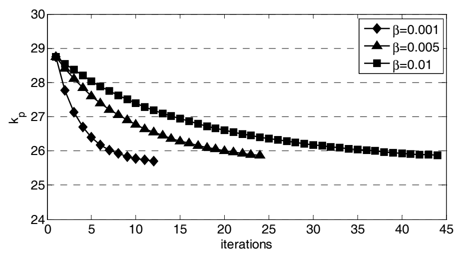

Figure 6: Convergence of Pasternak foundation modulus

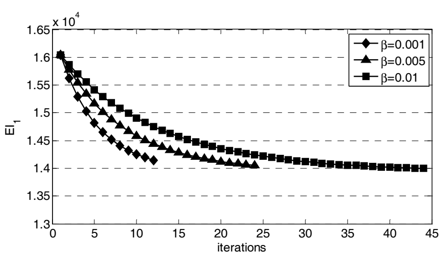

Figure 7: Convergence of bending stiffness for the first element

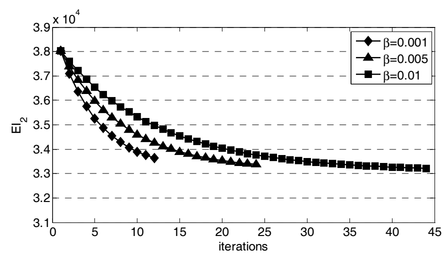

Figure 8: Convergence of bending stiffness for the second element

Another set of initial parameters was considered with the same coefficients of variance-covariance matrices and coefficient β. Underestimated initial parameters were selected in this case. It is important to note that also in this case the convergence was good for all parameters with differences between exact and identified values ranging between 0.16% and 3.36% . The number of iterations required was the same as in the previous case (see Tab. 3).

From these two cases we can conclude that there is a problem symmetry when the

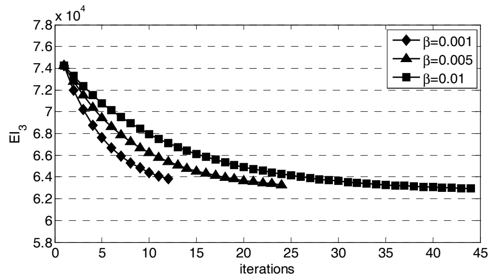

Figure 9: Convergence of bending stiffness for the third element

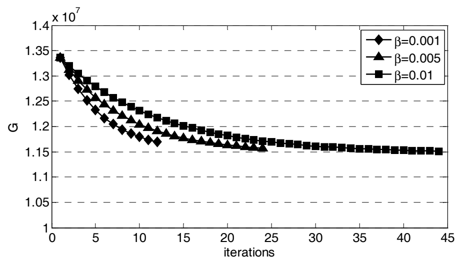

Figure 10: Convergence of shear modulus

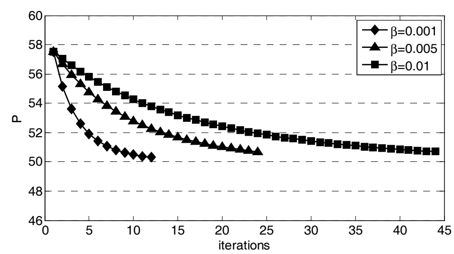

Figure 11: Convergence of axial force

initial values of the estimated parameters are all underestimated or overestimated.

Table 2: Final estimates and error (first set of initial parameters)

parameter	K	γ	k	k_p	EI_1
Actual values	300	250	21.7	25	13950
Final estimates	300.139	251.025	21.583	25.710	14146.499
Final error	0.05%	0.41%	0.54%	2.84%	1.41%
parameter	EI_2	EI_3	G	P	
Actual values	33067	64583	11625000	-50	
Final estimates	33648.966	63848.796	11692426.403	-50.316	
Final error	1.76%	1.14%	0.58%	0.63%	

6 Conclusions and remarks

A method for the statistical identification of a beam-column resting on two-parameter foundation has been proposed. It uses experimental response measurements of natural frequencies to improve some parameters of a finite element model. The finite

