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Exact Solutions for the Free Vibration of Extensional Curved Non-uniform Timoshenko Beams

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Abstract: The three coupled governing differential equations for the in-plane vibrations of curved non-uniform Timoshenko beams are derived via the Hamilton's principle. Three physical parameters are introduced to simplify the analysis. By eliminating all the terms with the axial displacement parameter, then reducing the order of differential operator acting on the flexural displacement parameter, one uncouples the three governing characteristic differential equations with variable coefficients and reduces them into a sixth-order ordinary differential equation with variable coefficients in term of the angle of the rotation due to bending for the first time. The explicit relations between the axial and the flexural displacements and the angle of the rotation due to bending are also revealed. It is shown that if the material and geometric properties of the beam are in arbitrary polynomial forms, then the exact solutions for the in-plane vibrations of the beam can be obtained. Several limiting studies are illustrated. Finally, limiting cases are studied and the influence of the taper ratio and the arc length on the first two natural frequencies of the beams is explored.

Keywords: curved non-uniform Timoshenko beam, free vibration, extensional beam

1 Introduction

Beams are one of the most commonly used structures. It can be widely found in all the engineering fields. Based on the Bernoulli-Euler and the Timoshenko beam theories, the studies on the static and dynamic response of straight beam structures are tremendous [Meirovitch (1967); Lee and Lin (1996); Iura, Suetake, and

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Atluri (2003); Beda (2003); Andreaus, Batra and Porfiri (2005); Vinod, Gopalakrishnan and Ganguli (2006); Lee and Hsu (2007); Huang and Shih (2007); Lin, Lee and Lin (2008), Lee, Lin, Lee, Lu and, Liu (2008), Lee, Lu, Liu and Huang (2008)]. For curved beams, an interesting review can be found in the review paper by Childamparam and Leissa (1993).

For two dimensional curved beam structures, the associated governing differential equations for in-plane vibrations of curved Bernoulli-Euler beams are two coupled differential equations in the flexural and the longitudinal displacements. If the curved beam is uniform, then the coefficients of the differential equations are constants. After some simple arithmetic operations, the coupled differential equations can be uncoupled and reduced into a sixth-order ordinary differential equation with constant coefficients. Hence the problem can be solved by different analytical methods, and the exact solutions can be obtained [Love (1944); Morley (1958)].

Based on Timoshenko beam theory, the associated governing differential equations for the in-plane vibrations of curved beams are three coupled differential equations in the longitudinal displacement, the flexural displacement and the angle of the rotation due to bending. The problems were studied by the numerical methods, such as: the transfer matrix approach [Irie, Yamada and Takahashi (1980)] and the dynamic stiffness matrix method [Wang and Issa (1987), Huang, Tseng and Lin (1998)]. The exact solutions for extensional curved uniform Timoshenko beams were developed by Lin and Lee (2001) and Lee, Lin and Hsu (2008).

When the beams are non-uniform, the coefficients of associated governing differential equations are variable coefficients. The exact solutions in general are not available. Hence, the problems were studied mainly by approximated methods such as the Rayleigh–Ritz method [Laura, Bambill, Filipich and Rossi (1988)], the Galerkin method [Lecoanet and Piranda (1983)], the transfer matrix method [Murthy and Nigam (1975)], the discrete Green function method [Kawakami, Sakiyama, Matsuda and Morita (1995)], and the asymptotic analysis of the equations of free vibrations [Tarnopolskaya, De Hoog, Fletcher and Thwaites (1996)].

Exact solutions for curved non-uniform Bernoulli-Euler beams are only found in the works done by Suzuki and Takahashi (1982) and Lee and Chao (2000a, 2000b, 2001). Suzuki and Takahashi (1982) gave an exact series solution to the beams with the same boundary conditions at both ends. Nevertheless, their method has difficulty in handing the problems with other kind of boundary conditions. Lee and Chao (2000a, 2000b, 2001) are the first one who uncoupled the two coupled differential equations with variable coefficients and developed the exact solutions for the beams with the physic properties in arbitrary polynomial forms.

For three dimensional space curved beams, a series developments can be found in the works by Reissner (1981), Iura and Atluri (1988), Atluri, Iura and Vasudevan (2001), Busool and Eisenberger (2002) and Zupan and Saje (2003, 2006).

From the existing literature, due to the complexity of the three coupled governing differential equations with variable coefficients, it can be found that the exact solutions for the free in-plane vibrations of extensional curved non-uniform Timoshenko beams had never been developed before.

In this paper, based on the two dimensional curved beam theory, one studies the free in-plane vibrations of extensional curved non-uniform Timoshenko beams. Three physical parameters are introduced to simplify the analysis. By eliminating all the terms with the non-dimensional axial displacement parameter, then reducing the order of differential operator acting on the non-dimensional flexural displacement parameter, one uncouples the three governing characteristic differential equations and reduces them into a sixth-order ordinary differential equation with variable coefficients in the angle of the rotation due to bending. The explicit relations between the axial and flexural displacements and the angle of the rotation due to bending are derived.

When the radius of a curved beam becomes infinite, the curved beam is reduced to a straight beam. Consequently, the sixth-order ordinary differential equation should be reduced to a fourth-order ordinary differential equation. However, the limiting study had never been successfully explored before. In this paper, a successful limiting study is revealed.

It is shown that if the material and geometric properties of the beam are in arbitrary polynomial forms, then the exact solutions for the in-plane vibrations of the beam can be obtained. Several limiting studies are revealed. Finally, the numerical results are compared with those in the existing literature. The influence of taper ratio and arc length on the first two natural frequencies of the beams is illustrated.

2 Coupled Governing Equations

Consider the in-plane motion of a curved non-uniform Timoshenko beam of constant radius R and doubly symmetric cross section, as shown in Figure 1. If the thickness of the beam is small in comparison with the radius of the beam, the displacement fields of the curved beam in cylindrical coordinates are:

$$u_r(r,s,z,t) = v(s,t), \tag{1}$$

$$u_\theta(r,s,z,t) = u(s,t) + \bar{r}\psi(s,t), \tag{2}$$

$$u_z(r,s,z,t) = 0, \tag{3}$$

where u_r, u_θ and u_z denote the displacements of the beam in the r, θ and z directions, respectively. $s = R\theta$. v and u are the neutral axis displacement of the beam in the r and θ directions, respectively. ψ is the angle of the rotation due to bending. $\bar{r}$ is measured outward from the neutral axis in the r direction.

