

A Fictitious Time Integration Method for the Numerical Solution of the Fredholm Integral Equation and for Numerical Differentiation of Noisy Data, and Its Relation to the Filter Theory

Chein-Shan Liu¹, Satya N. Atluri²

Abstract: The Fictitious Time Integration Method (FTIM) previously developed by Liu and Atluri (2008a) is employed here to solve a system of ill-posed linear algebraic equations, which may result from the discretization of a first-kind linear Fredholm integral equation. We rationalize the mathematical foundation of the FTIM by relating it to the well-known filter theory. For the linear ordinary differential equations which are obtained through the FTIM (and which are equivalently used in FTIM to solve the ill-posed linear algebraic equations), we find that the fictitious time plays the role of a regularization parameter, and its filtering effect is better than that of the Tikhonov and the exponential filters. Then, we apply this new method to solve the problem of numerical differentiation of noisy data [such as finding da/dN in fatigue, where a is the measured crack-length and N is the number of load cycles], and the inversion of the Abel integral equation under noise. It is established that the numerical method of FTIM is robust against the noise.

Keywords: Ill-posed linear equations, Regularization, Filter theory, Fictitious Time Integration Method (FTIM)

1 Introduction

In this paper we propose a robust and easy-to-implement FTIM-based method to solve the linear Fredholm integral equation of the first-kind:

$$\int_a^b K(s,t)x(t)dt = h(s), \quad s \in [c,d], \quad (1)$$

where $K(s,t)$ and $h(s)$ are known functions and $x(t)$ is an unknown function. We also suppose that $h(s)$ is perturbed by random noise. Some numerical methods

¹ Department of Civil Engineering, National Taiwan University, Taipei, Taiwan. E-mail: liu-uchein.nshan@msa.hinet.net

² Center for Aerospace Research & Education, University of California, Irvine

to solve the above equation are discussed in [Landweber (1951); Maleknejad and Mahmoudi (2004); Maleknejad et al. (2006); Wang (2006)], and in the references therein.

Also, in many applications, it is necessary to calculate the derivative of a function measured experimentally, i.e., to differentiate noisy data. Such a problem is well-known in fatigue-mechanics, wherein a relation for da/dN is often required to be obtained from the measured noisy-data for the crack-length a and the number of load-cycles N . The problem of numerical differentiation of noisy data is ill-posed: small changes in the data may result in large changes of the derivative [Ramm and Smirnova (2001); Ahn et al. (2006)]. We address this problem also in the present paper, and propose a new FTIM-based solution.

A possible application of the proposed new FTIM-based numerical method is to solve the following Abel integral equation under noise:

$$\int_0^s \frac{\phi(t)}{(s-t)^\eta} dt = h(s), \quad s > 0, \quad \eta \in (0, 1). \quad (2)$$

It is known that Eq. (2) has the exact solution:

$$\phi(s) = \frac{\sin(\eta\pi)}{\pi} \frac{d}{ds} \int_0^s \frac{h(t)}{(s-t)^{1-\eta}} dt. \quad (3)$$

Nevertheless, this exact solution fails in practical applications, when the input function $h(t)$ contains a random error, since the differential operator involved in Eq. (3) is ill-posed and unbounded.

There were many approaches for determining the numerical solution of the Abel integral equation [Gorenflo and Vessella (1991)]. Fettis (1964) has proposed a numerical solution of the Abel equation by using the Gauss-Jacobi quadrature rule. Kosarev (1973) proposed a numerical solution of the Abel equation by using the orthogonal polynomials expansion. Piessens and Verbaeten (1973) and Piessens (2000) developed an approximate solution of the Abel equation by means of the Chebyshev polynomials of the first-kind. When the input data contains noisy error, Hao (1985,1986) used the Tikhonov regularization technique, and Murio et al. (1992) suggested a stable numerical solution. Furthermore, Garza et al. (2001) and Hall et al. (2003) used the wavelet method, and Huang et al. (2008) used the Taylor expansion method to derive the inversion of noisy Abel equation.

Our strategy to tackle the problem with the numerical differentiation of noisy data is to recast it into Eq. (1) by utilizing the Laplace transform. Then, we solve the first-kind Fredholm integral equation by discretizing Eq. (1) into a linear system of algebraic equations:

$$\mathbf{Ax} = \mathbf{b}, \quad (4)$$

where \mathbf{A} is a given matrix, and \mathbf{x} is an unknown vector. Eq. (4) may represent the least-squares solution of a system of equations $\mathbf{R}\mathbf{x} = \mathbf{c}$, where \mathbf{R} is a rectangular matrix. In such a case, $\mathbf{A} = \mathbf{R}^T\mathbf{R}$ and $\mathbf{b} = \mathbf{R}^T\mathbf{c}$, where the superscript T signifies the transpose. The input data of \mathbf{b} may be corrupted by noise.

