Copyright © 2009 Tech Science Press CMES, vol.39, no.2, pp.125-154, 2009

A Fictitious Time Integration Method for Solving m-Point Boundary Value Problems

Chein-Shan Liu[1]

Abstract: We propose a new numerical method for solving the boundary value problems of ordinary differential equations (ODEs) under multipoint boundary conditions specified at $t = T_i,\ i = 1, \ldots, m$, where $T_1 < \ldots < T_m$. The finite difference scheme is used to approximate the ODEs, which together with the m-point boundary conditions constitute a system of nonlinear algebraic equations (NAEs). Then a Fictitious Time Integration Method (FTIM) is used to solve these NAEs. Numerical examples confirm that the new approach is highly accurate and efficient with a fast convergence. The FTIM can also be used to find the periods of nonlinear ODEs system and its corresponding periodic solutions, as the van der Pol and Duffing equations are investigated here. The numerical examples also include a vibration problem of the Euler-Bernoulli beam under three-point boundary conditions. The present method has a number advantages of easy implementation, easily to treat nonlinear multipoint boundary value problems, and easily to extend to a higher-dimensional first-order ODEs.

Keywords: Multi-point boundary value problems, Periodic solution, Euler-Bernoulli beam, Fictitious Time Integration Method (FTIM)

1 Introduction

The boundary value problems (BVPs) for differential equations appear in a variety of different areas of applied mathematics, physics and engineering, to name a few, Melnikov and Melnikov (2001), Ali Libre et al. (2008), Liu, Chang and Chang (2008), Chang, Chang and Liu (2008). Multipoint BVPs of ordinary differential equations (ODEs) arise, when the states of a dynamical system are measured at many points, and thus they are fundamental in many areas of engineering and science. For example, the determination of period and periodic solution of nonlinear dynamical system is one of the most important fields in the nonlinear research.

[1] Department of Civil Engineering, National Taiwan University, Taipei, Taiwan. E-mail: liuchei.nshan@msa.hinet.net

When the period is known, the finding of periodic solution can be achieved by solving a two-point BVP; when the period is unknown, the finding of period and periodic solution can be achieved by solving a multipoint BVP.

There are many computational methods that have been developed for solving two-point BVPs; see, e.g., Kubicek and Hlavacek (1983), Cash (1986, 1988), Cash and Wright (1998), Keller (1992), Ascher, Mattheij and Russell (1995), Deeba, Khuri and Xie (2000), Garg (1980), Ha (2001), Ha and Lee (2002), and Cuomo and Marasco (2008). However, for the m-point BVPs with $m \geq 3$ only a few computational methods have been reported [Ojika and Kasue (1979); Agarwal (1979); Cakir and Amiraliyev (2007); Marhoune (2006), Liu (2008a)]. Also, many numerical methods to determine periodic solutions and periods of nonlinear dynamical systems are classified into two main categories: frequency domain method and time domain method [Li and Xu (2004)]. Upon comparing them with the present approach of Fictitious Time Integration Method (FTIM), one may appreciate that the FTIM has a great flexibility by applying it to the determination of periodic solutions and periods for many nonlinear dynamical systems.

Multipoint BVPs have attracted much attention from researchers, e.g., Calvert and Gupta (2005), Gao and Pei (2008), An and Ma (2008), Sun and Zhang (2007), Jiang and Li (2007), Bai (2007), Luo and Ma (2005), and Henderson (2004). Gupta (1992, 1994) was the first who studied the solvability of three-point BVPs of second-order ODEs. The shooting technique was used by Kwong (2006) to study a certain three-point BVP of second-order ODE. Quasilinearization method is also used by Ahmad, Khan and Eloe (2002) to obtain a monotone sequence converging quadratically to a solution of three-point BVP of second-order ODE.

In this paper a new method is proposed for the numerical solution of the following second-order m-point BVP:

$$\ddot{x} = f(t,x,\dot{x}), \;\; T_1 < t < T_m, \tag{1}$$

$$H_1(x(T_1),\dot{x}(T_1),\ldots,x(T_m),\dot{x}(T_m)) = 0, \tag{2}$$

$$H_2(x(T_1),\dot{x}(T_1),\ldots,x(T_m),\dot{x}(T_m)) = 0, \tag{3}$$

where $x(T_1)$, $\dot{x}(T_1),\ldots,x(T_m),\dot{x}(T_m)$ are respectively the values of x and $\dot{x}$ at m different temporal points $T_1 < \ldots < T_m$. Here, $[T_1,T_m]$ is a time interval of our problem. Since the boundary conditions are specified at m distinct points, this problem is called an m-point BVP.

For an easier explanation of the present strategy to approach the m-point BVP, we first focus on the above Eqs. (1)-(3). Later, they will be written as a first-order ODEs system in Section 4, of which our approach by using the new method is delivered. Then, the calculation of the m-point BVPs for the higher-dimensional first-order ODEs is easily extended by a similar approach.

The group preserving scheme (GPS) was developed previously by Liu (2001) for initial value problems of ODEs. Recently, Liu (2006a, 2006b, 2006c) has extended the GPS technique to solve the two-point BVPs, and numerical results reveal that the Lie group shooting method (LGSM) is a rather promising technique to effectively solve the two-point BVPs. Chang, Liu and Chang (2007) have employed the LGSM to solve a backward heat conduction problem with a high performance. Liu (2008b, 2008c) has employed the LGSM technique to accurately solve the inverse heat conduction problems of identifying nonhomogeneous heat conductivity functions and time-dependent heat conductivity functions. More interestingly, as shown by Liu (2008d) the Lie-group method is also useful in the inverse Sturm-Liouville problem. Liu (2008a) has developed a two-stage Lie-group shooting method (TSLGSM) for the three-point BVP governed by Eqs. (1)-(3) with $m = 3$. The extension to a more points BVP is possible; however, many nonlinear algebraic equations (NAEs) are required to solve iteratively, which is computational expensive, as well as increases the complexity of such an extension.

This paper is arranged as follows. In Section 2 we transform the above m-point BVP into the NAEs by using the finite difference approximations, and we explain a mathematical basis of a ficititious time integration method (FTIM) for solving NAEs. In Section 3 we use some numerical examples to demonstrate the efficiency of the new method of FTIM. Then, in Section 4 we use the first-order differential equations system and the FTIM to solve the m-point BVPs. Numerical examples are given under this formulation and some computed results are shown. Periods and periodic solutions of the van der Pol and Duffing equations are given there. We also use a vibration problem of an Euler-Bernoulli beam under multipoint boundary conditions to evaluate the performance of the FTIM, which is a very high-dimensional multipoint BVP, as being a discretization of the fourth-order partial differential equation of wave propagation. Finally, we draw conclusions in Section 5.

2 A fictitious time integration method

2.1 Finite difference equations

We divide the time interval of $[T_1, T_m]$ into $n-1$ subintervals by using a constant time-step length $\Delta t = (T_m - T_1)/(n-1)$. At a temporal grid point $t_i = T_1 + (i-1)\Delta t$, x_i is used to approximate the true value of $x(t_i)$. Therefore, from Eqs. (1)-(3) by using a finite difference scheme we can derive the following NAEs:

$$\frac{x_{i+1}-2x_i+x_{i-1}}{(\Delta t)^2}-f\left(t_i,x_i,\frac{x_{i+1}-x_{i-1}}{2\Delta t}\right)=0,\ \ 2\leq i\leq n-1, \tag{4}$$

$$H_1\left(x_1,\frac{x_2-x_1}{\Delta t},\ldots,x_i,\frac{x_{i+1}-x_{i-1}}{2\Delta t},\ldots,x_n,\frac{x_n-x_{n-1}}{\Delta t}\right)=0, \tag{5}$$

$$H_2\left(x_1,\frac{x_2-x_1}{\Delta t},\ldots,x_i,\frac{x_{i+1}-x_{i-1}}{2\Delta t},\ldots,x_n,\frac{x_n-x_{n-1}}{\Delta t}\right)=0. \tag{6}$$

Here, for simplicity we assume that the interior point T_j, where the boundary value is specified, is coincident with a certain discretized time-point t_i. Otherwise, we may need to interpolate the values of x and $\dot{x}$ at T_j by using the values at its two neighbour grid points.

2.2 Transformation into an ODEs system

Eqs. (4)-(6) constitute a system of n nonlinear algebraic equations (NAEs), which can be used to solve the n unknowns of $x_i,\ i=1,\ldots,n$.

In order to apply our new method to solve the system of NAEs, let us demonstrate it by using a single NAE:

$$F(x)=0, \tag{7}$$

where we only have an independent variable x. We transform it into a first-order ODE by introducing a fictitious time-like variable τ into the following transformation of variables from x to y:

$$y(\tau)=(1+\tau)x. \tag{8}$$

Here, τ is a variable which is independent of x; hence, $y'=dy/d\tau=x$. If $\nu\neq 0$, Eq. (7) is equivalent to

$$0=-\nu F(x). \tag{9}$$

Adding the equation $y'=x$ to Eq. (9) we obtain:

$$y'=x-\nu F(x). \tag{10}$$

By using Eq. (8) we can derive

$$y'=\frac{y}{1+\tau}-\nu F\left(\frac{y}{1+\tau}\right). \tag{11}$$

This is a first-order ODE for $y(\tau)$. The initial condition for the above equation is $y(0)=x$, which is however an unknown and requires a guess.

