 CMES, vol.42, no.3, pp.187-215, 2009

Variational formulation and Nonsmooth Optimization Algorithms in Elastostatic Contact Problems for Cracked Body

V.V. Zozulya[1]

Abstract: The mathematical statement for contact problem with unilateral restrictions and friction is done in classical and weak forms. Different variational formulation of unilateral contact problems with friction based on principles of virtual displacements and virtual stresses are considered. Especially boundary variational functionals that are used with boundary integral equations have been established. Nonsmooth optimization algorithms of Udzawa type for solution of unilateral contact problem with friction have been developed. Some theoretical results of existence and uniqueness in elastostatic unilateral contact problem with friction are outlined.

Keywords: Unilateral contact, friction, principles of virtual displacements, principles of virtual virtual stresses, boundary variational functional, nonsmooth optimization algorithm.

1 Introduction

It is well known that the classical approach to the crack problem is characterized by the equality type boundary conditions considered at the crack faces; in particular, the crack faces are assumed to be stress-free (Anderson 1995). This means that displacements found as solutions of these boundary value problems do not provide a nonpenetration condition between crack faces. There are many practical examples showing that interpenetration of crack faces may occur and affect fracture mechanics criterions. Many examples of the crack faces contact interaction and friction influence on the fracture mechanics criterions have been considered in the book Guz and Zozulya 1993 and review papers Guz and Zozulya 2001, 2002, Zozulya and Gonzalez-Chi 2000 and also in the papers Guz and Zozulya 2007; Guz, at all (2007). In these cases the contact area is "a priori" unknown and the unilateral

[1] Centro de Investigacion Cientifica de Yucatan A.C., Calle 43, No 130, Colonia: Chuburná de Hidalgo, C.P. 97200, Mérida, Yucatán, México. E-mail: zozulya@cicy.mx

conditions have to be imposed on the relative displacements and the mutual tractions. The unilateral contact restriction with friction can be written as an inequality for the displacement and traction vectors. As a result a complete set of boundary conditions at crack faces is written as a system of equations and inequalities. The presence of inequality type boundary conditions implies the boundary problems to be nonlinear, which requires the investigation of corresponding boundary value problems.

In recent years a lot of interest has been devoted to the mathematical formulation of elastostatic contact problems involving unilateral constraints and their numerical solutions (Antes and Panagiotopoulos1992; Eck, Jarusek and Krbec 2005; Kikuchi and Oden 1987; Kravchuk and Neittaanmaki 2007; Panagiotopoulos 1985; Shillor, Sofonea and Telega 2004; Wriggers 2006). Mathematical formulation of the problem of crack faces contact interaction in the static case has been considered in Khludnev and Kovtunenko1999, and in the dynamic case in Guz and Zozulya 1993, 2001, 2002. It is worth noting that all the problems with unilateral constrains and friction can be naturally expressed in terms of variational inequalities stating nothing but the principle of virtual or complementary virtual work in its inequality form. As it is well known, these statements are fully equivalent to minimizing on a convex set the potential or the complementary energy respectively. Such approach imply formulation of the problem in whole domain and for numerical solution finite elememt methods (FEM) is usually been used. For details of such approach see Kikuchi and Oden 1987; Wriggers 2006; etc.

Since the constraints concern boundary variables only, it is natural to look for a numerical solution by means of boundary element method (BEM). A lot of papers on this subject have recently appeared in the technical literature (see for instance Antes and Panagiotopoulos1992 and references there). The first variational formulations defined on the contact area only, rather than on the whole domain, is based on the use of Green's functions (Antes and Panagiotopoulos1992; Kikuchi and Oden 1987; Theocaris and Panagiotopoulos 1992; etc.). More specifically, Theocaris and Panagiotopoulos 1992 derived the two dual boundary variational statements from the classical minimum potential and complementary energy principles by means of saddle point formulations using appropriate Lagrangian functions. The extremum formulations of the discretized problem are derived from the corresponding continuum problem by approximating Green's function or its inverse by means of standard collocations BEM procedures. Another approach is based on use of fundamental solutions (Polizzotto1991, 1993). In Polizzotto1991 BEM formulations via energy methods is based on boundary min-max principle, i.e. a principle expressed in terms of the boundary unknowns. Then in Polizzotto1993 developed boundary min-max principle was applied to unilateral frictionless problem in elastostatics.

First time boundary variational formulation of elastodynamic contact a problem with frication was proposed in Zozulya 1992 and then was extended and applied to elastodynamic problems for bodies with cracks with considering unilateral frictional contact of the crack faces.

