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An investigation on the regularized meshless method for irregular domain problems

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Abstract: The regularized meshless method (RMM) is a novel boundary-type meshless method but by now has mainly been tested successfully to the regular domain problems in reports. This note makes a further investigation on its solution of irregular domain problems. We find that the method fails to produce satisfactory results for some benchmark problems. The reason is due to the inaccurate calculation of the diagonal elements of the numerical discretization matrix in the original RMM, which have strong effect on the resulting solution accuracy. To overcome this severe drawback, this study introduces the weighted diagonal element approach. Our numerical experiments demonstrate the effectiveness and accuracy of the present RMM technique.

Keywords: Regularized meshless method, irregular domain, desingularization technique, weighted diagonal element approach, numerical integration.

1 Introduction

The partial differential equations (PDE) play an important role in science research and engineering. As known, most PDE solutions must be obtained approximately via numerical methods, for example, the finite difference method, the finite element method, the boundary element method, and the meshless method, etc. Due to the direct use of the geometry of the simulated object without replying on grid, the meshless methods now attract more and more attentions from the mathematicians and engineers, especially when studying large deformations, complex geometry, nonlinear material behavior, discontinuities and singularities. Generally, these meshless methods can be divided into the domain-type or boundary-type techniques, depending on if their basis functions satisfy the governing equation

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of interest. The method of fundamental solutions (MFS) is one of the collocation based boundary type meshless methods with the merit of easy programming, high accuracy, and fast convergence.

Since proposed by Kupradze and Aleksidze (1964), the MFS has been used to solve various problems. Two excellent review papers of MFS are [Golberg and Chen (1998); Fairweather and Karageorghis (1998)]. To list a few recent progresses, see [Tsai, Lin, Young, and Aturi (2006); Young, Chen, Chen, and Kao (2007); Ting, Hon, and Ling (2007); Liu (2008c); Marin (2008a,b)]. However, there are two main drawbacks delay the spread of MFS, namely, factitious boundary and ill-conditioned interpolation matrix. Many efforts have been done to overcome the shortcomings, for example, [Kang, Lee, and Kang (1999); Chen, Chang, Chen, and Lin (2002); Chen and Tanaka (2002); Chen and Hon (2003); Young, Chen, Chen, and Kao (2007); Liu (2008c)].

As seen, most problems studied by the MFS and its improved methods are the boundary value problem (BVP) of elliptic equations. This kind of problems are also solved by other methods, such as [Liu (2007a,b,c); Kim, Lee, and Shin (2003); Kim and Shin (2009)]. An interesting method proposed recently is the fictitious time integration method (FTIM) [Liu (2008a,b, 2009); Liu and Atluri (2008a,b)], which is verified to be efficient for the quasi-linear BVP, and especially the nonlinear boundary value problems. If only linear BVP of elliptic equations are considered, a troublesome difficulty is the irregular domain of problem, since the solution accuracy may be sensitive to the nodes distribution. In this paper, this aspect will be mainly considered together with the overcoming of the drawbacks of MFS.

Recently, Young et al [Young, Chen, and Lee (2005)] developed a regularized meshless method (RMM) based on the MFS, which removes the two major drawbacks of the latter as mentioned above, by the desingularization technique of subtracting and adding-back. Their numerical studies show that the RMM keeps all merits of the MFS and is very efficient in the solution of Laplace problems [Young, Chen, and Lee (2005); Chen, Kao, Chen, Young, and Lu (2006)], exterior acoustics problem [Young, Chen, and Lee (2006)], acoustic eigenproblem [Chen, Chen, and Kao (2006)], and anti-plane shear problem [Chen, Chen, and Kao (2008)]. But most of the RMM numerical solutions available in reports are concerned with regular domain problems, for which the RMM seems to outperform the other meshless boundary-type methods, e.g., the MFS. To our knowledge, very few irregular domain problems have been reported in RMM literatures [Young, Chen, and Lee (2005); Chen, Kao, Chen, Young, and Lu (2006)], and the verification of the method is inconclusive.

In this study, we investigate the RMM solution of Laplace equation in irregular domains, where the boundary nodes can not be uniformly placed as in regular domain cases. We find that the direct use of the diagonal elements calculation formulas for the RMM, given in [Young, Chen, and Lee (2005)], fails to produce satisfactory results in some arbitrary domain cases. And the diagonal elements are critical in determining the accuracy and convergence of the RMM solutions.