Multiplying Eq. (11) by an integrating factor of $1/(1+\tau)$ we can obtain

$$\frac{d}{d\tau}\left(\frac{y}{1+\tau}\right)=-\frac{\nu}{1+\tau}F\left(\frac{y}{1+\tau}\right). \tag{12}$$

Further using $y/(1+\tau)=x$, leads to

$$x'=-\frac{\nu}{1+\tau}F(x). \tag{13}$$

Therefore, we have transformed the algebraic Eq. (7) into a first-order nonautonomous ODE. Under certain condition we expect that the solution of Eq. (13) starting from an initial guess of $x(0)$ can approximate the true solution x of Eq. (7).

The above idea was first proposed by Liu (2008d) to treat an inverse Sturm-Liouville problem by transforming an ODE into a PDE. Then, Liu (2008e, 2008f, 2008g), and Liu, Chang, Chang and Chen (2008) extended this idea to develop new methods for estimating parameters in the inverse vibration problems. Liu and Atluri (2008a) have employed the technique of FTIM to solve a large system of nonlinear algebraic equations, and showed that high performance can be achieved by using the FTIM. More recently, Liu (2008h) has used the FTIM technique to solve the nonlinear complementarity problems, whose numerical results are very well. Then, Liu (2008i) used the FTIM to solve the boundary value problems of elliptic type partial differential equations. Liu and Atluri (2008b) also employed this technique of FTIM to solve mixed-complementarity problems and optimization problems. Then, Liu and Atluri (2008c) using the technique of FTIM solved the inverse Sturm-Liouville problem, for specified eigenvalues.

Now, applying Eq. (13) to Eqs. (4)-(6) we can obtain

$$x_i'=-\frac{\nu_1}{1+\tau}\left[\frac{x_{i+1}-2x_i+x_{i-1}}{(\Delta t)^2}-f\left(t_i,x_i,\frac{x_{i+1}-x_{i-1}}{2\Delta t}\right)\right],\ \ 2\le i\le n-1, \tag{14}$$

$$x_1'=-\frac{\nu_2}{1+\tau}H_1\left(x_1,\frac{x_2-x_1}{\Delta t},\ldots,x_i,\frac{x_{i+1}-x_{i-1}}{2\Delta t},\ldots,x_n,\frac{x_n-x_{n-1}}{\Delta t}\right), \tag{15}$$

$$x_n'=-\frac{\nu_3}{1+\tau}H_2\left(x_1,\frac{x_2-x_1}{\Delta t},\ldots,x_i,\frac{x_{i+1}-x_{i-1}}{2\Delta t},\ldots,x_n,\frac{x_n-x_{n-1}}{\Delta t}\right). \tag{16}$$

The different coefficients ν_1, ν_2 and ν_3 can be used to enhance the stability of numerical integrations of the above equations.

2.3 The GPS for ODEs system

We can write Eqs. (14)-(16) as

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}, \tau), \quad \mathbf{x} \in \mathbb{R}^n, \tag{17}$$

where $\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$.

Group-preserving scheme (GPS) can preserve the internal symmetry group of the considered ODE system. Although we do not know previously the symmetry group of differential equations system, Liu (2001) has embedded it into an augmented differential system, which concerns with not only the evolution of state variables themselves but also the evolution of the magnitude of the state variables vector. Let us note that

$$||\mathbf{x}|| = \sqrt{\mathbf{x}^{\mathrm{T}}\mathbf{x}} = \sqrt{\mathbf{x} \cdot \mathbf{x}}, \tag{18}$$

where the dot between two n-dimensional vectors denotes their inner product. Taking the derivatives of both the sides of Eq. (18) with respect to τ, we have

$$\frac{d||\mathbf{x}||}{d\tau} = \frac{\dot{\mathbf{x}}^{\mathrm{T}}\mathbf{x}}{\sqrt{\mathbf{x}^{\mathrm{T}}\mathbf{x}}}. \tag{19}$$

Then, by using Eqs. (17) and (18) we can derive

$$\frac{d||\mathbf{x}||}{d\tau} = \frac{\mathbf{f}^{\mathrm{T}}\mathbf{x}}{||\mathbf{x}||}. \tag{20}$$

It is interesting that Eqs. (17) and (20) can be combined together into a simple matrix equation:

$$\frac{d}{d\tau}\begin{bmatrix} \mathbf{x} \\ ||\mathbf{x}|| \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n\times n} & \frac{\mathbf{f}(\mathbf{x},\tau)}{||\mathbf{x}||} \\ \frac{\mathbf{f}^{\mathrm{T}}(\mathbf{x},\tau)}{||\mathbf{x}||} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ ||\mathbf{x}|| \end{bmatrix}. \tag{21}$$

It is obvious that the first row in Eq. (21) is the same as the original equation (17), but the inclusion of the second row in Eq. (21) gives us a Minkowskian structure of the augmented state variables of $\mathbf{X} := (\mathbf{x}^{\mathrm{T}}, ||\mathbf{x}||)^{\mathrm{T}}$, which satisfies the cone condition:

$$\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = 0, \tag{22}$$

where

$$\mathbf{g} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & -1 \end{bmatrix} \tag{23}$$

is a Minkowski metric, and $\mathbf{I}_n$ is the identity matrix of order n. In terms of $(\mathbf{x}, \|\mathbf{x}\|)$, Eq. (22) becomes

$$\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = \mathbf{x}\cdot\mathbf{x} - \|\mathbf{x}\|^2 = \|\mathbf{x}\|^2 - \|\mathbf{x}\|^2 = 0. \tag{24}$$

It follows from the definition given in Eq. (18), and thus Eq. (22) is a natural result. Consequently, we have an $n+1$-dimensional augmented system:

$$\mathbf{X}' = \mathbf{A}\mathbf{X} \tag{25}$$

with a constraint (22), where

$$\mathbf{A} := \begin{bmatrix} \mathbf{0}_{n\times n} & \frac{\mathbf{f}(\mathbf{x},\tau)}{\|\mathbf{x}\|} \\ \frac{\mathbf{f}^{\mathrm{T}}(\mathbf{x},\tau)}{\|\mathbf{x}\|} & 0 \end{bmatrix}, \tag{26}$$

satisfying

$$\mathbf{A}^{\mathrm{T}}\mathbf{g} + \mathbf{g}\mathbf{A} = \mathbf{0}, \tag{27}$$

is a Lie algebra $so(n,1)$ of the proper orthochronous Lorentz group $SO_o(n,1)$. This fact prompts us to devise the group-preserving scheme (GPS), whose discretized mapping $\mathbf{G}$ must exactly preserve the following Lie-group properties:

$$\mathbf{G}^{\mathrm{T}}\mathbf{g}\mathbf{G} = \mathbf{g}, \tag{28}$$

$$\det \mathbf{G} = 1, \tag{29}$$

$$G_0^0 > 0, \tag{30}$$

where G_0^0 is the 00-th component of $\mathbf{G}$.

Although the dimension of the new system is raised one more, it has been shown that the new system permits a GPS given as follows:

$$\mathbf{X}_{k+1} = \mathbf{G}(k)\mathbf{X}_k, \tag{31}$$

where $\mathbf{X}_k$ denotes the numerical value of $\mathbf{X}$ at τ_k, and $\mathbf{G}(k) \in SO_o(n,1)$ is the group value of $\mathbf{G}$ at τ_k. If $\mathbf{G}(k)$ satisfies the properties in Eqs. (28)-(30), then $\mathbf{X}_k$ satisfies the cone condition in Eq. (22).

The Lie group can be generated from $\mathbf{A} \in so(n,1)$ by an exponential mapping,

$$\mathbf{G}(k) = \exp[h\mathbf{A}(k)] = \begin{bmatrix} \mathbf{I}_n + \frac{(a_k-1)}{\|\mathbf{f}_k\|^2}\mathbf{f}_k\mathbf{f}_k^{\mathrm{T}} & \frac{b_k\mathbf{f}_k}{\|\mathbf{f}_k\|} \\ \frac{b_k\mathbf{f}_k^{\mathrm{T}}}{\|\mathbf{f}_k\|} & a_k \end{bmatrix}, \tag{32}$$

where

$$a_k := \cosh\left(\frac{h\|\mathbf{f}_k\|}{\|\mathbf{x}_k\|}\right), \tag{33}$$

$$b_k := \sinh\left(\frac{h\|\mathbf{f}_k\|}{\|\mathbf{x}_k\|}\right), \tag{34}$$

and $h = \tau_{k+1} - \tau_k$ is a constant step length of the fictitious time τ.
Substituting Eq. (32) for $\mathbf{G}(k)$ into Eq. (31), we obtain

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \eta_k \mathbf{f}_k, \tag{35}$$

$$\|\mathbf{x}_{k+1}\| = a_k\|\mathbf{x}_k\| + \frac{b_k}{\|\mathbf{f}_k\|}\mathbf{f}_k \cdot \mathbf{x}_k, \tag{36}$$

where

$$\eta_k := \frac{b_k\|\mathbf{x}_k\|\|\mathbf{f}_k\| + (a_k - 1)\mathbf{f}_k \cdot \mathbf{x}_k}{\|\mathbf{f}_k\|^2}. \tag{37}$$

This scheme is group properties preserved for all $h > 0$, and is called the group-preserving scheme.