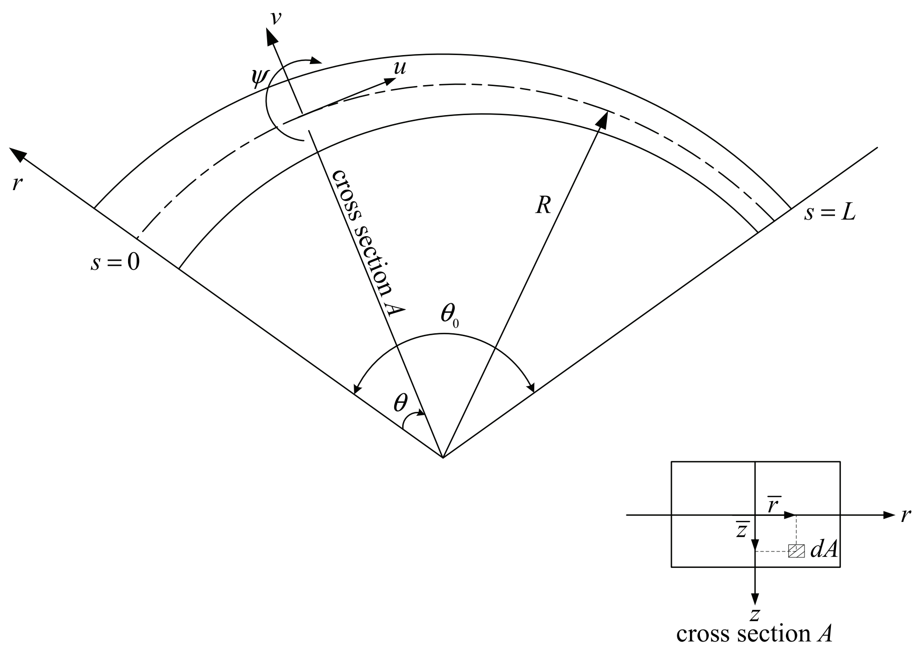

Figure 1: Geometry and coordinate system of a curved non-uniform beam of constant radius.

Substituting equations (1-3) into the strain-displacement relations in the cylindrical coordinate, the only two non-zero strains, $\varepsilon_{\theta\theta}$ and $\varepsilon_{r\theta}$, are

$$\varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = \frac{R}{R+\bar{r}}\left[\left(\frac{\partial u}{\partial s} + \frac{v}{R}\right) + \bar{r}\frac{\partial \psi}{\partial s}\right], \tag{4}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = \frac{1}{2}\frac{R}{R+\bar{r}}\left(\frac{\partial v}{\partial s} - \frac{u}{R} + \psi\right). \tag{5}$$

When $\bar{r}$ is small in comparison with R, the two strains are reduced to

$$\varepsilon_{\theta\theta} = \frac{\partial u}{\partial s} + \frac{v}{R} + \bar{r}\frac{\partial \psi}{\partial s}, \tag{6}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial v}{\partial s} - \frac{u}{R} + \psi\right). \tag{7}$$

Employing the two stress and strain relations, $\sigma_{\theta\theta} = E\varepsilon_{\theta\theta}$ and $\tau_{r\theta} = 2kG\varepsilon_{r\theta}$, the strain energy of the beam is

$$
\begin{aligned}
V^* =& \frac{1}{2}\int_0^L \int_A \sigma_{\theta\theta}\varepsilon_{\theta\theta} dAdS + \frac{1}{2}\int_0^L \int_A \tau_{r\theta}\varepsilon_{r\theta} dAds \\
=& \frac{1}{2}\int_0^L \int_A E\left[\left(\frac{\partial u}{\partial s} + \frac{v}{R}\right)^2 + 2\bar{r}\left(\frac{\partial u}{\partial s} + \frac{v}{R}\right)\frac{\partial \psi}{\partial s} + \bar{r}^2\left(\frac{\partial \psi}{\partial s}\right)^2\right] dAds \\
&+ \frac{1}{2}\int_0^L \int_A kG\left(\frac{\partial v}{\partial s} - \frac{u}{R} + \psi\right)^2 dAds,
\end{aligned} \tag{8}
$$

where L is the length of the neural axis. E, G, k, $I(s)$ and $A(s)$ denote the Young's modulus, the shear correction factor, the area moment of inertia and the cross section area of the beam, respectively. Since the cross section of the beam considered is doubly symmetric, the integral of the second term in the square bracket vanishes. The strain energy is simplified as

$$
V^* = \frac{1}{2}\int_0^L \left[EA\left(\frac{\partial u}{\partial s} + \frac{v}{R}\right)^2 + EI\left(\frac{\partial \psi}{\partial s}\right)^2 + kAG\left(\frac{\partial v}{\partial s} - \frac{u}{R} + \psi\right)^2\right] ds. \tag{9}
$$

The Kinetic energy of the system is

$$
T^* = \frac{1}{2}\int_0^L \left\{\rho A\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2\right] + \rho I\left(\frac{\partial \psi}{\partial t}\right)^2\right\} ds, \tag{10}
$$

where ρ is the mass per unit volume of the beam.