In a practical application of Eq. (4) in engineering problems, the data \mathbf{b} are rarely given exactly; instead, the noises are unavoidable due to measurement and modeling errors. Therefore, we may encounter the problem that the numerical solution of Eq. (4) may deviate from the exact one to a great extent, when \mathbf{A} is severely ill-conditioned and \mathbf{b} is perturbed by noise.

The solution of ill-posed linear equations is an important issue in many engineering problems. We are specially interested in the solution of Eq. (4) under noisy data, when the condition number of \mathbf{A} is large. Many numerical methods used in computational mechanics [Zhu et al. (1999); Atluri (2005); Atluri et al. (2006); Atluri and Shen (2002); Atluri and Zhu (1998a,1998b)] lead to the requirement of solving an ill-posed system of linear equations ultimately. Collocation methods, such as those discussed by Liu (2007a,2007b,2007c) for the Trefftz method of Laplace equation also need to solve a large system of often-ill-conditioned linear equations.

To account for the sensitivity to noise, it is customary to use a regularization method to solve an ill-posed problem [Kunisch and Zou (1998); Wang and Xiao (2001); Xie and Zou (2002); Resmerita (2005)], where a suitable regularization parameter is used to suppress the bias in the computed solution by a better balance of the approximation error and the propagated data error. Following the pioneering work of Tikhonov and Arsenin (1977), many regularization techniques were developed. For a large scale system the main choice is to use the iterative regularization algorithm, where a regularization parameter is represented by the number of iterations. The iterative method works if an early stopping criterion is used to prevent the reintroduction of noisy components in the approximated solutions.

In this paper we solve an ill-posed linear system of linear equations by the FTIM-based method, and show that it is equivalent to a new filter regularization. To rationalize the proposed methods we show the characteristics of the filtering effect. This paper is organized as follows. In Section 2 we briefly illustrate the Tikhonov regularization and the inherent exponential filter, by using the linear ordinary differential equations (ODEs). In Section 3, we explore the effect of the Fictitious Time Integration Method (FTIM) to solve the ill-posed linear equations system, and show its equivalence to the filter theory. Section 4 is devoted to the use of the FTIM to solve the first-kind linear Fredholm integral equation under noise, to finding the numerical derivative of noisy data, and to solve the Abel integral equation under noise. Some conclusions are drawn in Section 5.

2 The regularization due to Tikhonov, and the inherent exponential filter

We suppose that \mathbf{A} is nonsingular, and thus has a singular-value decomposition:

$$\mathbf{A} = \mathbf{U} \text{diag}\{s_i\} \mathbf{V}^T, \tag{5}$$

where s_i are the singular values of \mathbf{A} . Thus, Eq. (4) has an exact solution:

$$\mathbf{x} = \mathbf{A}^{-1} \mathbf{b} = \mathbf{V} \text{diag}\{s_i^{-1}\} \mathbf{U}^T \mathbf{b}. \tag{6}$$

However, this solution may be incorrect when the data of \mathbf{b} are noisy. The effect of regularization is to modify s_i^{-1} for those singular values which are very small, by

$$\omega(s_i^2) s_i^{-1},$$

where $\omega(s)$ is called a filter function. So, instead of Eq. (6) we can obtain a regularized solution:

$$\mathbf{x} = \mathbf{V} \text{diag}\{\omega(s_i^2) s_i^{-1}\} \mathbf{U}^T \mathbf{b}, \tag{7}$$

where $\omega(s_i^2) s_i^{-1} \rightarrow 0$ when $s_i \rightarrow 0$.

The Tikhonov regularization can be employed to solve Eq. (4), when \mathbf{A} is highly ill-conditioned. Hansen (1992) and Hansen and O’Leary (1993) have given an illuminating explanation that the Tikhonov regularization of ill-posed linear problems is a trade-off between the size of the regularized solution, and the quality to fit the given data, namely, seeking the following minimum:

$$\min_{\mathbf{x} \in \mathbb{R}^n} \varphi(\mathbf{x}) = \min_{\mathbf{x} \in \mathbb{R}^n} [\|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2 + \alpha \|\mathbf{x}\|^2]. \tag{8}$$

The necessary condition for the optimality of $\varphi(\mathbf{x})$ in Eq. (8) is

$$[\mathbf{A}^T \mathbf{A} + \alpha \mathbf{I}_n] \mathbf{x} - \mathbf{A}^T \mathbf{b} = \mathbf{0}. \tag{9}$$

Obviously, from the Tikhonov regularization, we can derive a filter function such that

$$\omega(s) = \frac{s}{s + \alpha}, \tag{10}$$

which is named the Tikhonov filter function, and α is a regularization parameter.