To remedy this drawback in the original RMM, this paper introduces the weighted diagonal elements of the RMM to guarantee its solution accuracy and convergence, in which the weights highly depend on the boundary curve length and are obtained via numerical integration. Our numerical experiments, reported in this paper, demonstrate the effectiveness of the present RMM technique for irregular domain problems.

2 Diagonal elements of RMM

Without loss of generality, we consider the following Laplace problem

$$\nabla^2 u(x,y) = 0, \quad \text{in } D, \tag{1}$$

$$u(x,y) = f, \quad \text{on } \Gamma_1, \tag{2}$$

$$\frac{\partial u}{\partial n}(x,y) = g, \quad \text{on } \Gamma_2, \tag{3}$$

where f and g are known functions, $\Gamma = \Gamma_1 \cup \Gamma_2 = \partial D$ denotes the boundary and D represents the computational domain.

By using the radial basis function (RBF) method [Wang and Liu (2002); Hu, Li, and Cheng (2005)], the solution of Eqs. 1–3 at $t = (x,y)$ can be approximated by

$$u(t) = \sum_{j=1}^{N} \alpha_j A(t,s_j), \tag{4}$$

$$\frac{\partial u}{\partial n}(t) = \sum_{j=1}^{N} \alpha_j B(t,s_j), \tag{5}$$

where $A(t,s_j)$ and $B(t,s_j)$ are chosen radial basis functions, $\{s_j\}_{j=1}^{N}$ are the source points, and $\{\alpha_j\}_{j=1}^{N}$ are the unknown coefficients. In the RMM, the RBF is chosen to satisfy the governing equation of interest, namely, the so-called double layer potentials. Thus, no inner nodes are required to discretize the governing equation. The RMM is thus of the boundary type method.

By the collocation technique, the Eqs. 4 and 5 are forced to satisfy Eqs. 2 and 3 on N boundary nodes $\{t_i\}_{i=1}^{N}$. Then the coefficients $\{\alpha_j\}_{j=1}^{N}$ can be solved from the resulting linear system.

In the RMM, the RBFs are the following double layer potentials,

$$A(t_i,s_j) = -\frac{\langle (t_i - s_j), n_j \rangle}{r_{ij}^2},$$
$$B(t_i,s_j) = \frac{\partial A(t_i,s_j)}{\partial \bar{n}_i} = 2\frac{\langle (t_i - s_j), n_j \rangle \langle (t_i - s_j), \bar{n}_i \rangle}{r_{ij}^4} - \frac{\langle n_j, \bar{n}_i \rangle}{r_{ij}^2},$$

where $r_{ij} = |s_j - t_i|$, the symbol $\langle , \rangle$ denotes the inner product of vectors, n_j is the outward normal vector at s_j, and $\bar{n}_i$ is the outward normal vector at t_i.

Since $A(t_i,s_j)$ and $B(t_i,s_j)$ are either singular or hypersingular when t_i approaches to s_j, the desingularization technique of subtracting and adding-back is used in the RMM to derive the diagonal elements. It is based on the discretization of the reduced null-fields equations [Young, Chen, and Lee (2005); Chen and Chen (2007)]

$$\int_\Gamma A^{(e)}(t_i,s)d\Gamma(s) = 0, \quad t_i \in D^e, \tag{6}$$
$$\int_\Gamma B^{(e)}(t_i,s)d\Gamma(s) = 0, \quad t_i \in D^e, \tag{7}$$

where $A^{(e)}$ and $B^{(e)}$ are related to A and B by the opposite normal direction, and D^e is the exterior domain of D. In the RMM, Eqs. 6 and 7 are discretized as

$$\sum_{j=1}^{N} A^{(e)}(t_i,s_j)|l_j| = 0, \quad t_i \in B, \tag{8}$$
$$\sum_{j=1}^{N} B^{(e)}(t_i,s_j)|l_j| = 0, \quad t_i \in B, \tag{9}$$

where $|l_j|$ is the half distance of the source nodes s_{j-1} and s_{j+1}.