Table 3: Final estimates and error (second set of initial parameters)

parameter	K	γ	k	k_p	EI_1
Actual values	300	250	21.7	25	13950
Final estimates	299.503	248.948	21.853	24.159	13728.14
Final error	0.16%	0.42%	0.70%	3.36%	1.59%

parameter	EI_2	EI_3	G	P
Actual values	33067	64583	11625000	-50
Final estimates	32443.841	65187.764	11540230.02	-49.66
Final error	1.88%	0.94%	0.73%	0.68%

element model includes end spring effects in order to detect end constraints. In the present study, the proposed method basically relies on three items, namely an accurate mathematical model of the structure, a set of reliable modal model data and a parameter estimation method based on the Bayesian sensitivity analysis.

The introduction of a confidence coefficient accelerating the convergence characterizes the identification technique which can be defined as an improved statistical method.

A numerical example is presented, where the sensitivity matrix is calculated using the first three natural frequencies. This worked-out example is based on pseudo-experimentally determined data.

In the statistical Bayesian estimation, priority is given to assessment of uncertainties. The method takes into account both the confidence associated with mathematical modelling and parameter estimates. The parameters that have to be estimated are considered as stochastic values of fixed probability distribution. Incorporating the uncertainties into the initial estimates, the scheme will lead to improved estimate values for the parameters. As in each inverse problems, the parameters in the mathematical model have to be adjusted repeatedly until its analytical response match satisfactorily with those associated with the physical structure. This is accomplished through two weighting matrices containing the confidences on the measured natural frequencies and on the initial estimates.

It is worthy of remark to point out that the present study has the innovative aspect of contemplating identification of all the parameters affecting the dynamic free vibration of a structure, that is to say the parameters describing its geometry, density, boundary conditions and elastic constants.

It is well known that investigations and monitoring of structures are essential tools to improve the knowledge of their structural behaviour. As far as the practicability of a procedure for parameter identification is concerned, some noteworthy points are related to the importance of structures under investigation. Firstly, measurements gathered by means of an automatic monitoring system installed in the structure can make the refinement of the mathematical model possible. Secondly, assumptions about the law of deformation of materials and hypotheses involving various boundary conditions have to be made. Thirdly, a mathematical reference model for the numerical simulation of the structures is always required. Usually, such a model is based on the finite element method.

A number of researchers have presented methods to improve the analytical model of structural systems and several non-destructive evaluations techniques have been proposed especially for the determination of material properties of damaged and non-damaged structures.

On the upshot, the identification techniques as useful tools in the structural analysis process are gaining more and more popularity. However, it should be mentioned that some aspects involving the inverse problems of structural systems are not completely resolved, such as the uniqueness of the results [Udwadia, Sharma and Shah (1978)], the incompleteness of the measured data [Berman and Flannelly (1971)], the ill-conditioned equations arising in structural system identification [Hasan and Viola (1997)], among others. In the latter paper, the singular value decomposition method is used to investigate the ill-conditioning of physical and modal idenification methods and the quantities which make the identification problem well-conditioned are pointed out.

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Appendix A Stiffness and mass matrices

$$\mathbf{k}_{eb} = \frac{EI}{(1+\Omega)^2 l^3} \begin{bmatrix} 12 & -6l & -12 & -6l \\ -6l & \left(4+2\Omega+\Omega^2\right) l^2 & 6l & \left(2-2\Omega-\Omega^2\right) l^2 \\ -12 & 6l & 12 & 6l \\ -6l & \left(2-2\Omega-\Omega^2\right) l^2 & 6l & \left(4+2\Omega+\Omega^2\right) l^2 \end{bmatrix} \tag{A1}$$

$$\mathbf{k}_{es}=\frac{G\Lambda\Omega^2}{4l\,(1+\Omega)^2}\begin{bmatrix}4 & -2l & -4 & -2l\\ -2l & l^2 & -2l & l^2\\ -4 & -2l & 4 & 2l\\ -2l & l^2 & 2l & l^2\end{bmatrix}\tag{A2}$$