On the other hand, we can also view Eq. (9) as the steady-state counterpart of a time-dependent differential equation:

$$\dot{\mathbf{x}} = - [\mathbf{A}^T \mathbf{A} + \alpha \mathbf{I}_n] \mathbf{x} + \mathbf{A}^T \mathbf{b}, \tag{11}$$

where $\dot{\mathbf{x}}$ denotes the differential of \mathbf{x} with respect to t . With $\mathbf{x}(0) = \mathbf{0}$, the above equation has a solution:

$$\mathbf{x}(t) = \mathbf{V} \text{diag}\{\omega(s_i^2) s_i^{-1}\} \mathbf{U}^T \mathbf{b}, \tag{12}$$

where

$$\omega(s) = \frac{1 - e^{-t(s+\alpha)}}{s + \alpha} s. \tag{13}$$

The wrinkle here is that typically one no longer intends to carry out the integration to a steady-state. Indeed, the present regularization is achieved by integrating the ODE in Eq. (11) to a finite time t , and t plays a regularization parameter. For the case $t \rightarrow \infty$, the filter function in Eq. (13) reduces to the Tikhonov regularization in Eq. (10). More interestingly, when $\alpha = 0$, Eq. (13) reduces to an exponential filter function:

$$\omega(s) = 1 - e^{-ts}. \tag{14}$$

Calvetti and Reichel (2002) gave further reasons for preferring Eq. (14) in the linear case. In fact, they never refer to an ODE giving rise to this filter function; however, Ascher et al. (2007) pointed this direction by deriving the above filter function from the ODE.

3 The relation of FTIM to a filter function

Liu and Atluri (2008a) have introduced a novel method to solve a system of non-linear algebraic equations $\mathbf{F}(\mathbf{x}) = \mathbf{0}$, by embedding them in a system of nonautonomous first order ODEs:

$$\mathbf{x}' = -\frac{v}{1 + \tau} \mathbf{F}(\mathbf{x}), \tag{15}$$

or in general,

$$\mathbf{x}' = -\frac{v}{q(\tau)} \mathbf{F}(\mathbf{x}), \tag{16}$$

where \mathbf{x}' denotes the differential of \mathbf{x} with respect to a fictitious time-like variable τ , and v is a damping constant.

This numerical technique has been labelled the Fictitious Time Integration Method (FTIM).

The above idea of introducing a fictitious time τ was first proposed by Liu (2008a) to treat an inverse Sturm-Liouville problem by transforming an ODE into a PDE.

Then, Liu and his coworkers [Liu (2008b, 2008c); Liu et al. (2008)] extended this idea to develop new methods for estimating the parameters in the inverse vibration problems. More recently, Liu (2008d) has used the FTIM technique to solve the nonlinear complementarity problems, whose numerical results are very accurate. Liu (2008e) used the FTIM to solve the boundary value problems of elliptic type partial differential equations. Then, Liu (2008f) used the FTIM to solve the m -point boundary value problems of differential equations. Liu and Atluri (2008b) also employed this technique of FTIM to solve mixed-complementarity problems and optimization problems. Then, Liu and Atluri (2008c) employed the FTIM to solve the inverse Sturm-Liouville problems under specified eigenvalues.

Here, we give a new interpretation of the FTIM, for an ill-posed linear system of equations, from the Tikhonov filter theory point of view. In order to first apply our presently proposed method to solve the system of algebraic equations, we demonstrate it by considering a single equation:

$$F(x) = 0, \quad (17)$$

where we only have one independent variable x . We transform it into a first-order ODE by introducing a fictitious time variable τ , using the following transformation of variables from x to y :

$$y(\tau) = q(\tau)x, \quad (18)$$

where $q(\tau)$ is a differentiable function, $q(0) = 1$ and $q(\infty) = \infty$ and the time-like function $q(\tau)$ is chosen by the user. Here, τ is a variable which is independent of x ; hence, $y' = dy/d\tau = q'(\tau)x$. Adding the equation $y' = q'(\tau)x$ to Eq. (17) we obtain:

$$y' = q'(\tau)x - F(x). \quad (19)$$

By using Eq. (18) we can derive

$$y' = \frac{q'(\tau)y}{q(\tau)} - F\left(\frac{y}{q(\tau)}\right). \quad (20)$$

This is a first-order ODE for $y(\tau)$. The initial condition for the above equation is $y(0) = x$, which is however an unknown and requires a guess.

Multiplying Eq. (20) by an integrating factor of $1/q(\tau)$, we can obtain

$$\frac{d}{d\tau} \left(\frac{y}{q(\tau)} \right) = -\frac{1}{q(\tau)} F\left(\frac{y}{q(\tau)}\right). \quad (21)$$

Further, using $y/q(\tau) = x$, Eq. (21) leads to

$$x' = -\frac{1}{q(\tau)} F(x). \quad (22)$$

Therefore, we have transformed the algebraic Eq. (17) into a first-order nonautonomous ODE. Under certain condition we expect that the solution of Eq. (22) starting from an initial guess of $x(0)$ can approximate the true solution x of Eq. (17).