Then the non-weighted diagonal elements of RMM are given by

$$A(t_i,s_i) = -A^{(e)}(t_i,s_i) = \sum_{j \neq i}^{N} A^{(e)}(t_i,s_j), \quad t_i \in \Gamma_1, \tag{10}$$
$$B(t_i,s_i) = B^{(e)}(t_i,s_i) = -\sum_{j \neq i}^{N} B^{(e)}(t_i,s_j), \quad t_i \in \Gamma_2, \tag{11}$$

based on the assumption that the source nodes are uniformly distributed. Namely, $|l_j|$ is constant for different j. The above formulas 10 and 11 were used in [Young, Chen, and Lee (2005); Chen, Kao, Chen, Young, and Lu (2006)] for various cases, particularly, regular domain problems.

In this study, we propose the weighted diagonal elements of the RMM, which are derived directly from Eqs. 8 and 9 as

$$A(t_i,s_i) = -A^{(e)}(t_i,s_i) = -\frac{1}{|l_i|}\sum_{j\neq i}^{N} A^{(e)}(t_i,s_j)|l_j|, \quad t_i \in \Gamma_1, \tag{12}$$

$$B(t_i,s_i) = B^{(e)}(t_i,s_i) = -\frac{1}{|l_i|}\sum_{j\neq i}^{N} B^{(e)}(t_i,s_j)|l_j|, \quad t_i \in \Gamma_2. \tag{13}$$

In practical calculations, the curve lengths $\{|l_j|\}_{j=1}^{N}$ are obtained by numerical integration, for example, the adaptive Newton-Cotes 8-panel rule [Forsythe, Malcolm, and Moler (1977)]. To our knowledge, this method has not been reported in the literatures and is developed for handling the arbitrary placement of boundary nodes.

In this paper, the RMM with Eqs. 10 and 11 is called as the non-weighted RMM, while the RMM with Eqs. 12 and 13 is called as the weighted RMM. Numerical comparisons of these two methods for arbitrary domain problems will be presented in the following Section 3.

3 Numerical results and discussions

This section will give numerical comparisons of the non-weighted and weighted RMMs in the solution of the two benchmark problems. The first one is the Dirichlet problem in the gear-shape irregular domain. The second example has an elliptic domain with the mixed-type boundary conditions and is given to show that the traditional non-weighted RMM even fails to a simple regular domain problem if the uniform distribution of boundary nodes is not employed.

In this study, the error at point (x_i, y_j) is defined as

$$E_{ij} = |u(x_i,y_j) - \hat{u}(x_i,y_j)|^2, \tag{14}$$

where u and $\hat{u}$ are the analytical and numerical solutions, respectively. The total average error on the whole domain is defined as

$$TE = \frac{1}{PQ}\sum_{i=1}^{Q}\sum_{j=1}^{P} E_{ij}, \tag{15}$$

where P and Q are the numbers of y_j and x_i in the domain.

Case 1: Dirichlet problem in gear-shape domain

The boundary node (r,θ) on the gear-shape domain is generated by

$$r = \frac{1}{n^2}(n^2 + 2n + 2 - 2(n+1)\cos(n\theta)). \tag{16}$$

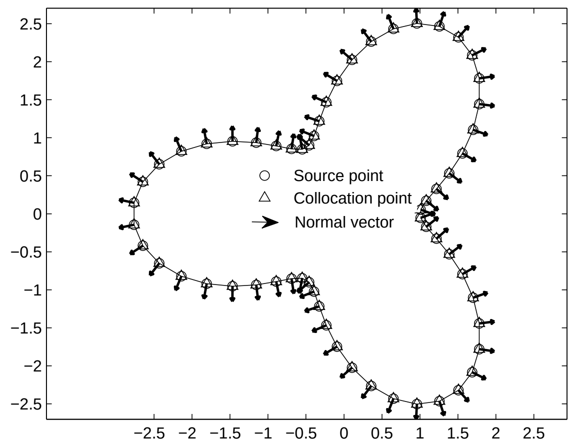


Figure 1: Nodes distribution of gear-shape domain ($N = 60$).

The case $n = 3$ is investigated. The exact solution of this problem is $u(x,y) = e^{0.5x}\sin(0.5y)$. The discrete nodes distribution is shown in Fig. 1.