2.4 Numerical procedure

Starting from an initial value of $\mathbf{x}(0)$, we can employ the above GPS to integrate Eqs. (14)-(16) from $\tau = 0$ to a selected final time τ_f. In the numerical integration process we can check the convergence of x_i at the k- and $k+1$-steps by

$$\sum_{i=1}^{n} (x_i^{k+1} - x_i^k)^2 \leq \varepsilon^2, \tag{38}$$

where ε is a selected criterion. If at a time $\tau_0 \leq \tau_f$ the above criterion is satisfied, then the solution of x_i is obtained. In practice, if a suitable τ_f is selected we find that the numerical solution also approaches very well to the true solution, even the above convergent criterion is not satisfied.

3 Numerical examples

In order to assess the performance of the newly developed method let us investigate the following examples. We first treat two linear cases but with different boundary conditions at the last two points of time. Then we consider some nonlinear cases, which are subjecting to complex boundary conditions. Numerical examples also cover a coupled second-order ODEs system.

3.1 Example 1

Let us consider the following three-point BVP [Ma (1998)]:

$$\ddot{x} = -2, \;\; x(0) = 0, \;\; x(1) = \alpha x(\xi). \tag{39}$$

The exact solution is

$$x(t) = \frac{1-\alpha\xi^2}{1-\alpha\xi} t - t^2. \tag{40}$$

By applying the FTIM we obtain

$$\begin{aligned} x_i' &= -\frac{\nu_1}{1+\tau}\left[\frac{x_{i+1}-2x_i+x_{i-1}}{(\Delta t)^2}+2\right], \;\; 2 \le i \le n-1, \\ x_n' &= -\frac{\nu_2}{1+\tau}[x_n - \alpha x_k], \end{aligned} \tag{41}$$

where we do not need the governing equation for x_1 because $x_1 = x(0) = 0$ is given. On the other hand, $k = \xi/\Delta t + 1$ is given when ξ is specified.

We calculate this three-point BVP using the following parameters: $\alpha = 3$, $\xi = 0.5$, $\Delta t = 0.02$, $h = 0.001$, $\nu_1 = -0.8$, $\nu_2 = 0.01$ and $\varepsilon = 10^{-6}$. The initial guess of x_i is given by $x_i = 1.5$. Through 3908 steps the solution is obtained. Comparing with the exact solution the numerical error of x is plotted in Fig. 1(a), of which the maximum error is 9.782×10^{-4}.

By using the exact solution (40) we form a nonlinear five-point BVP:

$$x(0) = 0, \;\; x(0.25) + x^2(0.5) - x^2(0.75) + x(1) = c, \tag{42}$$

where the value of c can be obtained by inserting Eq. (40) for $x(t)$ into the above equation. Under the following parameters: $\alpha = 3$, $\xi = 0.5$, $\Delta t = 0.02$, $h = 0.001$, $\nu_1 = -0.8$, $\nu_2 = 0.001$ and $\varepsilon = 10^{-6}$, and with the initial guess of $x_i = 1.5$, through 3870 steps the solution is obtained. The numerical error of x is plotted in Fig. 1(b), of which the maximum error is 3.445×10^{-3}.

3.2 Example 2

For the following three-point BVP [Zhao (2007)]:

$$\ddot{x} = -\cos t, \;\; x(0) = 0, \;\; 3x(1/3) + 2\dot{x}(1) = 0, \tag{43}$$

the exact solution is

$$x(t) = \frac{2}{3} t \sin 1 - t \cos\frac{1}{3} + t + \cos t - 1. \tag{44}$$

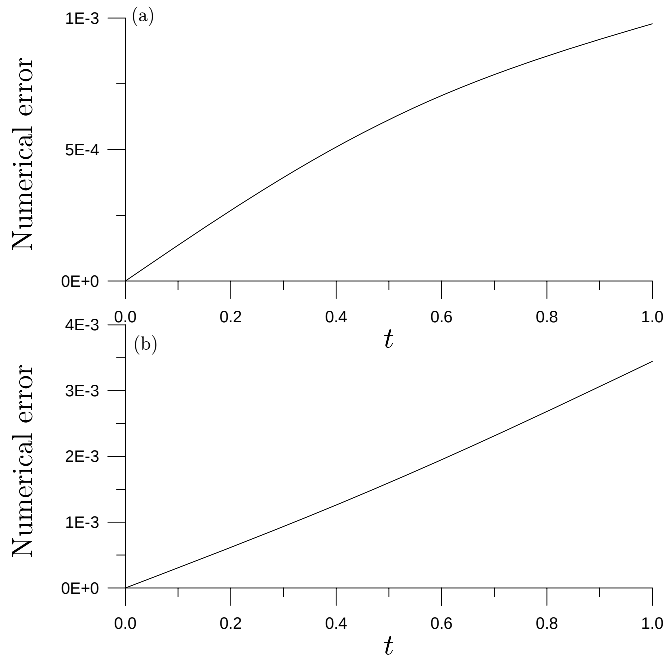


Figure 1: Displaying the numerical errors of Example 1 under different boundary conditions.

By applying the FTIM we have

$$
\begin{aligned}
x_i' &= -\frac{\nu_1}{1+\tau}\left[\frac{x_{i+1}-2x_i+x_{i-1}}{(\Delta t)^2}+\cos t_i\right], \quad 2\le i\le n-1,\\
x_n' &= -\frac{\nu_2}{1+\tau}[1.5\Delta t x_k + x_n - x_{n-1}],
\end{aligned}
\tag{45}
$$

where $\Delta t = 1/(3(k-1))$ with $k=19$ for this calculation.

We calculate this three-point BVP using the following parameters: $h=0.002$, $\nu_1=-0.2$, $\nu_2=-3$ and $\varepsilon=10^{-6}$. The initial guess of x_i is given by $x_i=0.15t_i$. Through 8342 steps the solution is obtained, whose numerical error is plotted in Fig. 2(a) by the solid line, of which the maximum error is 7.68×10^{-4}.

Next, we consider a more complex boundary condition with

$$
x(0)+x(1)=c, \quad x(0)+3x(1/3)+2\dot{x}(1)=0, \tag{46}
$$

where c can be calculated from the exact solution. The governing equations are

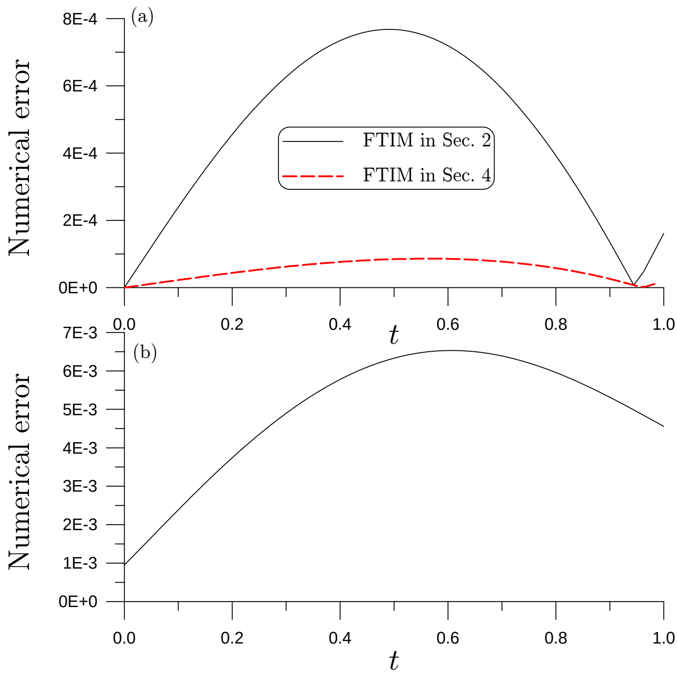


Figure 2: Displaying the numerical errors of Example 2 under different boundary conditions.

read as

$$
\begin{aligned}
x_i' &= -\frac{\nu_1}{1+\tau}\left[\frac{x_{i+1}-2x_i+x_{i-1}}{(\Delta t)^2}+\cos t_i\right], \quad 2\le i\le n-1,\\
x_1' &= -\frac{\nu_2}{1+\tau}[x_1+x_n-c],\\
x_n' &= -\frac{\nu_3}{1+\tau}[0.5\Delta t x_1+1.5\Delta t x_k+x_n-x_{n-1}].
\end{aligned}
\tag{47}
$$

We calculate this case using the following parameters: $h=0.002$, $\nu_1=-0.2$, $\nu_2=-0.1$, $\nu_3=-1$ and $\varepsilon=10^{-5}$. The initial guess of x_i is given by $x_i=0.15t_i$. Through 3175 steps the solution is obtained, whose numerical error is plotted in Fig. 2(b), of which the maximum error is 6.53×10^{-3}.

3.3 Example 3

For the following three-point BVP:

$$\ddot{x} = \frac{1}{8}(32 + 2t^3 - x\dot{x}),$$
$$x(1) = 17, \;\; \dot{x}(2) + x(3) = \frac{43}{3}, \tag{48}$$

the exact solution is

$$x(t) = t^2 + \frac{16}{t}. \tag{49}$$

The FTIM is given by

$$x_i' = -\frac{\nu_1}{1+\tau}\left[\frac{8(x_{i+1} - 2x_i + x_{i-1})}{(\Delta t)^2} - 32 - 2t_i^3 + x_i\frac{x_{i+1} - x_{i-1}}{2\Delta t}\right], \;\; 2 \le i \le n-1,$$
$$x_n' = -\frac{\nu_2}{1+\tau}\left[\frac{x_{k+1} - x_{k-1}}{2\Delta t} + x_n - \frac{43}{3}\right], \tag{50}$$

where $\Delta t = 1/(k-1)$ with $k = 31$.