Applying the concept in Eq. (22) to the following system of equations:

$$\mathbf{A}^T \mathbf{A} \mathbf{x} - \mathbf{A}^T \mathbf{b} = \mathbf{0}, \tag{23}$$

we can derive a FTIM for solving Eq. (23) through the system of ODEs:

$$\mathbf{x}' = -\frac{1}{q(\tau)} \mathbf{A}^T \mathbf{A} \mathbf{x} + \frac{1}{q(\tau)} \mathbf{A}^T \mathbf{b}. \tag{24}$$

By introducing the transformation:

$$t = \int_0^\tau \frac{1}{q(\xi)} d\xi, \tag{25}$$

Eq. (24) can be further reduced to

$$\dot{\mathbf{x}} = -\mathbf{A}^T \mathbf{A} \mathbf{x} + \mathbf{A}^T \mathbf{b}, \tag{26}$$

where $\dot{\mathbf{x}} = d\mathbf{x}/dt$. Thus, the time-varying linear system (24) is transformed into a linear system (26) with constant coefficients. Thus, with $\mathbf{x}(0) = \mathbf{0}$, the above equation has a solution:

$$\mathbf{x}(t) = \mathbf{V} \text{diag}\{\omega(s_i^2) s_i^{-1}\} \mathbf{U}^T \mathbf{b}, \tag{27}$$

where

$$\omega(s) = 1 - e^{-ts}. \tag{28}$$

Comparing Eq. (27), which results from the application of the FTIM [Liu and Atluri (2008a)] to Eq. (7), which results from the Tikhonov regularization, the interpretation of FTIM as a regularization method to solve ill-posed system of linear equations is apparent.

There are many choices of the time-like function $q(\tau)$; however, we only consider

$$q(\tau) = (1 + \tau)^\gamma, \quad 0 \leq \gamma \leq 1. \tag{29}$$

Thus, by Eq. (25) we have

$$t = \begin{cases} \ln(1 + \tau) & \gamma = 1, \\ \frac{1}{1-\gamma} [(1 + \tau)^{1-\gamma} - 1] & 0 \leq \gamma < 1. \end{cases} \tag{30}$$

Inserting them into Eq. (28) we can obtain the following equivalent filter functions:

$$\omega(s) = \begin{cases} 1 - \frac{1}{(1+\tau)^s} & \gamma = 1, \\ 1 - \exp\left(\frac{-s}{1-\gamma}[(1+\tau)^{1-\gamma} - 1]\right) & 0 \leq \gamma < 1. \end{cases} \quad (31)$$

Substituting Eq. (23) for \mathbf{F} into Eq. (15) we can obtain

$$\mathbf{x}' = -\frac{\nu}{1+\tau}[\mathbf{A}^T \mathbf{A} \mathbf{x} - \mathbf{A}^T \mathbf{b}]. \quad (32)$$

A similar derivation leads to another filter function:

$$\omega(s) = 1 - \frac{1}{(1+\tau)^{\nu s}}. \quad (33)$$

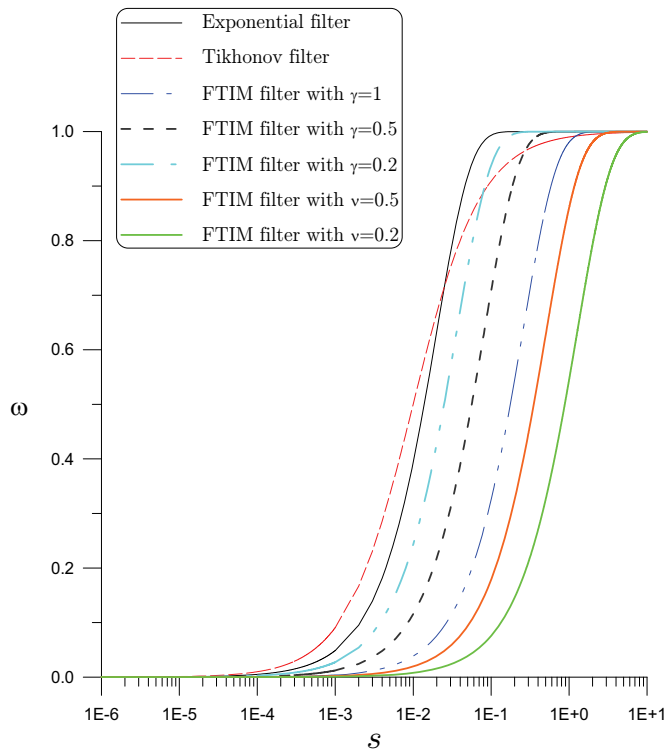


Figure 1: Comparison of different filters for $t = \tau = 50$ and $\alpha = 0.01$ arising from the FTIM, with the familiar Tikhonov filter.