The solution contours of the two methods are plotted in Fig. 2. Clearly, the non-weighted RMM solutions are distorted. In stark contrast, the weighted RMM solutions are very close to the exact ones. Fig. 3 shows the convergence curves of the two methods. The weighted RMM appears to converge quickly with the increasing node number N, while the non-weighted RMM solution errors oscillate at first and then converge slowly.

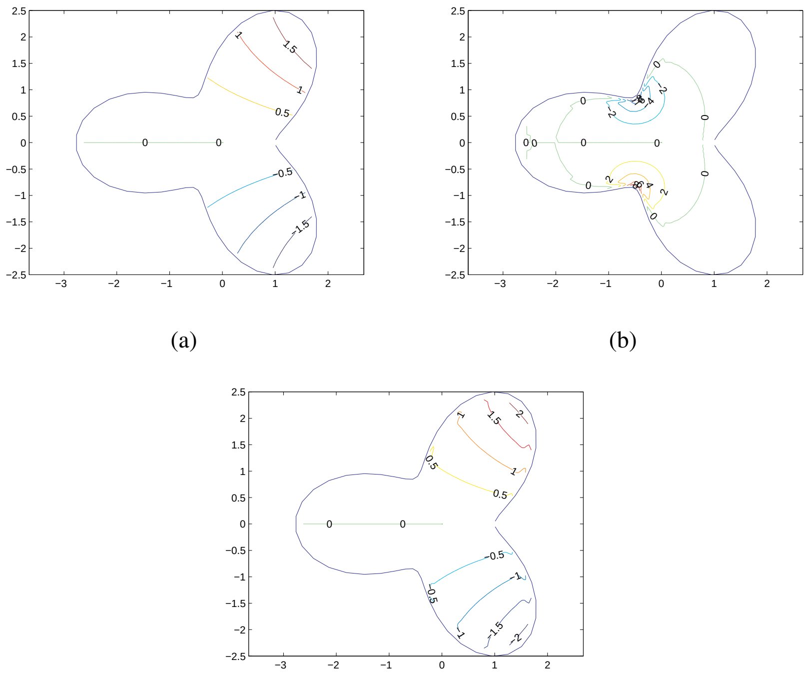
Case 2: Mixed-type boundary problem in elliptic domain

This case is a mixed-type boundary condition problem in the elliptic domain. The major and minor semi axes are $R_1 = 1.0$ and $R_2 = 0.5$, respectively. Its exact solution is $u(x,y) = x+y$ subject to the following boundary conditions:

$$f = R_1\cos(\theta)+R_2\sin(\theta), \quad \theta \in [0,\frac{3\pi}{2}),$$
$$g = \cos(\theta)+\sin(\theta), \quad \theta \in [\frac{3\pi}{2},2\pi],$$

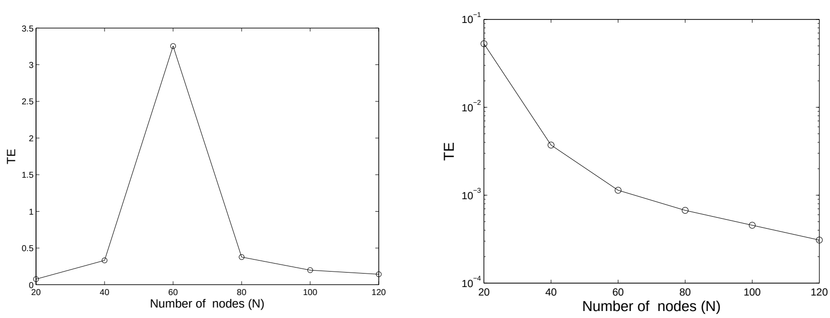
where f and g are the functions in Eqs. 2 and 3.

It is noted that the boundary nodes are of nonuniform distribution in this regular domain case. Fig. 4 shows the nodes configuration and Fig. 5 displays the



(a) (b) (c)

Figure 2: The solution contours of Case 1 ($N = 60$): (a) Exact solution, (b) Non-weighted RMM, (c) Weighted RMM.



(a) (b)

Figure 3: The convergence curves of Case 1: (a) Non-weighted RMM, (b) Weighted RMM.