$$\mathbf{k}_{ef}=\frac{kl}{(1+\Omega)^2}$$
$$\begin{bmatrix}\frac{13}{35}+\frac{7\Omega}{10}+\frac{\Omega^2}{3} & -\left(\frac{11}{210}+\frac{11\Omega}{120}+\frac{\Omega^2}{24}\right)l & \frac{9}{70}+\frac{3\Omega}{10}+\frac{\Omega^2}{6} & \left(\frac{13}{420}+\frac{3\Omega}{40}+\frac{\Omega^2}{24}\right)l\\ -\left(\frac{11}{210}+\frac{11\Omega}{120}+\frac{\Omega^2}{24}\right)l & \left(\frac{1}{105}+\frac{11\Omega}{60}+\frac{\Omega^2}{120}\right)l^2 & -\left(\frac{13}{420}+\frac{3\Omega}{40}+\frac{\Omega^2}{24}\right)l & -\left(\frac{1}{140}+\frac{\Omega}{60}+\frac{\Omega^2}{120}\right)l^2\\ \frac{9}{70}+\frac{3\Omega}{10}+\frac{\Omega^2}{6} & -\left(\frac{13}{420}+\frac{3\Omega}{40}+\frac{\Omega^2}{24}\right)l & \frac{13}{35}+\frac{7\Omega}{10}+\frac{\Omega^2}{3} & \left(\frac{11}{210}+\frac{11\Omega}{120}+\frac{\Omega^2}{24}\right)l\\ \left(\frac{13}{420}+\frac{3\Omega}{40}+\frac{\Omega^2}{24}\right)l & -\left(\frac{1}{140}+\frac{\Omega}{60}+\frac{\Omega^2}{120}\right)l^2 & \left(\frac{11}{210}+\frac{11\Omega}{120}+\frac{\Omega^2}{24}\right)l & \left(\frac{1}{105}+\frac{11\Omega}{60}+\frac{\Omega^2}{120}\right)l^2\end{bmatrix}\tag{A3}$$

$$\mathbf{k}_{eP}=\frac{k_p}{l\,(1+\Omega)^2}\cdot$$
$$\begin{bmatrix}\frac{6}{5}+2\Omega+\Omega^2 & -\frac{1}{10}l & -\left(\frac{6}{5}+2\Omega+\Omega^2\right) & -\frac{1}{10}l\\ -\frac{1}{10} & \left(\frac{2}{15}-\frac{\Omega}{6}+\frac{\Omega^2}{12}\right)l^2 & \frac{1}{10}l & \left(-\frac{1}{30}+\frac{\Omega^2}{12}\right)l^2\\ -\left(\frac{6}{5}+2\Omega+\Omega^2\right) & \frac{1}{10}l & \frac{6}{5}+2\Omega+\Omega^2 & \frac{1}{10}l\\ -\frac{1}{10} & \left(-\frac{1}{30}+\frac{\Omega^2}{12}\right)l^2 & \frac{1}{10}l & \left(\frac{2}{15}-\frac{\Omega}{6}+\frac{\Omega^2}{12}\right)l^2\end{bmatrix}\tag{A4}$$

$$\mathbf{k}_{eg}=\frac{P}{l\,(1+\Omega)^2}\cdot$$
$$\begin{bmatrix}\frac{6}{5}+2\Omega+\Omega^2 & -\frac{1}{10}l & -\left(\frac{6}{5}+2\Omega+\Omega^2\right) & -\frac{1}{10}l\\ -\frac{1}{10} & \left(\frac{2}{15}-\frac{\Omega}{6}+\frac{\Omega^2}{12}\right)l^2 & \frac{1}{10}l & \left(-\frac{1}{30}+\frac{\Omega^2}{12}\right)l^2\\ -\left(\frac{6}{5}+2\Omega+\Omega^2\right) & \frac{1}{10}l & \frac{6}{5}+2\Omega+\Omega^2 & \frac{1}{10}l\\ -\frac{1}{10} & \left(-\frac{1}{30}+\frac{\Omega^2}{12}\right)l^2 & \frac{1}{10}l & \left(\frac{2}{15}-\frac{\Omega}{6}+\frac{\Omega^2}{12}\right)l^2\end{bmatrix}\tag{A5}$$