Fig. 1 compares these filter functions appearing in Eqs. (10), (14), (31) and (33), where we let $t = \tau = 50$ and $\alpha = 0.01$. Of course they both tend to 0 as $s \rightarrow 0$

and to 1 as $s \rightarrow 1$. But even for mid-range values of s there is a similarity in the general shape. It appears that the choice of $t \approx 1/(2\alpha)$ makes the Tikhonov filter and the exponential filter particularly close, and that the exponential filter switches from 0 to 1 more sharply. The new filters have a more significant effect on the filtering of low singular values. When $\gamma \rightarrow 0$, the new filter is close to the exponential filter, and indeed, the exponential filter is a special case of the new filter with $\gamma = 0$. The damping constant ν also influences the filtering effect profoundly. Thus, we conclude that the general FTIM method of solving an ill-posed system of linear equations may be interpreted as leading, as a special case, to the Tikhonov regularization method.

4 Applications

4.1 The Fredholm integral equation

To demonstrate the applications of the FTIM, we first consider the first-kind Fredholm integral equation, because it is known to be severely ill-posed:

$$\int_a^b K(s,t)x(t)dt = h(s), \quad s \in [c, d]. \tag{34}$$

Let us discretize the intervals of $[a, b]$ and $[c, d]$ into m_1 and m_2 subintervals by noting $\Delta t = (b - a)/m_1$ and $\Delta s = (d - c)/m_2$. Let $x_j := x(t_j)$ be a numerical value of x at a grid point t_j , and let $K_{i,j} = K(s_i, t_j)$ and $h_i = h(s_i)$, where $t_j = a + (j - 1)\Delta t$ and $s_i = c + (i - 1)\Delta s$. Through a trapezoidal rule, Eq. (34) can be discretized into

$$\frac{\Delta t}{2}K_{i,1}x_1 + \Delta t \sum_{j=2}^{m_1} K_{i,j}x_j + \frac{\Delta t}{2}K_{i,m_1+1}x_{m_1+1} = h_i, \quad i = 1, \dots, m_2 + 1, \tag{35}$$

which are algebraic equations denoted by:

$$\mathbf{R}\mathbf{x} = \mathbf{h}, \tag{36}$$

where \mathbf{R} is a rectangular matrix with dimensions $(m_2 + 1) \times (m_1 + 1)$. Here, $\mathbf{h} = (h_1, \dots, h_{m_2+1})^T$, and

$$\mathbf{x} = (x_1, \dots, x_{m_1+1})^T \tag{37}$$

is the vector of unknowns. The data h_j may be corrupted by noise, such that:

$$\hat{h}_j = h_j + \sigma R(j), \tag{38}$$

where $R(j)$ are random numbers in $[-1, 1]$.

Then the FTIM can be used to solve the following least-squares error algebraic equation:

$$\mathbf{Ax} = \mathbf{b}, \tag{39}$$

where

$$\mathbf{A} := \mathbf{R}^T \mathbf{R}, \quad \mathbf{b} := \mathbf{R}^T \mathbf{h}. \tag{40}$$

We test our approach above, by considering the numerical solution of the following first-kind Fredholm integral equation:

$$\int_0^\pi e^{s \cos t} x(t) dt = \frac{2}{s} \sinh s, \quad s \in [0, \pi/2], \tag{41}$$

which has an exact solution $x(t) = \sin t$. By fixing $m_1 = m_2 = 50$ and under a noise with $\sigma = 0.1$ the numerical results are shown in Fig. 2. For the FTIM the ODEs are integrated by the group-preserving scheme (GPS) [Liu (2001)] with a time stepsize 0.001, while for the exponential filter the time stepsize must be decreased to 0.0003. When $m_1 = 100$ is increased we found that the exponential filter cannot be applied with a time stepsize 0.0001, but the FTIM is still applicable under $\nu = 0.1$, of which the numerical solution has a maximum error 0.051.

As a second example, we consider the problem of finding $x(t)$ in the following equation:

$$\int_0^1 [\sin(s+t) + e^t \cos(s-t)] x(t) dt = 1.4944 \cos s + 1.4007 \sin s, \quad s \in [0, 1], \tag{42}$$

where $x(t) = \cos t$ is the exact solution. We use the following parameters $m_1 = m_2 = 50$, $\nu = 0.1$, $\gamma = 0.1$ and $\varepsilon = 10^{-3}$ in the FTIM to calculate the numerical solution under a noise with $\sigma = 0.01$. Through 25 steps it is convergent to the true solution with a maximum error 0.029 as shown in Fig. 3. Even under a large noise our calculated result is better than that calculated by Maleknejad et al. (2006).

4.2 Finding the derivative of noisy data

In many applications it may be necessary to calculate the derivative of a function measured experimentally, i.e., to differentiate noisy data, such as determining da/dN from the measured crack-growth a , versus the number of cycles, N . The problem of numerical differentiation of noisy data is ill-posed: small changes of the data may result in large changes of the derivative.