Applying the Hamilton's principle, one obtains the following three coupled governing differential equations

$$
\frac{\partial}{\partial s}\left[EA\left(\frac{\partial u}{\partial s} + \frac{v}{R}\right)\right] + \frac{kGA}{R}\left(\frac{\partial v}{\partial s} - \frac{u}{R} + \psi\right) = \rho A\frac{\partial^2 u}{\partial t^2}, \tag{11}
$$

$$
\frac{\partial}{\partial s}\left[kGA\left(\frac{\partial v}{\partial s} - \frac{u}{R} + \psi\right)\right] - \frac{EA}{R}\left(\frac{\partial u}{\partial s} + \frac{v}{R}\right) = \rho A\frac{\partial^2 v}{\partial t^2}, \tag{12}
$$

$$
\frac{\partial}{\partial s}\left(EI\frac{\partial \psi}{\partial s}\right) - kGA\left(\frac{\partial v}{\partial s} - \frac{u}{R} + \psi\right) = \rho I\frac{\partial^2 \psi}{\partial t^2}, \tag{13}
$$

and the associated boundary conditions are at $s = 0$ and L:

$$
EA\left(\frac{\partial u}{\partial s} + \frac{v}{R}\right) = 0 \text{ or } u = 0 \tag{14}
$$

$$kGA\left(\frac{\partial v}{\partial s}-\frac{u}{R}+\psi\right)=0 \text{ or } v=0 \tag{15}$$

$$EI\frac{\partial \psi}{\partial s}=0 \text{ or } \psi=0 \tag{16}$$

For time harmonic vibration with angular frequency Ω, one assumes

$$u(s,t)=U(s)e^{i\Omega t}, \tag{17}$$

$$v(s,t)=V(s)e^{i\Omega t}, \tag{18}$$

$$\psi(s,t)=\Psi(s)e^{i\Omega t}. \tag{19}$$

The coupled governing characteristic differential equations for the in-plane vibrations of a curved non-uniform beam are

$$\left[EA\left(U'+\frac{V}{R}\right)\right]'+\frac{kGA}{R}\left(V'-\frac{U}{R}+\Psi\right)+\rho A\Omega^2 U=0, \tag{20}$$

$$\left[kGA\left(V'-\frac{U}{R}+\Psi\right)\right]'-\frac{EA}{R}\left(U'+\frac{V}{R}\right)+\rho A\Omega^2 V=0, \tag{21}$$

$$\left(EI\Psi'\right)'-kGA\left(V'-\frac{U}{R}+\Psi\right)+\rho I\Omega^2\Psi=0, \tag{22}$$

where the primes denote differentiation with respect to the s variable.

The associated boundary conditions are at s= 0 and L:

$$EA\left(\frac{\partial U}{\partial s}+\frac{V}{R}\right)=0 \text{ or } U=0 \tag{23}$$

$$kGA\left(\frac{\partial V}{\partial s}-\frac{U}{R}+\Psi\right)=0 \text{ or } V=0 \tag{24}$$

$$EI\frac{\partial \Psi}{\partial s}=0 \text{ or } \Psi=0 \tag{25}$$

3 Uncoupled Governing Equation in Terms of the Angle of Rotation due to Bending

3.1 Curved Non-uniform Timoshenko Beams

To uncouple the governing characteristic differential equations (20-22), one defines the following three physical parameters:

$$F_{\theta\psi}=\frac{kGA}{R}\Psi, \tag{26}$$

$$F_{r\psi} = (kGA\Psi)', \tag{27}$$

$$M_{z\psi} = \left(EI\Psi'\right)' - \left(kGA - \rho I\Omega^2\right)\Psi, \tag{28}$$

where $F_{\theta\psi}$, $F_{r\psi}$ and $M_{z\psi}$ are the forces per unit arc length in the θ, r directions and the bending moment per unit arc length in the z direction, caused by the angle of the rotation due to bending Ψ, respectively. In terms of $F_{\theta\psi}$, $F_{r\psi}$ and $M_{z\psi}$, The three coupled governing characteristic differential equations (20-22), can be rewritten as

$$\frac{U''}{R} + \gamma_1\frac{U'}{R} - \left(\frac{kGA}{R^2EA} - \gamma_2\Omega^2\right)\frac{U}{R} = -\frac{1}{R^2}\left[\left(1+\frac{kGA}{EA}\right)V' + \gamma_1 V\right] - \frac{F_{\theta\psi}}{REA}, \tag{29}$$

$$\frac{U'}{R} + \frac{(kGA)'}{EA+kGA}\frac{U}{R} = \frac{kGAV'' + (kGA)'V' - EA\left(\frac{1}{R^2} - \gamma_2\Omega^2\right)V + F_{r\psi}}{EA+kGA}, \tag{30}$$

$$\frac{U}{R} = V' - \frac{M_{z\psi}}{kGA}, \tag{31}$$

where γ_1 and γ_2 are defined in the appendix.

3.1.1 Coupled Differential Equations in terms of V and Ψ

Substituting equation (31) and its first and second derivatives into equations (29-30), one can eliminate all the terms with parameter U. Therefore, equations (29-30) can be expressed in terms of V and Ψ and rewritten as

$$LV = \left(D^2 + \gamma_1 D + \gamma_2\Omega^2\right)\left(\frac{M_{z\psi}}{kGA}\right) - \frac{V_S}{R^2EA}, \tag{32}$$

$$L_1 V = D\left(\frac{M_{z\psi}}{kGA}\right) + \frac{D(V_S)}{EA} \tag{33}$$

Here, D^n denotes the nth order differential operator with respect to s,

$$L = D^3 + \gamma_1 D^2 + \left(\frac{1}{R^2} + \gamma_2\Omega^2\right)D + \frac{\gamma_1}{R^2}, \tag{34}$$

$$L_1 = D^2 + \left(\frac{1}{R^2} - \gamma_2\Omega^2\right), \tag{35}$$

$$V_S = M_{Z\psi} + RF_{\theta\psi} = \left(EI\Psi'\right)' + \rho I\Omega^2\Psi, \tag{36}$$

and γ_3 is defined in the appendix.

It should be mentioned that the operators at the left hand side of equations (32) and (33) are third-order and second-order differential equations with variable coefficients in terms of V, respectively, the operators at the right hand side of equations (32) and (33) are fourth-order and third-order differential equations with variable coefficients in terms of Ψ, respectively. It can be easily shown that V_S is the shear force in the beam.

3.1.2 Uncoupled Governing Differential Equations in terms of Ψ

To further uncouple differential equations (32-33), one reduces the order of differential operators on V. Now one defines a first-order differential operator R_1 first,

$$\mathrm{R}_1 = D + \gamma_1. \tag{37}$$

After applying operator R_1 on equation (33), then subtracting it from equation (32), one obtains a first order differential equation

$$L_2 V = -\frac{1}{2\rho A\Omega^2}\left(D^2 + \frac{1}{R^2}\right)V_S + \frac{1}{2}\frac{M_{Z\psi}}{kGA}, \tag{38}$$

where

$$L_2 = D + \frac{1}{2}\gamma_3, \tag{39}$$

and γ_3 is defined in the appendix.