We calculate this case using the following parameters: $h = 3 \times 10^{-5}$, $\nu_1 = -3$, $\nu_2 = 100$, and $\varepsilon = 10^{-6}$. The initial guess of x_i is given by $x_i = t_i$. Through 16993 steps the solution is obtained. In Fig. 3(a) we plot the numerical error of x, which is in the order of 10^{-4}.

Next, we consider a more complex boundary condition:

$$x(1) + x(2) = 29, \;\; x(2) + x(3) + \dot{x}(3) = \frac{275}{9}. \tag{51}$$

In addition the first equation in Eq. (50) the FTIM is given by

$$x_1' = -\frac{\nu_2}{1+\tau}[x_1 + x_k - 29],$$
$$x_n' = -\frac{\nu_2}{1+\tau}\left[x_k + x_n + \frac{x_n - x_{n-1}}{\Delta t} - \frac{275}{9}\right]. \tag{52}$$

We calculate this case using the following parameters: $h = 2 \times 10^{-6}$, $\nu_1 = -8$, $\nu_2 = 2000$, and $\varepsilon = 10^{-6}$. The initial guess of x_i is given by $x_i = 5t_i$. In Fig. 3(b) we plot the numerical error of x by the solid line, which is in the order of 10^{-2}.

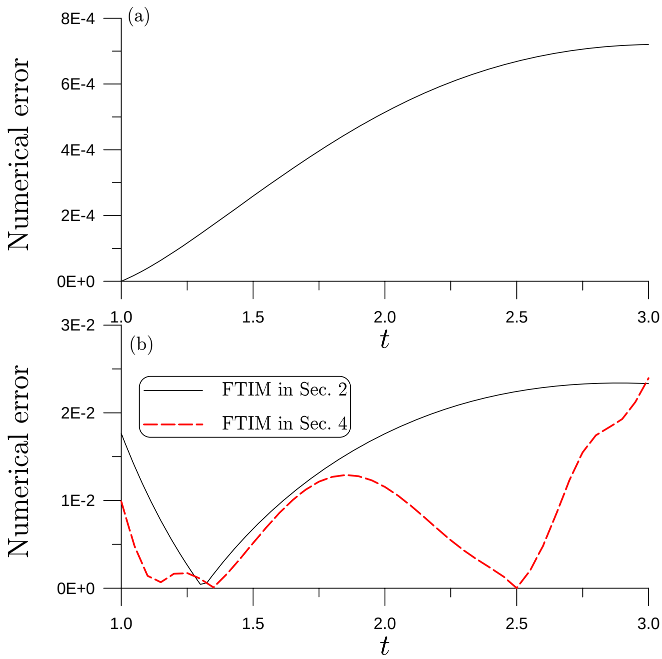


Figure 3: Displaying the numerical errors of Example 3 under different boundary conditions.

3.4 Example 4

We adopt an example from Kwong and Wong (2007):

$$\ddot{x}+\frac{x^2}{1+x}=0, \tag{53}$$

under the following boundary conditions:

$$x(0)-\dot{x}(0)=0,\ \ x(1)-\frac{1}{3}x(0.5)=1. \tag{54}$$

When applying the FTIM we can get

$$x_i'=-\frac{\nu_1}{1+\tau}\left[\frac{x_{i+1}-2x_i+x_{i-1}}{(\Delta t)^2}+\frac{x_i^2}{1+x_i}\right],\ \ 2\le i\le n-1,$$

$$x_1'=-\frac{\nu_2}{1+\tau}\left[x_1-\frac{x_2-x_1}{\Delta t}\right],$$

$$x_n'=-\frac{\nu_3}{1+\tau}\left[x_n-\frac{x_k}{3}-1\right], \tag{55}$$

where $k = 1/(2\Delta t) + 1$.

We calculate this case using the following parameters: $\Delta t = 1/60$, $h = 10^{-5}$, $\nu_1 = -10$, $\nu_2 = 20$, $\nu_3 = 25$ and $\varepsilon = 10^{-6}$. The initial guess of x_i is given by $x_i = 1.2$. The solution is compared with the solution obtained from Liu (2008a) by using the two-stage Lie-group shooting method (TSLGSM) in Fig. 4. It can be seen that these two solutions are very close.

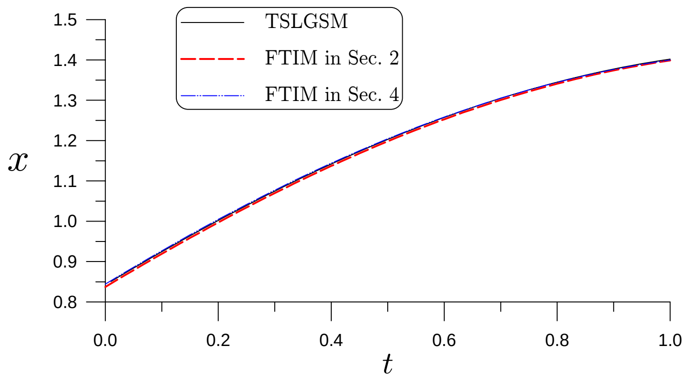


Figure 4: Comparing numerical solutions of Example 4 by using different numerical methods.

3.5 *Example 5*

As an application of FTIM to the two coupled second-order ODEs, we adopt an example from Bellman, Kagiwada and Kalaba (1962) by determining the orbit of a heavenly body:

$$\ddot{x} + \frac{x}{\sqrt[3]{x^2+y^2}} = 0, \quad \ddot{y} + \frac{y}{\sqrt[3]{x^2+y^2}} = 0. \tag{56}$$

At times t_i, $i = 1, \ldots, 4$ we are given that

$$y(t_i) = [x(t_i) - 1] \tan\theta(t_i). \tag{57}$$

Consider the observational data:

$$\theta(0.5) = 0.251297, \ \theta(1) = 0.51024, \ \theta(1.5) = 0.78369, \ \theta(2) = 1.07654, \tag{58}$$

which are obtained by assuming that at the zero time we have $x(0) = 2$, $\dot{x}(0) = 0$, $y(0) = 0$ and $\dot{y}(0) = 0.5$.

When applying the FTIM we can get

$$
\begin{aligned}
x_i' &= -\frac{\nu_1}{1+\tau}\left[\frac{x_{i+1}-2x_i+x_{i-1}}{(\Delta t)^2}+\frac{x_i}{\sqrt[3]{x_i^2+y_i^2}}\right], \quad 2\le i\le n-1,\\
y_i' &= -\frac{\nu_1}{1+\tau}\left[\frac{y_{i+1}-2y_i+y_{i-1}}{(\Delta t)^2}+\frac{y_i}{\sqrt[3]{x_i^2+y_i^2}}\right], \quad 2\le i\le n-1,\\
x_1' &= -\frac{\nu_2}{1+\tau}\left[y_{k_1}-x_{k_1}\tan\theta_1\right],\\
y_1' &= -\frac{\nu_2}{1+\tau}\left[y_{k_3}-x_{k_3}\tan\theta_3\right],\\
x_n' &= -\frac{\nu_3}{1+\tau}\left[y_{k_2}-x_{k_2}\tan\theta_2\right],\\
y_n' &= -\frac{\nu_3}{1+\tau}\left[y_{k_4}-x_{k_4}\tan\theta_4\right],
\end{aligned} \tag{59}
$$

where $k_1 = 1/(2\Delta t)+1$ with $\Delta t = 0.05$, $k_2 = 2k_1$, $k_3 = 3k_1$, $k_4 = 4k_1$, $n = 5k_1$, and $\theta_i = \theta(t_i)$.

We calculate this case using the following parameters: $h = 0.01$, $\nu_1 = -0.2$, $\nu_2 = 0.15$, $\nu_3 = 0.08$ and $\varepsilon = 10^{-4}$. Through 1826 steps the solution is obtained, of which the values of $x(0)$, $y(0)$, $x(2.5)$ and $y(2.5)$ are listed in Table 1 to compare with that obtained by Bellman, Kagiwada and Kalaba (1962).

Table 1: For Example 5 comparing the numerical results with that of Bellman, Kagiwada and Kalaba (1962), shortened as BKK.