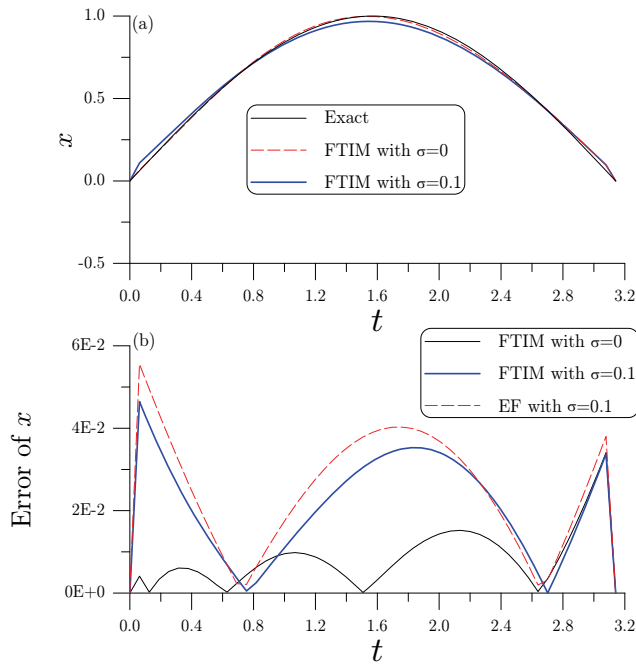


Figure 2: The numerical results obtained by applying the FTIM to a Fredholm integral equation.

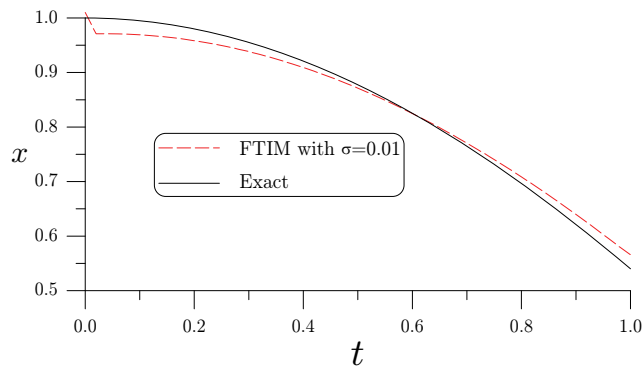


Figure 3: Comparison of the numerical and exact solutions of a Fredholm integral equation under noise.

We begin with the following identity:

$$\int_0^x y(s)ds = f(x) - f(0), \quad 0 \leq x \leq a, \tag{43}$$

where $y(x)$ is the differential of $f(x)$. To regularize the above first-kind Volterra integral equation we add a regularization term $\alpha y(x)$, such that

$$\alpha y(x) + \int_0^x y(s)ds = f(x) - f(0), \tag{44}$$

which is a second-kind Volterra integral equation.

Let us discretize the interval of $[0, a]$ into n subintervals by noting $\Delta x = a/n$, and let $y_i := y(x_i)$ be a numerical value of y at a grid point x_i . Then, through a trapezoidal rule, Eq. (44) can be discretized to

$$\alpha y_i + \frac{\Delta x}{2} y_1 + \Delta x \sum_{j=2}^{i-1} y_j + \frac{\Delta x}{2} y_i = f_i - f(0), \quad i = 2, \dots, n + 1. \tag{45}$$

Under a noise of $\hat{f}_i = f_i + \sigma R(i)$ with $\sigma = 0.1$, we apply the FTIM to solve the above system of linear equations by using $n = 50$, $\nu = 200$ and $\alpha = 0.005$, which, as shown by the solid line fitted with symbols \times is compared with the exact solution $f'(x) = 3x^2$ as shown in Fig. 4 with a dashed line. Also we calculate the derivative by a backward finite difference scheme, whose results are shown in Fig. 4 by the solid line fitted with solid circles. It is obvious that the FTIM is much better than that calculated by the finite difference scheme.

From the numerical results shown in Fig. 4 it can be seen that the above two methods are not accurate enough. We propose a third approach by considering the Laplace transform of $f'(x)$:

$$\int_0^\infty e^{-sx} f'(x) dx = s \int_0^\infty e^{-sx} f(x) dx - f(0), \tag{46}$$

where both $f(x)$ and $f'(x)$ are assumed to be defined in a finite interval of $[0, a]$. Outside this interval we let $f = f' = 0$. Let $y(x) = f'(x)$, then we have

$$\int_0^a e^{-sx} y(x) dx = s \int_0^a e^{-sx} f(x) dx - f(0). \tag{47}$$

Inserting the discretized noised data on the right-hand side, as that done in Section 4.1, we can derive the following algebraic equations:

$$\frac{\Delta x}{2} K_{i,1} y_1 + \Delta x \sum_{j=2}^m K_{i,j} y_j + \frac{\Delta x}{2} K_{i,m+1} y_{m+1} = h_i, \quad i = 1, \dots, m + 1, \tag{48}$$

where $y_i = y(x_i)$, $x_i = (i - 1)\Delta x = (i - 1)a/m$, $s_i = (i - 1)\Delta s = (i - 1)a/m$, $K_{ij} = \exp(-s_i x_j)$, and

$$h_i = s_i \left[\frac{\Delta x}{2} K_{i,1} \hat{f}_1 + \Delta x \sum_{j=2}^m K_{i,j} \hat{f}_j + \frac{\Delta x}{2} K_{i,m+1} \hat{f}_{m+1} \right] - \hat{f}_1. \tag{49}$$

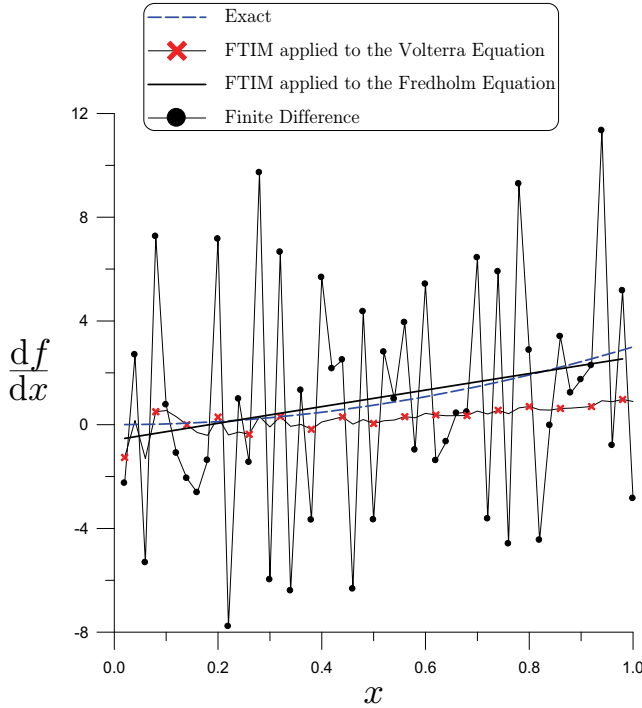


Figure 4: Comparison of different numerical methods for the numerical derivatives of noisy data, with the exact result.

The data are given by $\hat{f}_i = f(x_i) + \sigma R(i)$. Then, similarly we apply the FTIM to solve Eq. (48). Under the following parameters $m = 50$, $\Delta\tau = 10^{-4}$, $\nu = 0.01$, $\gamma = 1$, and $\varepsilon = 0.01$ we calculate y_i , whose results are plotted in Fig. 4 by the thick solid line. It is much better than the results obtained from the above two approaches under the same noise level of $\sigma = 0.1$.

Next, we consider a crack propagation problem, to estimate the crack propagating rate da/dN , where a is the measured crack-length and N is the number of load cycles [Broek (1982); Sih (1991)]. Theoretically, the crack propagating rate da/dN versus a has a power law relation: $da/dN = ca^\beta$. We suppose that the measured data of a are scattered along an unknown curve as shown in Fig. 5(a) with

$$\hat{a}_i = a(N_i) + \sigma R(i), \tag{50}$$

where for definiteness and for the purpose of comparison we take $a(N) = 0.02N^{1.2}$ and $\sigma = 10$. Usually, it is very difficult to estimate the rate da/dN versus a by using the scattered noisy data. However, when we apply the FTIM to solve Eq. (48) under

the following parameters $m = 20000$, $\Delta\tau = 10^{-4}$, $\nu = 0.05$, $\gamma = 1$, and $\varepsilon = 0.01$, we can calculate da/dN versus a at the measured points, whose results are plotted in Fig. 5(b) by the dashed line. It can be seen that the estimated rate is close to the exact one of $da/dN = ca^\beta$, where $\beta = 1/6$ and $c = 0.024/0.02^\beta$. The estimated curve is rather smooth, because the Laplace transform used in Eq. (47) and the FTIM used in the integration of Eq. (48) can filter the random noise.