Secondly, one defines the other first-order differential operator R_2

$$R_2 = D - \frac{1}{2}\gamma_3 \tag{40}$$

After applying operator R_2 on equation (38), then subtracting it from equation (33), one finally can uncouple the V and Ψ parameters and express V explicitly in terms of Ψ and its first five order derivatives.

$$V = \frac{-1}{2\rho A\Omega^2 a_0}\left[\left(D - \frac{3\gamma_3}{2}\right)\left(D^2 + \frac{1}{R^2}\right) + 2\gamma_2\Omega^2 D\right]V_S + \frac{1}{2a_1}\left(D + \frac{1}{2}\gamma_3\right)\frac{M_{Z\psi}}{kGA}, \tag{41}$$

where

$$a_0 = \frac{1}{R^2} - \gamma_2\Omega^2 - \left(\frac{\gamma_3}{2}\right)' + \left(\frac{\gamma_3}{2}\right)^2. \tag{42}$$

Substituting equation (41) and its differentiation into equation (38), one obtains the uncoupled sixth-order governing characteristic differential equation in terms of the angle of rotation due to bending

$$\begin{aligned}&\left\{\left[\left(D - \frac{\gamma_3}{2} - \frac{a_0'}{a_0}\right)\left(D - \frac{3\gamma_3}{2}\right) + a_1\right]\left(D^2 + \frac{1}{R^2}\right)\right.\\&\left.+2\gamma_2\Omega^2\left(D + \frac{\gamma_3}{2} - \gamma_1 - \frac{a_0'}{a_0}\right)D\right\}V_S\\&\qquad + \rho A\Omega^2\left[\left(D + \frac{\gamma_3}{2} - \frac{a_0'}{a_0}\right)\left(D + \frac{\gamma_3}{2}\right) - a_0\right]\frac{M_{Z\Psi}}{kGA} = 0.\end{aligned} \tag{43}$$

3.1.3 Explicit Relations

Equation (41) shows the explicit relation between V and Ψ and its first five order derivatives. The other explicit relations are derived in the following.

When equation (41) is substituted into equation (38), one can obtain an explicit relation between $V'$and Ψ and its first five order derivatives

$$V' = L_3 V_S + L_4 \left(\frac{M_{Z\psi}}{kGA}\right), \tag{44}$$

where L_3 and L_4 are differential operators ($A2$) and ($A3$), defined in the Appendix, respectively.

When equations (41) and (44) are substituted into equation (33), one has the explicit relation between $V''$and Ψ and its first fifth order differentiations

$$V'' = L_5 V_S + L_6 \left(\frac{M_{Z\psi}}{kGA}\right), \tag{45}$$

where L_5 and L_6 are differential operators ($A4$) and ($A5$), defined in the Appendix, respectively.

One can obtain the explicit relation between U and Ψ and its first fifth order differentiations by substituting equation (44) into equation (31)

$$\frac{U}{R} = L_3 V_S + (L_4 - 1)\left(\frac{M_{Z\psi}}{kGA}\right). \tag{46}$$

Substituting equations (41, 44-46) into equation (30), it yields the explicit relation between U' and Ψ and its first five order derivatives

$$\frac{U'}{R} = L_5 V_S + (L_6 - D)\left(\frac{M_{Z\psi}}{kGA}\right). \tag{47}$$

The associated boundary conditions, in terms of Ψ and its first five order derivatives, can be obtained by substituting explicit relations, equations (41, 44, 46-47), into equations (23-25).

3.2 Curved Uniform Timoshenko Beams

For uniform beams

$$a_0 = \frac{1}{R^2} - \gamma_2\Omega^2, \quad a_0' = 0, \quad \gamma_1 = 0, \quad \gamma_3 = 0, \tag{48}$$

$M_{z\phi}$ and V_S are reduced to

$$M_{z\phi} = EI\Psi'' - \left(kGA - \rho I\Omega^2\right)\Psi, \tag{49}$$

$$V_S = EI\Psi'' + \rho I \Omega^2 \Psi. \tag{50}$$

Consequently, the governing equation (41) becomes

$$\Psi'''''' + q_4 \Psi'''' + q_2 \Psi'' + q_0 \Psi = 0, s \in (0,\ L) \tag{51}$$

where

$$q_4 = \frac{2}{R^2} + \Omega^2 \frac{\rho A}{EI} \left(\frac{\rho I}{\rho A} + \frac{EI}{EA} + \frac{EI}{kGA} \right), \tag{52}$$

$$q_2 = \Omega^4 \left(\frac{\rho A}{EI} \right)^2 \left[\frac{EI}{kGA} \frac{\rho I}{\rho A} + \frac{EI}{EA} \left(\frac{EI}{kGA} + \frac{\rho I}{\rho A} \right) \right] \\ - \Omega^2 \frac{\rho A}{EI} \left[\frac{1}{R^2} \left(\frac{EI}{kGA} - 2\frac{\rho I}{\rho A} + \frac{EI}{EA} \right) + 1 \right] + \frac{1}{R^4}, \tag{53}$$

$$q_0 = -\left\{ -\Omega^6 \left(\frac{\rho A}{EI} \right)^3 \frac{EI}{EA} \frac{EI}{kGA} \frac{\rho I}{\rho A} + \Omega^4 \left(\frac{\rho I}{\rho A} \right)^2 \left[\frac{1}{R^2} \frac{EI}{kGA} \frac{\rho I}{\rho A} \right. \right. \\ \left. \left. + \frac{EI}{EA} \left(1 + \frac{1}{R^2} \frac{\rho I}{\rho A} \right) \right] - \Omega^2 \frac{\rho A}{EI} \left(\frac{1}{R^2} \frac{\rho I}{\rho A} \right) \right\}, \tag{54}$$

This equation is exactly the same as that given by Lee and Lin (2001).