	$x(0)$	$y(0)$	$x(2.5)$	$y(2.5)$
BKK	$\times$	$\times$	1.193610	1.060700
present paper	2.000748	-0.01683	1.196662	1.061995

4 The FTIM for the first-order ODEs system

The above Examples 2-4 show that the boundary conditions may include the differential terms of x. In order to directly apply the FTIM to these problems and easily extend to the higher-dimensional ODEs systems, let us return to Eqs. (4)-(6) and use $\dot{x} = y$ to obtain:

$$\dot{x} = y, \tag{60}$$

$$\dot{y} = f(t,x,y), \tag{61}$$

$$H_1(x(T_1), y(T_1), \ldots, x(T_m), y(T_m)) = 0, \tag{62}$$

$$H_2(x(T_1), y(T_1), \ldots, x(T_m), y(T_m)) = 0. \tag{63}$$

Then we propose the following governing equations of FTIM:

$$x_i' = -\frac{\nu_1}{1+\tau}\left[\frac{x_i - x_{i-1}}{\Delta t} - y_i\right], \; i = 2, \ldots, n, \tag{64}$$

$$y_i' = -\frac{\nu_2}{1+\tau}\left[\frac{y_{i+1} - y_i}{\Delta t} - f(t_i, x_i, y_i)\right], \; i = 2, \ldots, n-1, \tag{65}$$

$$x_1' = -\frac{\nu_3}{1+\tau} H_1(x_1, y_1, \ldots, x_n, y_n), \tag{66}$$

$$y_n' = -\frac{\nu_4}{1+\tau} H_2(x_1, y_1, \ldots, x_n, y_n). \tag{67}$$

Sometimes, it may be convenient by using

$$x_i' = -\frac{\nu_1}{1+\tau}\left[\frac{x_i - x_{i-1}}{\Delta t} - y_i\right], \; i = 2, \ldots, n-1, \tag{68}$$

$$y_i' = -\frac{\nu_2}{1+\tau}\left[\frac{y_i - y_{i-1}}{\Delta t} - f(t_i, x_i, y_i)\right], \; i = 2, \ldots, n-1, \tag{69}$$

$$x_1' = -\frac{\nu_3}{1+\tau} H_1(x_1, y_1, \ldots, x_n, y_n), \tag{70}$$

$$y_1' = -\frac{\nu_4}{1+\tau} H_2(x_1, y_1, \ldots, x_n, y_n). \tag{71}$$

This formulation is used when the initial conditions are absent, which imposes the evolutions on both the initial conditions of x and y. Numerical examples will be given below to employ the above two different formulations.

4.1 Example 6

We revisit Example 2 in Section 3.2 again, but use

$$x_i' = -\frac{\nu_1}{1+\tau}\left[\frac{x_i - x_{i-1}}{\Delta t} - y_i\right], \; i = 2, \ldots, n,$$

$$y_i' = -\frac{\nu_2}{1+\tau}\left[\frac{y_{i+1} - y_i}{\Delta t} + \cos t_i\right], \; i = 2, \ldots, n-1,$$

$$y_n' = -\frac{\nu_3}{1+\tau}[3x_k + 2y_n], \tag{72}$$

where we do not need x_1' because x_1 is given, and $\Delta t = 1/(3(k-1))$ with $k = 31$. We calculate this case using the following parameters: $h = 0.02$, $\nu_1 = 2$, $\nu_2 = -2$, $\nu_3 = 0.4$, and $\varepsilon = 10^{-5}$. The initial guesses of x_i and y_i are given by $x_i = y_i = 0.1$. Through 2748 steps the solution is obtained. In Fig. 2(a) we plot the numerical error of x by the dashed line, which is in the order of 10^{-5}. It can be seen that the accuracy of the present formulation is better than that in Section 2.

4.2 Example 7

We consider again Example 3 in Section 3.3 under condition (51), of which we have

$$\begin{aligned}
x_i' &= -\frac{\nu_1}{1+\tau}\left[\frac{x_i - x_{i-1}}{\Delta t} - y_i\right], \ i = 2,\ldots,n,\\
y_i' &= -\frac{\nu_2}{1+\tau}\left[\frac{y_{i+1} - y_i}{\Delta t} + 4 - \frac{t_i^3}{4} + \frac{x_i y_i}{8}\right], \ i = 2,\ldots,n-1,\\
x_1' &= -\frac{\nu_3}{1+\tau}[x_1 + x_k - 29],\\
y_n' &= -\frac{\nu_3}{1+\tau}\left[x_k + x_n + y_n - \frac{275}{9}\right].
\end{aligned} \tag{73}$$

We calculate this case using the following parameters: $h = 0.01$, $\nu_1 = 2$, $\nu_2 = -0.06$, $\nu_3 = 10$, and $\varepsilon = 10^{-4}$. Through 11894 steps the solution is obtained. In Fig. 3(b) we plot the numerical error of x by the dashed line, which is better than the result given in Section 3.3.

4.3 Example 8

We consider again Example 4 in Section 3.4 by

$$\begin{aligned}
x_i' &= -\frac{\nu_1}{1+\tau}\left[\frac{x_i - x_{i-1}}{\Delta t} - y_i\right], \ i = 2,\ldots,n,\\
y_i' &= -\frac{\nu_1}{1+\tau}\left[\frac{y_i - y_{i-1}}{\Delta t} + \frac{x_i^2}{1+x_i}\right], \ i = 2,\ldots,n,\\
x_1' &= -\frac{\nu_2}{1+\tau}[x_1 - y_1],\\
y_1' &= -\frac{\nu_2}{1+\tau}\left[x_n - \frac{x_k}{3} - 1\right].
\end{aligned} \tag{74}$$

We calculate this case using the following parameters: $h = 0.01$, $\nu_1 = 5$, $\nu_2 = 0.5$, and $\varepsilon = 10^{-5}$. Starting from the initial guesses of $x_i = y_i = 0.83$, through 1987 steps the solution is obtained. In Fig. 4 we plot the numerical solution of x by the dashed-dotted line, which is very close to other two solutions.

4.4 Example 9

For the unforced van der Pol equation:

$$\dot{x} = y,$$
$$\dot{y} = 0.01(1-x^2)y - x, \tag{75}$$

we subject it to the initial condition of $(x(0), y(0)) = (1, 2)$. By using the fourth-order Runge-Kutta method we can calculate the solutions in the interval of $[0, 2]$, and the following relations are denoted:

$$g_1 = x(0) + ax(0.5) + y^2(1) - b = 0,$$
$$g_2 = cx^2(0) + y(1) - d = 0, \tag{76}$$

where $a = -0.712710841715850$, $b = -0.258887329600159$, $c = 0.427909747385296$, and $d = 0.648930342563872$.

The FTIM for this example is

$$x_i' = -\frac{\nu_1}{1+\tau}\left[\frac{x_i - x_{i-1}}{\Delta t} - y_i\right], \ i = 2, \ldots, n,$$
$$y_i' = -\frac{\nu_2}{1+\tau}\left[\frac{y_i - y_{i-1}}{\Delta t} - 0.01(1-x_i^2)y_i + x_i\right], \ i = 2, \ldots, n,$$
$$x_1' = -\frac{\nu_3}{1+\tau}g_1,$$
$$y_1' = -\frac{\nu_4}{1+\tau}g_2. \tag{77}$$

We calculate this case using the following parameters: $h = 0.01$, $\nu_1 = 1.5$, $\nu_2 = 2.5$, $\nu_3 = 0.001$, $\nu_4 = 3$ and $\varepsilon = 10^{-5}$. Through 2592 steps the solution is obtained, of which the values of $x(0)$, $y(0)$, $x(0.5)$, $y(0.5)$, $x(1)$ and $y(1)$ are listed in Table 2, which are compared with the solutions obtained by Ojika (1987).

Table 2: For Example 9 comparing the numerical results with that of Ojika (1987).

	$x(0)$	$y(0)$	$x(0.5)$	$y(0.5)$	$x(1)$	$y(1)$
Ojika (1987)	0.999999	2.000000	1.834878	1.267030	2.214366	0.221021
present paper	0.999186	2.002273	1.829238	1.265275	2.200455	0.221721

4.5 Example 10

We calculate some periodic solutions of the periodically forced van der Pol equation:

$$\ddot{x}+\mu(x^2-1)\dot{x}+x=p_0\cos(\omega t), \tag{78}$$

where we fix $p_0=9$, $\omega=\pi$ and $\mu=5.25$, and the period T of periodic solution is unknown; however, we have

$$\begin{aligned} &x(0)=x(T),\\ &\dot{x}(0)=\dot{x}(T). \end{aligned} \tag{79}$$

For the above case, Xu and Jiang (1996) had proven its periodic solution theoretically.

Let $s=t/T$; we have

$$\begin{aligned} &x''(s)+\mu T(x^2-1)x'(s)+T^2x=T^2p_0\cos(\omega Ts),\\ &x(0)=x(1),\\ &x'(0)=x'(1). \end{aligned} \tag{80}$$

Furthermore, by using $Ty=x'$ we have the following system:

$$\begin{aligned} &x'(s)=Ty,\\ &y'(s)+\mu T(x^2-1)y+Tx-Tp_0\cos(\omega Ts)=0,\\ &x(0)=x(1),\\ &y(0)=y(1). \end{aligned} \tag{81}$$

Because T is an unknown, we require a governing equation of T. From Eq. (80) by integrating the first equation one period from $s=0$ to $s=1$, we can get

$$\frac{p_0\sin(\omega T)}{\omega T}=\int_0^1 x(\xi)d\xi. \tag{82}$$

Suppose that the interval of $s\in[0,1]$ is discretized into $n-1$ subintervals with $\Delta s=1/(n-1)$ and that we use $x_i=x(s_i)$ and $y_i=y(s_i)$ where $s_i=(i-1)\Delta s$, then from Eqs. (81) and (82) it follows that

$$\begin{aligned}
&\frac{x_i - x_{i-1}}{\Delta s} - T y_i = 0,\ i = 2,\dots,n,\\
&\frac{y_i - y_{i-1}}{\Delta s} + \mu T (x_i^2 - 1) y_i + T x_i - T p_0 \cos(\omega T s_i) = 0,\ i = 2,\dots,n,\\
&\Delta s \sum_{i=1}^{n-1} x_i - \frac{p_0 \sin(\omega T)}{\omega T} = 0,\\
&x_1 = x_n,\\
&y_1 = y_n.
\end{aligned} \tag{83}$$

The above equations constitute a nonlinear system with dimensions $2n+1$ to solve the $2n+1$ unknowns of $x_i, y_i,\ i = 1,\dots,n$, and T.