$$\mathbf{m}_{ev} = \frac{\rho A l}{(1+\Omega)^2} \cdot$$

$$\begin{bmatrix} \frac{13}{35}+\frac{7\Omega}{10}+\frac{\Omega^2}{3} & -\left(\frac{11}{210}+\frac{11\Omega}{120}+\frac{\Omega^2}{24}\right)l & \frac{9}{70}+\frac{3\Omega}{10}+\frac{\Omega^2}{6} & \left(\frac{13}{420}+\frac{3\Omega}{40}+\frac{\Omega^2}{24}\right)l \\ -\left(\frac{11}{210}+\frac{11\Omega}{120}+\frac{\Omega^2}{24}\right)l & \left(\frac{1}{105}+\frac{11\Omega}{60}+\frac{\Omega^2}{120}\right)l^2 & -\left(\frac{13}{420}+\frac{3\Omega}{40}+\frac{\Omega^2}{24}\right)l & -\left(\frac{1}{140}+\frac{\Omega}{60}+\frac{\Omega^2}{120}\right)l^2 \\ \frac{9}{70}+\frac{3\Omega}{10}+\frac{\Omega^2}{6} & -\left(\frac{13}{420}+\frac{3\Omega}{40}+\frac{\Omega^2}{24}\right)l & \frac{13}{35}+\frac{7\Omega}{10}+\frac{\Omega^2}{3} & \left(\frac{11}{210}+\frac{11\Omega}{120}+\frac{\Omega^2}{24}\right)l \\ \left(\frac{13}{420}+\frac{3\Omega}{40}+\frac{\Omega^2}{24}\right)l & -\left(\frac{1}{140}+\frac{\Omega}{60}+\frac{\Omega^2}{120}\right)l^2 & \left(\frac{11}{210}+\frac{11\Omega}{120}+\frac{\Omega^2}{24}\right)l & \left(\frac{1}{105}+\frac{11\Omega}{60}+\frac{\Omega^2}{120}\right)l^2 \end{bmatrix} \tag{A6}$$

$$\mathbf{m}_{e\varphi} = \frac{\rho A l}{(1+\Omega^2)} \left(\frac{\zeta}{l}\right)^2 \cdot$$

$$\begin{bmatrix} \frac{6}{5} & -\left(\frac{1}{10}-\frac{\Omega}{2}\right)l & \frac{6}{5} & -\left(\frac{1}{10}-\frac{\Omega}{2}\right)l \\ -\left(\frac{1}{10}-\frac{\Omega}{2}\right)l & \left(\frac{2}{15}+\frac{\Omega}{6}+\frac{\Omega^2}{3}\right)l^2 & -\left(\frac{13}{420}+\frac{3\Omega}{40}+\frac{\Omega^2}{24}\right)l & -\left(\frac{1}{30}+\frac{\Omega}{6}-\frac{\Omega^2}{6}\right)l^2 \\ \frac{6}{5} & \left(\frac{1}{10}-\frac{\Omega}{2}\right)l & \frac{6}{5} & \left(\frac{1}{10}-\frac{\Omega}{2}\right)l \\ -\left(\frac{1}{10}-\frac{\Omega}{2}\right)l & -\left(\frac{1}{30}+\frac{\Omega}{6}-\frac{\Omega^2}{6}\right)l^2 & \left(\frac{1}{10}-\frac{\Omega}{2}\right)l & \left(\frac{2}{15}+\frac{\Omega}{6}+\frac{\Omega^2}{3}\right)l^2 \end{bmatrix} \tag{A7}$$