3.3 Curved Non-uniform Bernoulli-Euler Beams

For Bernoulli-Euler beams, both shear deformation and rotary inertia are not considered. By letting $\rho I = 0$ and kGA being infinite, $M_{z\Psi}$ and V_S now are

$$\lim_{\substack{kGA \to \infty \\ \rho I \to 0}} \frac{M_{z\psi}}{kGA} = -\Psi, \tag{55}$$

$$\lim_{\rho I \to 0} V_S = \left(EI\Psi' \right)'. \tag{56}$$

Consequently, the governing characteristic differential equation (43) is reduced to

$$\left\{ \left[\left(D - \frac{\gamma_3}{2} - \frac{a_0'}{a_0} \right) \left(D - \frac{3\gamma_3}{2} \right) + a_0 \right] \left(D^2 + \frac{1}{R^2} \right) \right. \\ \left. + 2\gamma_2 \Omega^2 \left(D + \frac{\gamma_3}{2} - \gamma_1 - \frac{a_0'}{a_0} \right) D \right\} \left(EI\Psi' \right)' \\ - \rho A \Omega^2 \left[\left(D + \frac{\gamma_3}{2} - \frac{a_0'}{a_0} \right) \left(D + \frac{\gamma_3}{2} \right) - a_0 \right] \Psi = 0. \tag{57}$$

3.4 Curved Uniform Bernoulli-Euler Beams

For curved uniform Bernoulli-Euler beams, equation (57) is further reduced to

$$\Psi'''''' + \left(\frac{2}{R^2} + \frac{\rho\Omega^2}{E}\right)\Psi'''' + \left[\frac{1}{R^4} - \frac{\rho\Omega^2}{E}\left(\frac{1}{R^2} + \frac{A}{I}\right)\right]\Psi'' + \frac{\rho A\Omega^2}{EI}\left(\frac{1}{R^2} - \frac{\rho\Omega^2}{E}\right)\Psi = 0. \quad (58)$$

This equation is exactly the same as that given by Lee (1975).

3.5 Straight Non-Uniform Timoshenko Beams

By letting R being infinite, equations (42-43) and (47) are reduced to

$$a_0 = -\gamma_2\Omega^2 - \left(\frac{\gamma_3}{2}\right)' + \left(\frac{\gamma_3}{2}\right)^2, \quad (59)$$

$$\left\{\left[\left(D - \frac{\gamma_3}{2} - \frac{a_0'}{a_0}\right)\left(D - \frac{3\gamma_3}{2}\right) + a_0\right]D^2 + 2\gamma_2\Omega^2\left(D + \frac{\gamma_3}{2} - \gamma_1 \ - \frac{a_0'}{a_0}\right)D\right\}V_S + \rho A\Omega^2\left[\left(D + \frac{\gamma_3}{2} - \frac{a_0'}{a_0}\right)\left(D + \frac{\gamma_3}{2}\right) - a_0\right]\frac{M_{z\psi}}{kGA} = 0, \quad (60)$$

$$\left[D^3 - \frac{3\gamma_3}{2}D^2 + \left(2a_0 + 2\gamma_2\Omega^2\right)D\right]V_S + \rho A\left[\Omega^2 D + \frac{\gamma_3}{2}\Omega^2\right]\frac{M_{z\psi}}{kGA} = 0. \quad (61)$$

It is well known that the uncoupled governing differential equation for a Timoshenko beam is a fourth order differential equation. However, one can observe that equations (60-61) are a sixth-order and a fifth-order differential equations, respectively. To reduce the order of differential equation, one define a first-order differential operator R_3

$$R_3 = D - \left(\frac{a_0'}{a_0} + \frac{\gamma_3}{2}\right). \quad (62)$$

After applying operator R_3 on equation (61), then subtracting it from equation (60), one has

$$\left(D^2 - \gamma_3 D\right)V_S + \rho A\Omega^2\left(\frac{M_{z\psi}}{kGA}\right) = 0. \quad (63)$$

It is a fourth-order governing differential equation in terms of Ψg and is the governing differential equation for the flexural vibration of a straight non-uniform Timoshenko beam

$$kGA\left\{\frac{-1}{\rho A\Omega^2}\left[\left(EI\Psi'\right)'+\rho I\Omega^2\Psi\right]'\right\}'-\left(EI\Psi'\right)'+\left(-\rho I\Omega^2+kGA\right)\Psi=0. \quad (64)$$

This equation is exactly the same as given by Lee and Lin (1995).

3.6 Uniform Straight Timoshenko Beams

For uniform straight Timoshenko beams, equation (64) is further reduced to

$$\Psi''''+\Omega^2\frac{\rho A}{EI}\left(\frac{\rho I}{\rho A}+\frac{EI}{kGA}\right)\Psi''+\left[\frac{EI}{kGA}\frac{\rho I}{\rho A}\Omega^4\left(\frac{\rho A}{EI}\right)^2-\Omega^2\frac{\rho A}{EI}\right]\Psi=0. \quad (65)$$

This equation is exactly the same as that given by Huang (1961).

4 Exact Fundamental Solutions

The uncoupled governing characteristic differential equation, in terms of Ψ, for the in-plane vibration of curved non-uniform Timoshenko beams can be expressed as a sixth-order differential equation with variable coefficients in the form of

$$\sum_{i=0}^{6} e_i(s)\Psi^{(6-i)}(s)=f(s), \quad s\in(0,\, L)\,. \quad (66)$$

If all the coefficients are in the polynomial forms, i.e.,

$$e_p(s)=\sum_{j=0}^{m_p} a_{p,j}\,(s-s_0)^j, \quad p=0\sim 6, \quad (67)$$

where s_0 is a constant and $0<s_0<L$ and m_p, $p=0\sim 6$, are integers representing the numbers of the terms in the series, then one can assume the six fundamental solutions of the differential equation in the form:

$$\Psi_i(s)=\frac{1}{i!}\,(s-s_0)^i+\sum_{q=n}^{\infty}A_{q,i}\,(s-s_0)^q, \quad i=0\sim 5,\ n=6. \quad (68)$$

Substituting equations (67-68) into equation (66), collecting the coefficients of like powers, the following recurrence formula can be obtained:

$$A_{q,i}=-\frac{1}{n!a_{0,0}}\sum_{l=0}^{n-1} l!a_{n-l,0}A_{l,i}, \quad q=n, \quad (69)$$

$$A_{q,i} = -\frac{(q-n)!}{q!a_{0,0}}\left[\sum_{l=0}^{n-1}\frac{(q-n+l)!}{(q-n)!}a_{n-l,0}A_{q-n+l,i}\right.$$
$$\left.+\sum_{l=0}^{n}\sum_{j=1}^{q-n}\frac{(q-n+l-j)!}{(q-n-j)!}a_{n-l,j}A_{q-n+l-j,i}\right], \quad q>n, \quad (70)$$

With this recurrence formula, one can generate the six exact fundamental equations of the governing characteristic differential equation.