By applying the FTIM to the above equations we can get

$$\begin{aligned}
&x_i'(\tau) = \frac{-\nu_1}{1+\tau}\left[\frac{x_i - x_{i-1}}{\Delta s} - T y_i\right],\ i = 2,\dots,n,\\
&y_i'(\tau) = \frac{-\nu_2}{1+\tau}\left[\frac{y_i - y_{i-1}}{\Delta s} + \mu T (x_i^2 - 1) y_i + T x_i - T p_0 \cos(\omega T s_i)\right],\ i = 2,\dots,n,\\
&T'(\tau) = \frac{-\nu_3}{1+\tau}\left[\Delta s \sum_{i=1}^{n-1} x_i - \frac{p_0 \sin(\omega T)}{\omega T}\right],\\
&x_1'(\tau) = \frac{-\nu_4}{1+t}[x_1 - x_n],\\
&y_1'(\tau) = \frac{-\nu_4}{1+t}[y_1 - y_n].
\end{aligned} \tag{84}$$

Under the following parameters: $\Delta s = 10^{-3}$, $h = 10^{-3}$, $\nu_1 = 0.34$, $\nu_2 = 0.1$, $\nu_3 = -30$ and $\nu_4 = 1500$, we obtain a numerical solution where the values of T, x_1, x_n, y_1 and y_n are recorded in Table 3. We also obtain another solution by using $\nu_1 = \nu_2 = 0.1$ and $\nu_3 = -4$, when other parameters are kept unchanged.

Table 3: For Example 10 comparing the numerical results with different $\nu' s$.

T	x_1	x_n	y_1	y_n
19.99983	1.45345	1.45418	0.960997	0.961413
9.99924	1.73801	1.73970	0.62296	0.62161

When the initial values and period are available, we go back to Eq. (81) by integrating the first two equations to obtain the periodic orbit. These two solutions are plotted in Fig. 5, which are different periodic solutions.

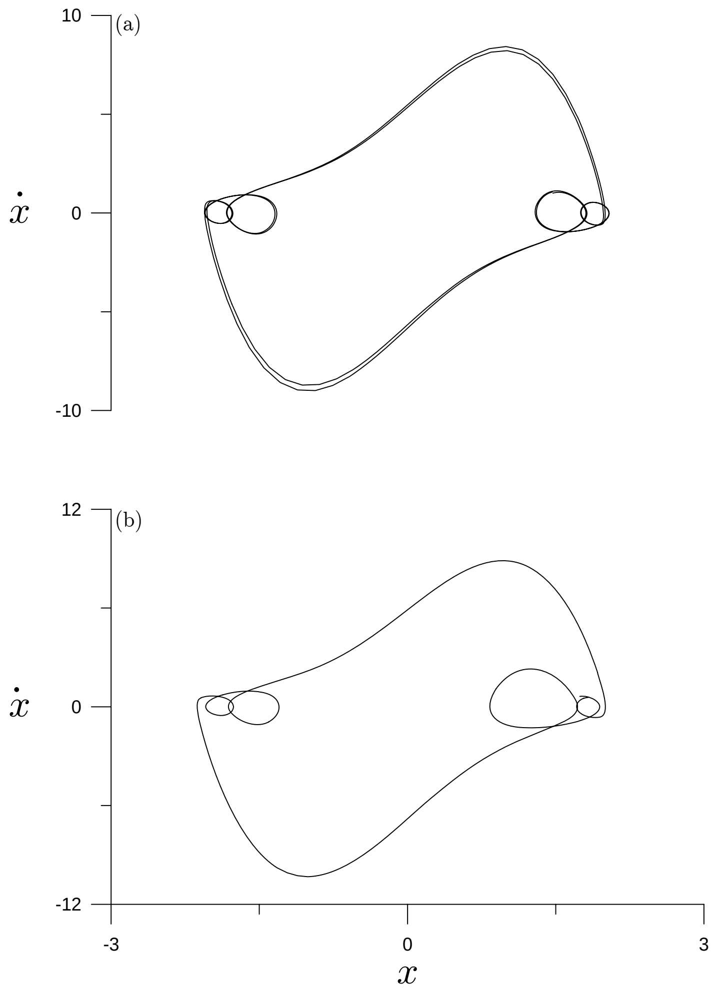


Figure 5: Comparing two different periodic solutions of Example 10.

4.6 Example 11

Then, we calculate periodic solutions of the periodically forced Duffing equation:

$$\ddot{x}+\gamma\dot{x}+\alpha x+\beta x^3 = p_0\cos(\omega t), \tag{85}$$

where we fix $p_0 = 0.2$, $\omega = 1.2$, $\alpha = -0.2$, $\beta = 0.1$ and $\gamma = 0.3$.

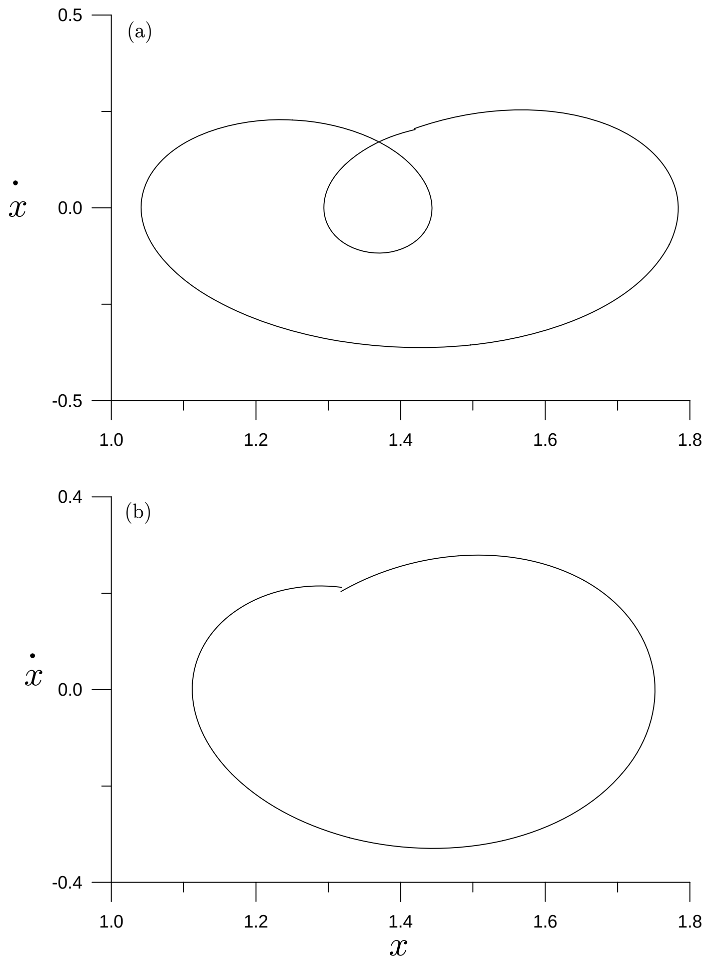


Figure 6: Two periodic solutions of Example 11.

By applying the FTIM to the above equation we can get

$$
\begin{aligned}
x_i'(\tau) &= \frac{-\nu_1}{1+\tau}\left[\frac{x_i - x_{i-1}}{\Delta s} - Ty_i\right],\ i = 2,\ldots,n,\\
y_i'(\tau) &= \frac{-\nu_2}{1+\tau}\left[\frac{y_i - y_{i-1}}{\Delta s} + \gamma T y_i + \alpha T x_i + \beta T x_i^3 - T p_0 \cos(\omega T s_i)\right],\ i = 2,\ldots,n,\\
T'(\tau) &= \frac{-\nu_3}{1+\tau}\left[\alpha \Delta s \sum_{i=1}^{n-1} x_i + \beta \Delta s \sum_{i=1}^{n-1} x_i^3 - \frac{p_0 \sin(\omega T)}{\omega T}\right],\\
x_1'(\tau) &= \frac{-\nu_4}{1+t}[x_1 - x_n],\\
y_1'(\tau) &= \frac{-\nu_4}{1+t}[y_1 - y_n]. \qquad (86)
\end{aligned}
$$

Under the following parameters: $\Delta s = 2\times 10^{-3}$, $h = 10^{-3}$, $\nu_1 = 0.31$, $\nu_2 = 0.05$, $\nu_3 = 15$ and $\nu_4 = 1000$, we obtain a numerical solution with the values of $T = 11.07718$, $x_1 = 1.4189$, $x_n = 1.41925$, $y_1 = 0.2053151$ and $y_n = 0.2053637$ and the periodic orbit is plotted in Fig. 6(a).

Similarly, under the following parameters: $\Delta s = 10^{-3}$, $h = 10^{-3}$, $\nu_1 = 0.3295$, $\nu_2 = 0.1$, $\nu_3 = 15$ and $\nu_4 = 1000$, we obtain a numerical solution with the values of $T = 6.5994$, $x_1 = 1.317459$, $x_n = 1.317812$, $y_1 = 0.203511$ and $y_n = 0.203517$ and the periodic orbit is plotted in Fig. 6(b).