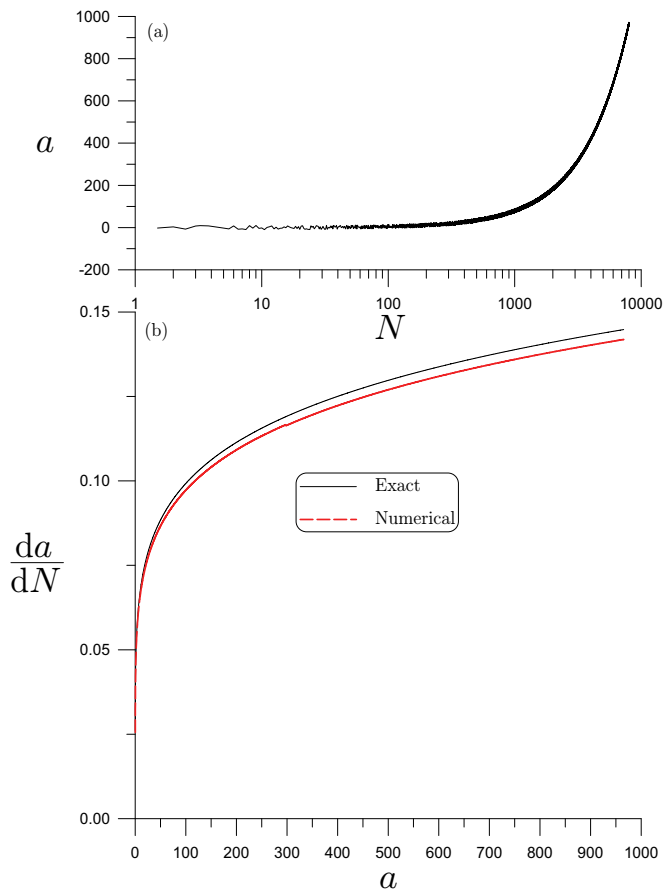


Figure 5: Comparison of the numerical and exact solutions for obtaining da/dN versus a from the measured noisy data for crack-length a versus the number of cycles, N .

Finally, we consider the Abel integral equation (2) of the following case: $\eta = 1/3$,

$\phi(s) = 10s/9$, $h(s) = s^{5/3}$, $0 < s \leq 1$. Let

$$f(s_i) = \int_0^{s_i} \frac{\hat{h}(t)}{(s_i - t)^{1-\eta}} dt \tag{51}$$

be the discretized data of $f(s)$, and $\hat{h}(t_i) = h(t_i) + \sigma R(i)$ be the discretized data of $\hat{h}(t)$. Such that through some calculations we have $f(s_i) = 5\pi s_i^2/[9 \sin(\eta\pi)] + s_i^\eta \sigma R(i)/\eta$.

The numerical solution of the Abel equation is obtained by

$$\phi(s_i) = \frac{\sin(\eta\pi)}{\pi} f'(s_i), \tag{52}$$

where we apply the FTIM method in Eq. (48) to calculate $f'(s_i)$. We use $m = 100$, $\Delta\tau = 10^{-4}$, $\nu = 0.001$, $\gamma = 1$, and $\varepsilon = 10^{-4}$ in this calculation. Even under a large noise with $\sigma = 0.1$, the numerical result as shown in Fig. 6 by the dashed line is close to the exact solution.

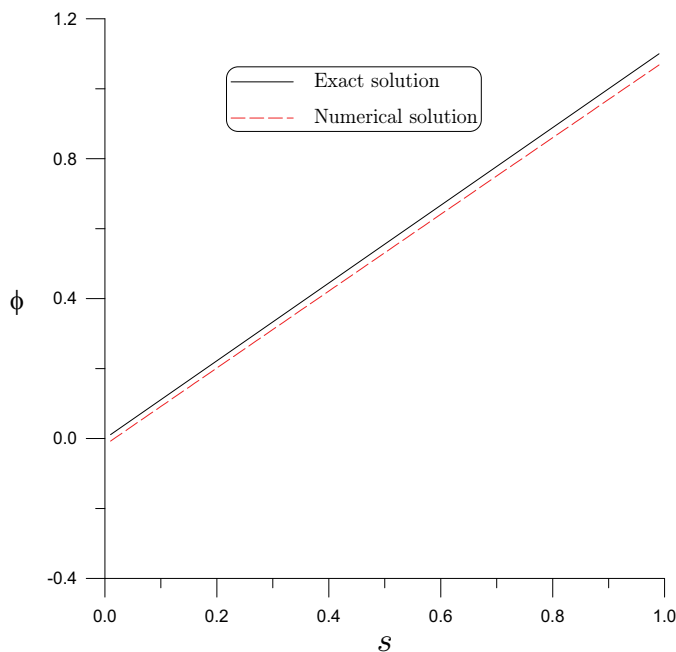


Figure 6: Comparison of the numerical and exact solutions of an Abel integral equation under noise.

5 Conclusions

In this paper we have applied the Fictitious Time Integration Method (FTIM) to solve a system of ill-posed linear equations. The mathematical foundation of the FTIM was explored by comparing it to the filter theory. We found that the fictitious time plays the role of a regularization parameter, and its filtering effect is better than that of the Tikhonov and the exponential filters. The influence of other two parameters in the time-like function q was also discussed. We applied this new filter to solve the first-kind Fredholm integral equation under noise, to solve the problem of numerical differentiation of noisy data, and to solve the Abel integral equation under noise. Numerical examples showed that the present theory is robust against the noise, when solving ill-posed linear problems.

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