After substituting the homogenous solution which is a linear combination of the six fundamental solutions into the associated boundary conditions, the frequency equation and the natural frequencies of the beam can be obtained, consequently.

5 Numerical Results and Discussion

To illustrate the previous analysis, the following non-dimensional parameters will be used in the following numerical analysis:

$$\xi = \frac{s}{L}, \quad U^* = \frac{U}{L}, \quad V^* = \frac{V}{L}, \quad \Psi^* = \Psi, \quad \theta_0 = \frac{L}{R}, \quad L_z = \frac{L}{\gamma_z(0)},$$
$$a(\xi) = \frac{E(s)A(s)}{E(0)A(0)}, \quad m(\xi) = \frac{\rho(s)A(s)}{\rho(0)A(0)}, \quad \delta(\xi) = \frac{kG(s)A(s)}{kG(0)A(0)},$$
$$b(\xi) = \frac{E(s)I(s)}{E(0)I(0)}, \quad J(\xi) = \frac{\rho(s)I(s)}{\rho(0)I(0)}, \quad \mu = \frac{2(1+\nu)}{k}\frac{1}{L_z^2}, \quad \eta = \frac{1}{L_z^2},$$
$$\omega = \Omega L^2\sqrt{\frac{\rho(0)A(0)}{E(0)I(0)}}, \quad \Lambda = \Omega L\sqrt{\frac{\rho(0)}{E(0)}},$$

where $\gamma_z(\xi)$ is the radius of gyration about z-axis, θ_0 is the centre angle, ω is the non-dimension angular natural frequency of flexural vibration and Λ is the non-dimension angular natural frequency of longitudinal vibration.

In the following, the natural frequencies of curved linearly tapered beams of rectangular cross-section are studied. The material and geometric properties of the beams with taper ratio ε are $a(\xi) = \delta(\xi) = m(\xi) = 1-\varepsilon\xi$ and $b(\xi) = J(\xi) = (1-\varepsilon\xi)^3$, respectively.

In Table 1, the first five non-dimensional angular natural frequencies ω of curved Timoshenko beams with hinged-hinged boundary conditions determined in the present analysis are compared with those in the existing literature. It shows the results are very consistent.

In Figure 2, the influence of the taper ratio ε and the non-dimensional arc length L_z on the first non-dimensional natural frequencies Λ of curved linearly tapered

Table 1: The first five dimensionless angular natural frequencies ω of curved Timoshenko beams with hinged-hinged boundary conditions. $[1/(\theta_0\sqrt{\eta}) = 100,\ \nu = 0.3,\ k = 5/6,\ \mu = 2(1+\nu)\eta/k,\ L_Z = 100\theta_0]$

θ_0	Mode	Present study			*	**	* * *
		$\varepsilon = 0.8$	$\varepsilon = 0.4$	$\varepsilon = 0.0$	$\varepsilon = 0.0$	$\varepsilon = 0.0$	$\varepsilon = 0.0$
150°	1	13.7415	20.7851	26.4079	26.43	26.4079	26.4079
	2	38.6852	57.3028	72.5588	72.71	72.5587	72.5588
	3	75.6215	112.597	142.593	143.1	142.5925	142.5931
	4	120.964	180.271	227.935	229.2	227.9351	227.9352
	5	178.478	266.216	236.476	339.2	336.4950	336.4755
180°	1	11.5353	17.573	22.3497	22.37	22.3497	22.3497
	2	36.3591	53.8146	68.164	68.27	68.1644	68.1644
	3	72.8014	108.454	137.429	137.8	137.4288	137.4288
	4	118.402	176.696	223.742	224.6	223.7427	223.7416
	5	175.646	262.372	332.071	334.0	332.0705	332.0712

* : Wolf (1971); * * : Tufekci and Arpaci (1998); * * * : Lee and Lin (2001)

Bernoulli-Euler and Timoshenko beams with clamped-free ends is shown. It can be observed that with constant central angle, $\theta_0 = 10^0$, the first non-dimensional natural frequencies Λ decrease as the arc length L_z is increased.

In Figures 3 and 4, the relative displacements of the first mode in the radial and the axial directions are shown. It can be found that the first mode vibration is mainly dominated by the flexural (radial) motion. Since the natural frequency of a flexural vibration will decrease as the total length of the beam with the same cross section is increased, this explains the observation revealed in Figure 2. In addition, the first non-dimensional natural frequencies Λ increase as the taper ratio is increased.

In Figure 2, one can find that the difference between the first natural frequencies evaluated via two different beam theories will decrease as the arc length L_z of the beam is increased. It is due to the fact revealed in Figures 3 and 4. Since the first mode vibration is mainly dominated by the flexural motion, the shear deformation and the rotary inertia effects will be important when the arc length L_z is small.