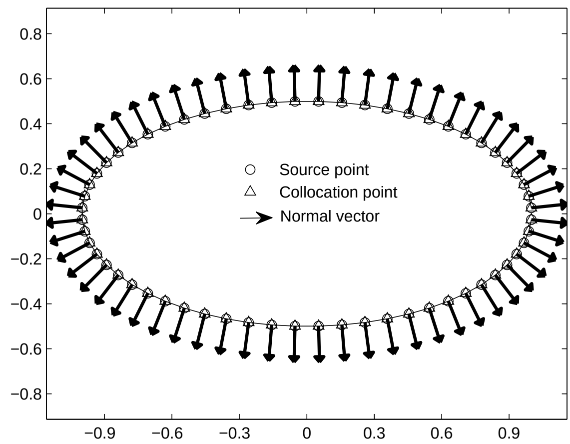


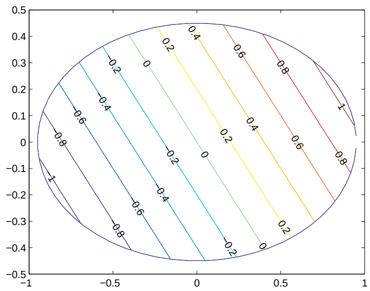
Figure 4: Nodes distribution of elliptic domain ($N = 60$).

solution contours. It is clearly observed from Fig. 5 that the weighted RMM dramatically outperforms the non-weighted RMM in comparison with the analytical solutions. Fig. 6 shows the convergence curves, from which we can see that the weighted RMM converges more quickly than the non-weighted RMM. In fact, the non-weighted RMM fails in this regular domain case, since the nonuniform boundary nodes are used.

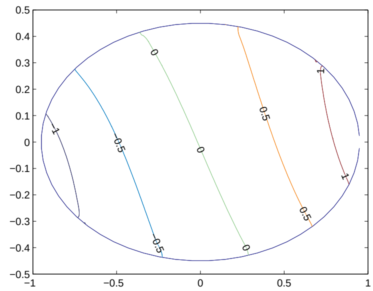
4 Concluding remarks

Our numerical experiments demonstrate the effectiveness of the weighted RMM, proposed by this paper, in the numerical solution of irregular domain problems. The present weighted diagonal elements conserve the discrete accuracy of the null-fields equation, which is lost in the non-weighted RMM for arbitrary domain problems. On the other hand, the non-weighted RMM dose not succeed even in regular domain problems with nonuniform distribution of boundary nodes, let alone in irregular domain problems. It is noted that, for regular domains with equally-spaced boundary nodes, the weighted RMM can be reduced to the non-weighted RMM.

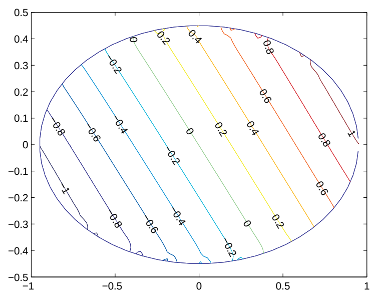
Acknowledgement: The work described in this paper was partially supported by Natural Science Foundation of China (Project No. 10672051). The first author was then a visiting research fellow to the Engineering Mechanics Department of Hohai University.



(a)



(b)



(c)

Figure 5: The solution contours of Case 2 ($N = 60$): (a) Exact, (b) Non-weighted RMM, (c)Weighted RMM.

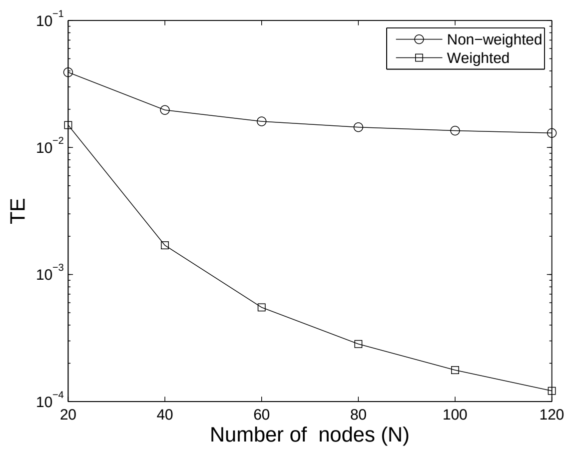


Figure 6: The convergence curves of Case 2.

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