Appendix B Elements of the matrix K

$$a = 12\theta_b + 4\theta_s + \left(\frac{13}{35}+\frac{7\Omega}{120}+\frac{\Omega^2}{24}\right)\theta_f + \left(\frac{6}{5}+2\Omega+\Omega^2\right)\theta_p - \left(\frac{6}{5}+2\Omega+\Omega^2\right)\theta_g \tag{B1}$$

$$b = \left[-6\theta_b - 2\theta_s - \left(\frac{11}{210}+\frac{11\Omega}{120}+\frac{\Omega^2}{24}\right)\theta_f - \frac{1}{10}\theta_p + \frac{1}{10}\theta_g\right] l \tag{B2}$$

$$c = -12\theta_b - 4\theta_s + \left(\frac{9}{70}+\frac{3\Omega}{10}+\frac{\Omega^2}{6}\right)\theta_f - \left(\frac{6}{5}+2\Omega+\Omega^2\right)\theta_p + \left(\frac{6}{5}+2\Omega+\Omega^2\right)\theta_g \tag{B3}$$

$$d = \left[-6\theta_b - 2\theta_s - \left(\frac{11}{210}+\frac{11\Omega}{120}+\frac{\Omega^2}{24}\right)\theta_f - \frac{1}{10}\theta_p + \frac{1}{10}\theta_g\right] l \tag{B4}$$

$$e = \Bigg[\left(4 + 2\Omega + \Omega^2\right) \theta_b + \theta_s + \left(\frac{1}{105} + \frac{\Omega}{60} + \frac{\Omega^2}{120} \right) \theta_f + \\ + \left(\frac{2}{15} - \frac{\Omega}{6} + \frac{\Omega^2}{12} \right) \theta_p - \left(\frac{2}{15} - \frac{\Omega}{6} + \frac{\Omega^2}{12} \right) \theta_g \Bigg] l^2 \quad \text{(B5)}$$

$$f = \left[6\theta_b + 2\theta_s - \left(\frac{13}{420} + \frac{3\Omega}{40} + \frac{\Omega^2}{24} \right) \theta_f + \frac{1}{10}\theta_p - \frac{1}{10}\theta_g \right] l \quad \text{(B6)}$$

$$g = \Bigg[\left(2 - 2\Omega - \Omega^2\right) \theta_b + 2\theta_s - \left(\frac{1}{140} + \frac{\Omega}{60} + \frac{\Omega^2}{120} \right) \theta_f + \\ + \left(-\frac{1}{30} + \frac{\Omega^2}{12} \right) \theta_p - \left(-\frac{1}{30} + \frac{\Omega^2}{12} \right) \theta_g \Bigg] l^2 \quad \text{(B7)}$$

$$h = \left[6\theta_b + 2\theta_s + \left(\frac{11}{210} + \frac{11\Omega}{120} + \frac{\Omega^2}{24} \right) \theta_f + \frac{1}{10}\theta_p - \frac{1}{10}\theta_g \right] l \quad \text{(B8)}$$

$$a' = 12\theta_b + 4\theta_s + \left(\frac{13}{35} + \frac{7\Omega}{120} + \frac{\Omega^2}{24} \right) \theta_f + \\ + \left(\frac{6}{5} + 2\Omega + \Omega^2 \right) \theta_p - \left(\frac{6}{5} + 2\Omega + \Omega^2 \right) \theta_g + K_1 \quad \text{(B9)}$$

$$e' = \Bigg[\left(4 + 2\Omega + \Omega^2\right) \theta_b + \theta_s + \left(\frac{1}{105} + \frac{\Omega}{60} + \frac{\Omega^2}{120} \right) \theta_f + \\ + \left(\frac{2}{15} - \frac{\Omega}{6} + \frac{\Omega^2}{12} \right) \theta_p - \left(\frac{2}{15} - \frac{\Omega}{6} + \frac{\Omega^2}{12} \right) \theta_g \Bigg] l^2 + \gamma_1 \quad \text{(B10)}$$