In Figure 5, the influence of the taper ratio ε and the non-dimensional arc length L_z on the second non-dimensional natural frequencies Λ of the Bernoulli-Euler and Timoshenko beams is shown. It can be found that when L_z is small, the second non-dimensional natural frequencies of the beams with high taper ratio are greater than those with low taper ratio. However, when the taper ratio is greater than the associated critical value, the second non-dimensional natural frequencies of the

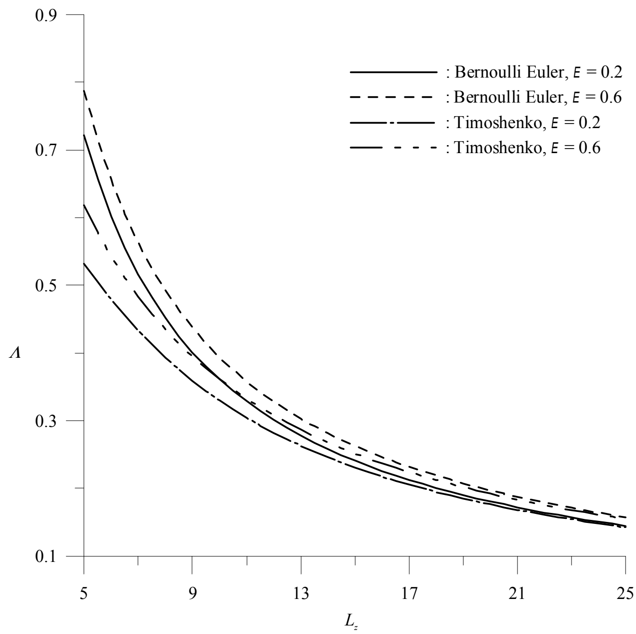

Figure 2: Influence of the taper ratio ε and the non-dimensional arc length L_z on the first non-dimensional natural frequencies Λ of curved linearly tapered beams with clamped-free ends [$\theta_0 = 10^0$].

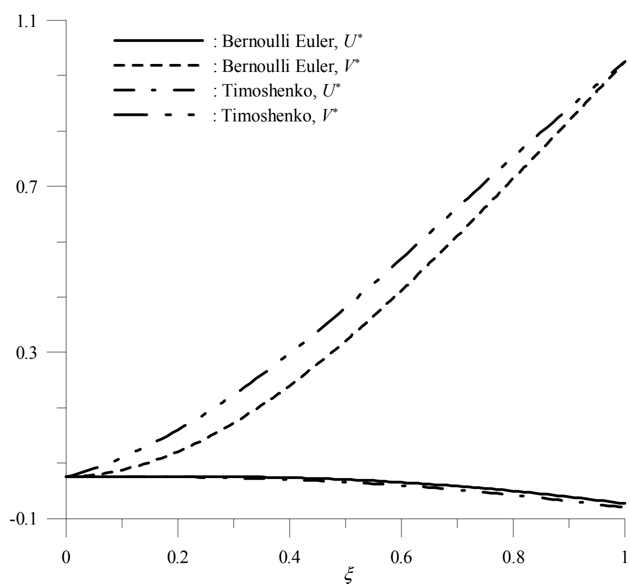

Figure 3: Relative displacements of the first mode in the radial and the axial directions [L_z = 5, ε = 0.2, $\theta_0 = 10^0$].

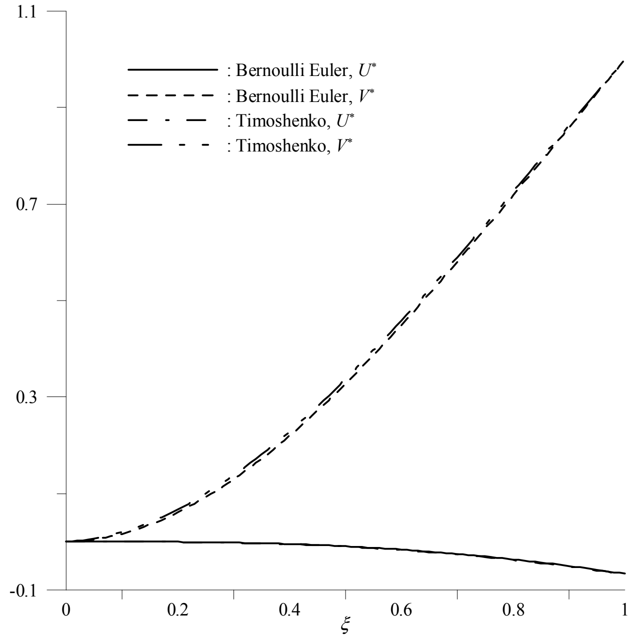

Figure 4: Relative displacements of the first mode in the radial and the axial directions [L_z = 15, ε = 0.2, $\theta_0 = 10^0$].

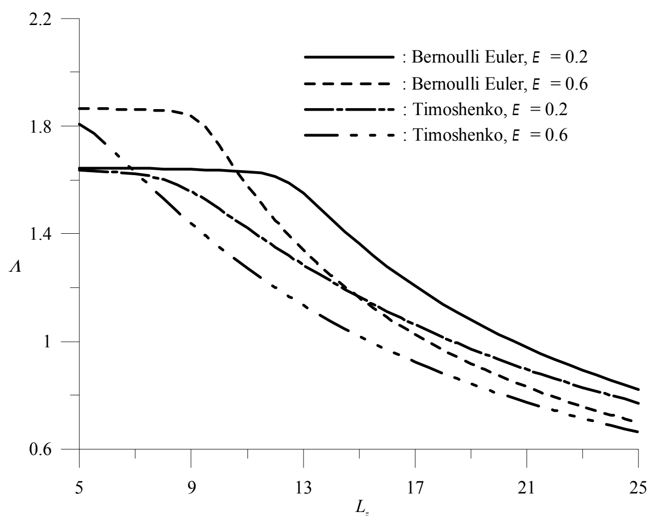

Figure 5: Influence of the taper ratio ε and the non-dimensional arc length L_z on the second non-dimensional natural frequencies Λ of curved linearly tapered beams with clamped-free ends [$\theta_0 = 10^0$].

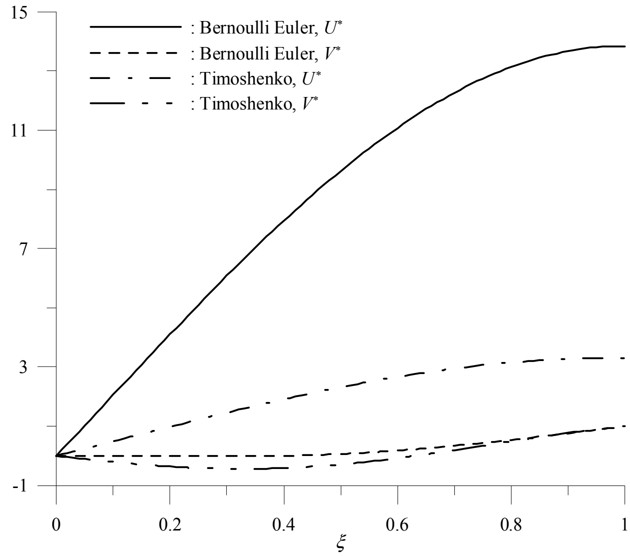

Figure 6: Relative displacements of the second mode in the radial and the axial directions [L_z = 5, ε = 0.2, $\theta_0 = 10^0$].