4.7 Example 12

In this example we consider a higher-dimensional first-order ODEs system, obtained from the dimensionless formulation of the Euler-Bernoulli beam:

$$\frac{\partial^4 y(x,t)}{\partial x^4} + \frac{\partial^2 y(x,t)}{\partial t^2} = 0, \tag{87}$$

where we subject it to a three-point boundary conditions with

$$\begin{aligned} &y(0,t) = \frac{\partial y(0,t)}{\partial x} = 0, \\ &\frac{\partial^2 y(x_0,t)}{\partial x^2} = M(t), \\ &\frac{\partial^3 y(1,t)}{\partial x^3} = F(t). \end{aligned} \tag{88}$$

$0 < x_0 < 1$ is an internal point wherein a time-varying moment $M(t)$ is applied. For simplicity, the initial conditions are given by

$$y(x,0) = \frac{\partial y(x,0)}{\partial t} = 0. \tag{89}$$

Let $y_{i,j}$ be the discretized value of $y(x_i,t_j)$, where $x_i = (i-1)/(m_1-1)$ and $t_j = (j-1)t_f/(m_2-1)$ are respectively the discretized coordinates of $0 < x < 1$ and $0 < t < t_f$. A fully discretization of Eq. (87) by using the new variables $u_1 = y$, $u_2 = y_x$, $u_3 = y_{xx}$ and $u_4 = y_{xxx}$ is

$$\begin{aligned} &\frac{u_1^{i,j} - u_1^{i-1,j}}{\Delta x} - u_2^{i,j} = 0, \\ &\frac{u_2^{i,j} - u_2^{i-1,j}}{\Delta x} - u_3^{i,j} = 0, \\ &\frac{u_3^{i+1,j} - u_3^{i,j}}{\Delta x} - u_4^{i,j} = 0, \\ &-\frac{u_1^{i,j+1} - 2u_1^{i,j} + u_1^{i,j-1}}{\Delta t} - \frac{u_4^{i+1,j} - u_4^{i,j}}{\Delta x} = 0. \end{aligned} \tag{90}$$

By applying the FTIM to the above equations we can get

$$
\begin{aligned}
\frac{du_1^{i,j}}{d\tau} &= \nu\left[u_2^{i,j} - \frac{u_1^{i,j} - u_2^{i-1,j}}{\Delta x}\right],\\
\frac{du_2^{i,j}}{d\tau} &= \nu\left[u_3^{i,j} - \frac{u_2^{i,j} - u_3^{i-1,j}}{\Delta x}\right],\\
\frac{du_3^{i,j}}{d\tau} &= \nu\left[u_4^{i,j} - \frac{u_3^{i+1,j} - u_4^{i,j}}{\Delta x}\right],\\
\frac{du_4^{i,j}}{d\tau} &= \nu\left[\frac{u_1^{i,j+1} - 2u_1^{i,j} + u_1^{i,j-1}}{\Delta t} + \frac{u_4^{i+1,j} - u_4^{i,j}}{\Delta x}\right].
\end{aligned} \tag{91}
$$

Because $u_1^{i,1} = u_1^{i,2} = 0$, $u_1^{1,j} = u_1^{2,j} = 0$, $u_3^{m_3,j} = M(t_j)$, where $m_3 = (m_1-1)x_0 + 1$, and $u_4^{m_1,j} = F(t_j)$ are given, the differentials of these terms with respect to τ are set equal to zeros. Under the following parameters: $m_1 = 51$, $m_2 = 101$, $x_0 = 0.9$, $\nu = 0.01$, and $h = 0.005$ we integrate the above equations by the FTIM with an initial guess of all $u_k^{i,j}$, $k = 1,\ldots,4$ to be 0.01. Through 1000 steps the results are obtained, which is plotted in Fig. 7(a), where we use

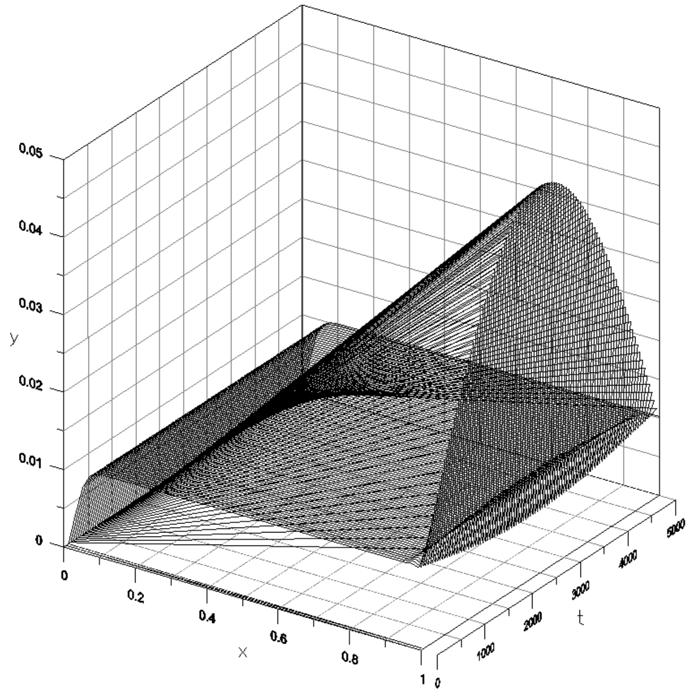
$$
\begin{aligned}
F(t) &= 50000\sin\left(\frac{\pi t}{t_f}\right),\\
M(t) &= -\left[\sin\left(\frac{\pi t}{t_f}\right) + 0.5\sin\left(\frac{2\pi t}{t_f}\right)\right]
\end{aligned} \tag{92}
$$

with $t_f = 5000$. Because the moment is applied at one point $x = 0.9$, the projection of the solution surface as shown in Fig. 7(a) along the t-axis as shown in Fig. 7(b) appears an N-type kink near to the point $x = 0.9$.

5 Conclusions

The multipoint BVPs are discretized by the finite difference method. The present paper simply transformed the resulting nonlinear algebraic equations into an evolutionary system of equations by introducing a fictitious time, and had adding different coefficients ν_i to enhance the stability of numerical integration of the resulting ODEs and to speed up the convergence of solutions. Because no inverse of a matrix is required, the present method is very time efficient. Several numerical examples were worked out. Some are compared with exact solutions revealing that high accuracy can be achieved by the FTIM. The new method is also applicable to the solutions of multipoint BVPs of the first-order ODEs system, which is simple and

(a)



(b)

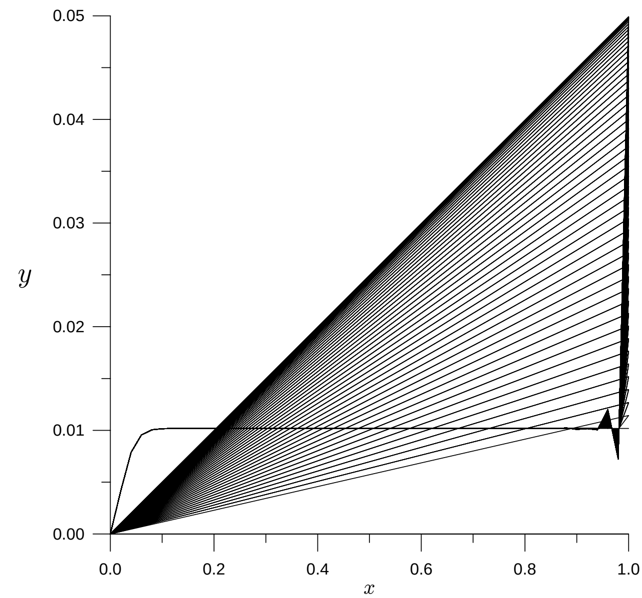


Figure 7: The solution of Example 12 by using the FTIM.

has a great advantage to easily extend to the higher-dimensional ODEs with nonlinear multipoint BVPs. The FTIM was used to find the periods of nonlinear ODEs system and its corresponding periodic solutions, like as the van der Pol and Duffing equations. Its easy implementation and efficiency is over previous numerical methods. The numerical examples also included a vibration problem of the Euler-Bernoulli beam under a three-point boundary condition.

Acknowledgement: Taiwan's National Science Council project NSC-97-2221-E-019-009-MY3 granted to the author is highly appreciated.

References

Agarwal, R. P. (1979): The numerical solution of multipoint boundary value problems. *Journal of Computational and Applied Mathematics*, vol. 5, pp. 17-24.

Ahmad, B.; Ali Khan, R.; Eloe, P. W. (2002): Generalized quasilinearization method for a second-order three point boundary-value problem with nonlinear boundary conditions. *Electric Journal of Differential Equations*, vol. 2002, pp. 1-12.

Ali Libre, N.; Emdadi, A.; Kansa, E. J.; Rahimian, M.; Shekarchi, M. (2008): A stabilized RBF collocation scheme for Neumann type boundary value problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 24, pp. 61-80.

An, Y.; Ma, R. (2008): Global behavior of the components for the second order m-point boundary value problems. *Boundary Value Problems*, vol. 2008, Article ID 254593.

Ascher, U. M.; Mattheij, R. M. M.; Russell, R. D. (1995): Numerical Solution of Boundary Value Problems for Ordinary Differential Equations. SIAM, Philadelphia.

Bai, C. (2007): Existence of positive solutions for fourth-order three-point boundary value problems. *Boundary Value Problems*, vol. 2007, Article ID 68758.

Bellman, R.; Kagiwada, H.; Kalaba, R. (1962); Orbit determination as a multipoint boundary-value problem and quasilinearization. *Mathematics*, vol. 48, pp. 1327-1329.