$$a'' = 12\theta_b + 4\theta_s + \left(\frac{13}{35} + \frac{7\Omega}{120} + \frac{\Omega^2}{24} \right) \theta_f + \\ + \left(\frac{6}{5} + 2\Omega + \Omega^2 \right) \theta_p - \left(\frac{6}{5} + 2\Omega + \Omega^2 \right) \theta_g + K_2 \quad \text{(B11)}$$

$$e'' = \Bigg[\left(4 + 2\Omega + \Omega^2\right) \theta_b + \theta_s + \left(\frac{1}{105} + \frac{\Omega}{60} + \frac{\Omega^2}{120} \right) \theta_f + \\ + \left(\frac{2}{15} - \frac{\Omega}{6} + \frac{\Omega^2}{12} \right) \theta_p - \left(\frac{2}{15} - \frac{\Omega}{6} + \frac{\Omega^2}{12} \right) \theta_g \Bigg] l^2 + \gamma_2 \quad \text{(B12)}$$

where

$$\theta_b = \frac{1}{(1+\Omega)^2}\frac{EI}{l^3} \tag{B13}$$

$$\theta_s = \frac{1}{(1+\Omega)^2}\frac{G\Lambda}{4l}\Omega^2 \tag{B14}$$

$$\theta_f = \frac{1}{(1+\Omega)^2}kl \tag{B15}$$

$$\theta_p = \frac{k_p}{l(1+\Omega)^2} \tag{B16}$$

$$\theta_g = \frac{P}{l(1+\Omega)^2} \tag{B17}$$

$$\theta_r = \frac{\rho A l}{(1+\Omega)^2}\left(\frac{\zeta}{l}\right)^2 \tag{B18}$$

$$\theta_v = \frac{\rho A l}{(1+\Omega)^2} \tag{B19}$$

Appendix C Elements of the matrix M

$$A = \frac{6}{5}\theta_r + \left(\frac{13}{35} + \frac{7\Omega}{120} + \frac{\Omega^2}{24}\right)\theta_v \tag{C1}$$

$$B = \left[-\left(\frac{1}{10} - \frac{\Omega}{2}\right)\theta_r - \left(\frac{11}{210} + \frac{11\Omega}{120} + \frac{\Omega^2}{24}\right)\theta_v\right] l \tag{C2}$$

$$C = \frac{6}{5}\theta_r + \left(\frac{9}{70} + \frac{3\Omega}{10} + \frac{\Omega^2}{6}\right)\theta_v \tag{C3}$$

$$D = \left[-\left(\frac{1}{10} - \frac{\Omega}{2}\right)\theta_r + \left(\frac{13}{420} + \frac{3\Omega}{40} + \frac{\Omega^2}{24}\right)\theta_v\right] l \tag{C4}$$

$$E = \left[\left(\frac{2}{15} + \frac{\Omega}{6} + \frac{\Omega^3}{3}\right)\theta_r + \left(\frac{1}{105} + \frac{\Omega}{60} + \frac{\Omega^2}{120}\right)\theta_v\right] l^2 \tag{C5}$$

$$F = \left[\left(\frac{1}{10} - \frac{\Omega}{2}\right)\theta_r - \left(\frac{13}{420} + \frac{3\Omega}{40} + \frac{\Omega^2}{24}\right)\theta_v\right] l \tag{C6}$$

$$G = \left[-\left(\frac{1}{30} + \frac{\Omega}{6} - \frac{\Omega^2}{6}\right)\theta_r - \left(\frac{1}{140} + \frac{\Omega}{60} + \frac{\Omega^2}{120}\right)\theta_v\right] l^2 \tag{C7}$$

$$H = \left[\left(\frac{1}{10} - \frac{\Omega}{2}\right)\theta_r + \left(\frac{11}{210} + \frac{11\Omega}{120} + \frac{\Omega^2}{24}\right)\theta_v\right] l \tag{C8}$$