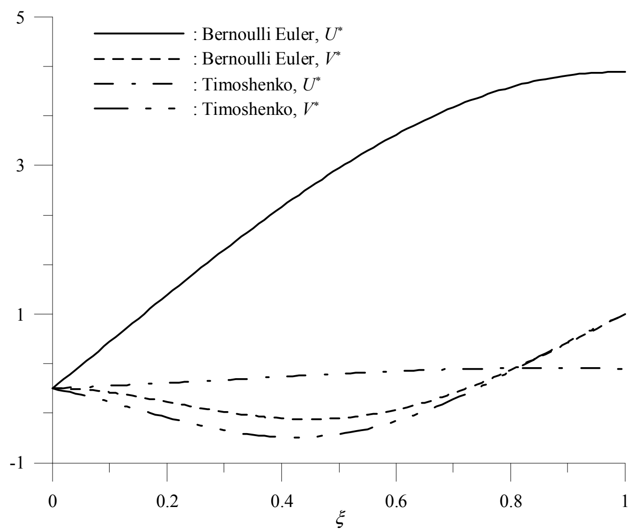

Figure 7: Relative displacements of the second mode in the radial and the axial directions [L_z = 10, ε = 0.2, $\theta_0 = 10^0$].

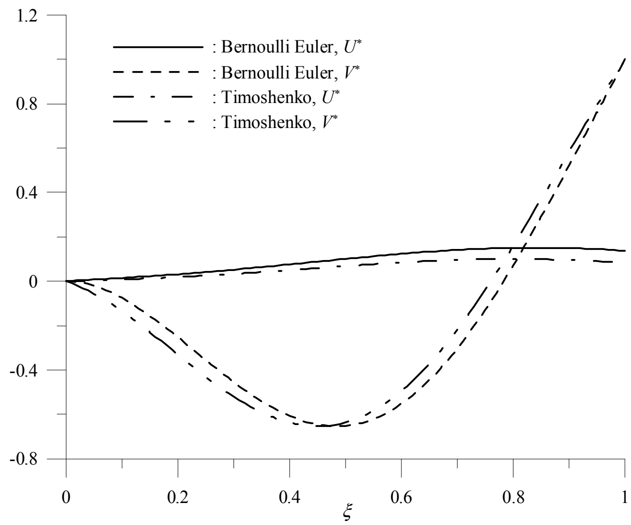

Figure 8: Relative displacements of the second mode in the radial and the axial directions [L_z = 15, ε = 0.2, $\theta_0 = 10^0$].

beams with high taper ratio will be lower than those with low taper ratio.

In Figures 6 $\sim$ 8, the second mode shapes of beams with taper ratio ε = 0.2 and different non-dimensional arc length L_z are shown. It can be observed that when L_z = 5 $\sim$ 10, the second mode vibration of Bernoulli-Euler beams is mainly dominated by the longitudinal (axial) motion. Since the influence of the total length on the natural frequency of a longitudinal vibration is not significant, the second natural frequency of Bernoulli-Euler beams with ε = 0.2 will be almost the same, when L_z = 5 $\sim$ 10. The conclusion is consistent with the curve that corresponds to the second natural frequency of Bernoulli-Euler beams with ε = 0.2 in Figure 5.

From Figure 5, one can observed that the difference between the natural frequencies evaluated via two different theories is small, when the arc length L_z is small. It is due to the fact that when the non-dimensional arc length L_z is small, the second mode vibration of the beams is mainly dominated by the longitudinal (axial) motion. In longitudinal (axial) vibration, the shear deformation and the rotary inertia effects will be negligible.

6 Conclusions

In this paper, three physical parameters are introduced to simplify the analysis. By eliminating all the terms with the axial displacement parameter, then reducing the order of differential operator acting on the flexural displacement parameter, one successfully uncouples the three governing characteristic differential equations with variable coefficients and reduces them into a sixth-order ordinary differential equation with variable coefficients in term of the angle of the rotation due to bending for the first time. The explicit relations between the axial and the flexural displacements and the angle of the rotation due to bending are also revealed. The exact solutions for the in-plane vibrations of the beams with material and geometric properties in arbitrary polynomial forms are obtained. Several limiting studies are illustrated. The influence of the taper ratio and the arc length on the first two natural frequencies of the beams is explored. The mode shapes are given to explain the observations.

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Appendix

$$\gamma_1 = \frac{(EA)'}{EA}, \quad \gamma_2 = \frac{\rho A}{EA}, \quad \gamma_3 = \frac{(\rho A)'}{\rho A} \tag{A1}$$

$$L_3 = \left[\frac{\gamma_3}{2a_0}\left(D - \frac{\gamma_3}{2}\right) + 1\right]\left[\frac{-1}{2\rho A\Omega^2}\left(D^2 + \frac{1}{R^2}\right)\right] - \frac{\gamma_3}{2a_0 EA} D \tag{A2}$$

$$L_4 = \frac{1}{2} - \frac{\gamma_3}{4a_0}\left(D + \frac{\gamma_3}{2}\right) \tag{A3}$$

$$\begin{aligned} L_5 = \frac{1}{a_0}\left(\frac{1}{R^2} - \gamma_2\Omega^2\right)\left(D - \frac{\gamma_3}{2}\right)\left[\frac{-1}{2\rho A\Omega^2}\left(D^2 + \frac{1}{R^2}\right)\right] \\ + \left[1 - \frac{1}{a_0}\left(\frac{1}{R^2} - \gamma_2\Omega^2\right)\right]\frac{1}{EA} D \end{aligned} \tag{A4}$$

$$L_6 = D - \frac{1}{2a_0}\left(\frac{1}{R^2} - \gamma_2\Omega^2\right)\left(D + \frac{\gamma_3}{2}\right) \tag{A5}$$