Cakir, M.; Amiraliyev, G. M. (2007): Numerical solution of a singularly perturbed three-point boundary value problem. *International Journal of Computer Mathematics*, vol. 84, pp. 1465-1481.

Calvert, B.; Gupta, C. P. (2005): Existence and uniqueness of solutions to a superlinear three-point boundary-value problem. *Electric Journal of Differential Equations*, vol. 2005, pp. 1-21.

Cash, J. R. (1986): On the numerical integration of nonlinear two-point boundary value problems using iterated deferred corrections, Part 1: a survey and comparison of some one-step formulae. *Computer and Mathematics with Applications*, vol. 12, pp. 1029-1048.

Cash, J. R. (1988): On the numerical integration of nonlinear two-point boundary value problems using iterated deferred corrections, Part 2: the development and analysis of highly stable deferred correction formulae. *SIAM Journal on Numerical Analysis*, vol. 25, pp. 862-882.

Cash, J. R.; Wright, R. W. (1998): Continuous extensions of deferred correction schemes for the numerical solution of nonlinear two-point boundary value problems. *Applied Numerical Mathematics*, vol. 28, pp. 227-244.

Chang, C. W.; Chang, J. R.; Liu, C.-S. (2008): The Lie-group shooting method for solving classical Blasius flat-plate problem. *CMC: Computers, Materials & Continua*, vol. 7, pp. 139-153.

Chang, J. R.; Liu, C.-S.; Chang, C. W. (2007); A new shooting method for quasi-boundary regularization of backward heat conduction problems. *International Journal of Heat and Mass Transfer*, vol. 50, pp. 2325-2332.

Cuomo, S.; Marasco, A. (2008): A numerical approach to nonlinear two-point boundary value problems for ODEs. *Computer and Mathematics with Applications*, vol. 55, pp. 2476-2489.

Deeba, E.; Khuri, S. A.; Xie, S. (2000): An algorithm for solving boundary value problems. *Journal of Computational Physics*, vol. 159, pp. 125-138.

Gao, Y.; Pei, M. (2008): Solvability for two classes of higher-order multi-point boundary value problems at resonance. *Boundary Value Problems*, vol. 2008, Article ID 723828.

Garg, V. K. (1980): Improved shooting techniques for linear boundary value problems. *Computer Methods in Applied Mechanics and Engineering*, vol. 22, pp. 87-99.

Gupta, C. P. (1992): Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equation. *Journal of Mathematical Analysis and Applications*, vol. 168, pp. 540-551.

Gupta, C. P. (1994): A note on a second order three-point nonlinear boundary value problem. *Journal of Mathematical Analysis and Applications*, vol. 186, pp. 277-281.

Ha, S. N. (2001): A nonlinear shooting method for two-point boundary value problems. *Computer and Mathematics with Applications*, vol. 42, pp. 1411-1420.

Ha, S. N.; Lee, C. R. (2002): Numerical study for two-point boundary value problems using Green's functions. *Computer and Mathematics with Applications*, vol.

44, pp. 1599-1608.

Henderson, J. (2004): Double solutions of three-point boundary-value problems for second-order differential equations. *Electric Journal of Differential Equations*, vol. 2004, pp. 1-7.

Jiang, W.; Li, F. (2007): Several existence theorems of monotone positive solutions for third-order multipoint boundary value problems. *Boundary Value Problems*, vol. 2007, Article ID 17951.

Keller, H. B. (1992): Numerical Methods for Two-point Boundary Value Problems. Dover, New York.

Kubicek, M.; Hlavacek,, V. (1983): Numerical Solution of Nonlinear Boundary Value Problems with Applications. Prentice-Hall, New York.

Kwong, M. K. (2006): The shooting method and multiple solutions of two/multi-point BVPs of second order ODE. *Electric Journal of Qualitative Theory of Differential Equations*, vol. 2006, pp. 1-14.

Kwong, M. K.; Wong, J. S. (2007): The shooting method and nonhomogeneous multipoint BVPs of second-order ODE. *Boundary Value Problems*, vol. 2007, Article ID 64012.

Li, D.; Xu, J. (2004): A method to determine the periodic solution of the non-linear dynamics system. *Journal of Sound and Vibration*, vol. 275, pp. 1-16.

Liu, C.-S. (2001): Cone of non-linear dynamical system and group preserving schemes. *International Journal of Non-Linear Mechanics*, vol. 36, pp. 1047-1068.

Liu, C.-S. (2006a): The Lie-group shooting method for nonlinear two-point boundary value problems exhibiting multiple solutions. *CMES: Computer Modeling in Engineering & Sciences*, vol. 13, pp. 149-163.

Liu, C.-S. (2006b): Efficient shooting methods for the second order ordinary differential equations. *CMES: Computer Modeling in Engineering & Sciences*, vol. 15, pp. 69-86.

Liu, C.-S. (2006c): The Lie-group shooting method for singularly perturbed two-point boundary value problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 15, pp. 179-196.

Liu, C.-S. (2008a): A two-stage LGSM for three-point BVPs of second-order ODEs. *Boundary Value Problems*, vol. 2008, Article ID 963753, 22 pages.

Liu, C.-S. (2008b): An LGSM to identify nonhomogeneous heat conductivity functions by an extra measurement of temperature. *International Journal of Heat and Mass Transfer*, vol. 51, pp. 2603-2613.

Liu, C.-S. (2008c): An LGEM to identify time-dependent heat conductivity function by an extra measurement of temperature gradient. *CMC: Computers, Materials & Continua*, vol. 7, pp. 81-95.

Liu, C.-S. (2008d): Solving an inverse Sturm-Liouville problem by a Lie-group method. *Boundary Value Problems*, vol. 2008, Article ID 749865, 18 pages.

Liu, C.-S. (2008e): Identifying time-dependent damping and stiffness functions by a simple and yet accurate method. *Journal of Sound and Vibration*, vol. 318, pp. 148-165.

Liu, C.-S. (2008f): A Lie-group shooting method for simultaneously estimating the time-dependent damping and stiffness coefficients. *CMES: Computer Modeling in Engineering & Sciences*, vol. 27, pp. 137-149.

Liu, C.-S. (2008g): A Lie-group shooting method estimating nonlinear restoring forces in mechanical systems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 35, pp. 157-180.

Liu, C.-S. (2008h): A time-marching algorithm for solving non-linear obstacle problems with the aid of an NCP-function. *CMC: Computers, Materials & Continua*, vol. 8, pp. 53-65.

Liu, C.-S. (2008i): A fictitious time integration method for two-dimensional quasilinear elliptic boundary value problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 33, pp. 179-198.

Liu, C.-S.; Atluri, S. N. (2008a): A novel time integration method for solving a large system of non-linear algebraic equations. *CMES: Computer Modeling in Engineering & Sciences*, vol. 31, pp. 71-83.

Liu, C.-S.; Atluri, S. N. (2008b): A fictitious time integration method (FTIM) for solving mixed complementarity problems with applications to non-linear optimization. *CMES: Computer Modeling in Engineering & Sciences*, vol. 34, pp. 155-178.

Liu, C.-S.; Atluri, S. N. (2008c): A novel fictitious time integration method for solving the discretized inverse Sturm-Liouville problems, for specified eigenvalues. Modified Potentials as a Tool for Computing Green's Functions in Continuum Mechanics

Liu, C.-S.; Chang, C. W.; Chang, J. R. (2008): A new shooting method for solving boundary layer equation in fluid mechanics. *CMES: Computer Modeling in Engineering & Sciences*, vol. 32, pp. 1-15.

Liu, C.-S.; Chang, J. R.; Chang, K. H.; Chen, Y. W. (2008): Simultaneously estimating the time-dependent damping and stiffness coefficients with the aid of vibrational data. *CMC: Computers, Materials & Continua*, vol. 7, pp. 97-107.

Luo, H.; Ma, Q. (2005): Positive solutions to a generalized second-order three-point boundary value problem on time scales. *Electric Journal of Differential Equations*, vol. 2005, pp. 1-14.

Ma, R. (1998): Positive solutions of a nonlinear three-point boundary-value problem. *Electric Journal of Differential Equations*, vol. 1998, pp. 1-8.

Marhoune, A. L. (2006): A three-point boundary value problem with an integral two-space-variables condition for parabolic equations. *Computer and Mathematics with Applications*, vol. 53, pp. 940-947.

Melnikov, Y. A.; Melnikov, M. Y. (2001): Modified potentials as a tool for computing Green's functions in continuum mechanics. *CMES: Computer Modeling in Engineering & Sciences*, vol. 2, pp. 291-306.

Ojika, T. (1987): Modified deflation algorithm for the solution of singular problems. II. nonlinear multipoint boundary value problems. *Journal of Mathematical Analysis and Applications*, vol. 123, pp. 222-237.

Ojika, T.; Kasue, Y. (1979): Initial-value adjusting method for the solution of nonlinear multipoint boundary-value problems. *Journal of Mathematical Analysis and Applications*, vol. 69, pp. 359-371.

Sun, Y.; Zhang, X. (2007): Existence of symmetric positive solutions for an m-point boundary value problem. *Boundary Value Problems*, vol. 2007, Article ID 79090.

Xu, J.; Jiang, J. (1996): The global bifurcation characteristics of the forced van der Pol oscillator. *Chaos Solitons and Fractals*, vol. 7, pp. 3-19.

Zhao, Z. (2007): Positive solutions for singular three-point boundary-value problems. *Electric Journal of Differential Equations*, vol. 2007, pp. 1-8.