There are many algorithms for unilateral contact problems with friction. Because of nonlinearity of the problem most of them include discretization and iterative procedure to satisfy unilateral constrains. In the case if varational formulation is based on principles formulated for whole domain the FEM is usually used. Detailed description of such approach and algorithms based in FEM discretization can be find in Kikuchi and Oden 1987; Wriggers 2006. In the case if varational formulation is based on principles formulated only for boundary the BEM is usually used. In this case several BEM based discretization methods can be used. For example: standard (i.e. based on collocations) direct boundary element procedures (Balas; Sladek; Sladek 1989; Karlis, Tsinopoulos, Polyzos, Beskos 2008; Sanz, Solis, Dominguez 2007; Zhou, Li, Dehao 2008; Zozulya 1992; Yueting Zhou, Xing Li, Dehao Yu (2008)); symmetric based on Galerkin method procedure (Polizzotto1991, 1993; Han and Atluri 2007). Also recently developed meshless local Petrov-Galerkin methods can be used for domain and boundary discretization (see Atluri, Liu, Han 2006a, 2006b; Sladek, Sladek, Zhang 2007, Shu, Atluri (2008a); Shu, Atluri (2008b); Zhang, Chen (2008); Tan, Shiah, Lin, (2009)). When BEM is applied for boundary discretization divergent integrals of various type appear. Methods of such integrals calculation are developed in (Balas; Sladek; Sladek 1989; Han and Atluri 2007; Sanz, Solis, Dominguez 2007; Zozulya 2006a, 2006b; Young, Chen, Liu, Shen, and Wu (2009)).

There are many iterative procedures to satisfy unilateral constrains. In this paper will be used iterative procedures that are based on projection on the set of unilateral restrictions and friction. Iterative algorithms of such type are named in Cea 1971; Ekeland, Temam 1975 Uzawa's type algorithms. For the first time Uzawa's type algorithm was proposed in Zozulya 1990 for solution elastodynamic frictional contact problem for body with crack. Then the approach was developed in Zozulya 1992; Zozulya Menshykov 2003. In Zozulya 2001a, 2001d were proposed more algorithms based on saddle point finding and projection on the set of unilateral restrictions and friction respectively. In Zozulya and Menshykova 2002 it was shown that algorithms are convergent and studies peed of convergence. Some mathematical problems related existence and uniqueness of the problem of unilateral contact problem with friction were studied in Zozulya 2002, 2003; Zozulua and Rivera 2000.

The aim of this paper is to present various variational formulation of elastostatic problem for body with crack with considering possibility for unilateral crack faces contact interaction and friction in the for that permit easy extension to elastodynamic problems. Variational formutations based on principles of virtual displacements and virtual stresses and also boundary variational principle based on fundamental solutions are presented. Nonsmooth functionals that correspond to unilareral frictional contact conditions are constructed. The case of the crack in infinite elastic media is considered in more details.

2 Classical formulation of the problem

Let us consider a homogeneous, lineally elastic body, which contain arbitrarily oriented crack. The crack is described by a corresponding oriented middle surface Ω since we suppose that only small deformations occur. The body occupies an open bounded region $V$with $C^{0,1}$ Lipschitzian regular boundary ∂V. The boundary contain two parts ∂V_u and ∂V_p such that $\partial V_u \cap \partial V_p = \emptyset$ and $\partial V_u \cup \partial V_p = \partial V$. On the part ∂V_u displacements $\mathbf{u}(\mathbf{x}) = u_i(\mathbf{x})\mathbf{e}_i$ of the body points and on the part ∂V_p tractions $\mathbf{p}(\mathbf{x}) = p_i(\mathbf{x})\mathbf{e}_i$ are prescribed respectively. The body may by affected by volume forces$b_i(\mathbf{x})$. We assume that displacements of body points and their gradients are small.

In this case in $V\backslash\Omega$ the differential equations of equilibrium in displacement may be presented in the form

$$A_{ij}u_j + b_i = 0, \quad A_{ij} = \mu\delta_{ij}\partial_k\partial_k + (\lambda+\mu)\partial_i\partial_j. \quad \forall \mathbf{x} \in V\backslash\Omega \tag{2.1}$$

where λ and μ are Lame constants, $\mu > 0$ and $\lambda > -\mu$, δ_{ij} is a Kronecker's symbol, $\partial_i = \partial/\partial x_i$ denotes the partial derivatives with respect to space. Throughout this paper we use the Einstein summation convention.

If the body occupy a finite region V, it is necessary to establish boundary conditions. Mixed boundary conditions are

$$u_i(\mathbf{x}) = \varphi_i(\mathbf{x}), \quad \forall \mathbf{x} \in \partial V_u, \tag{2.2a}$$

$$p_i(\mathbf{x}) = \sigma_{ij}(\mathbf{x})n_j(\mathbf{x}) = P_{ij}[u_j(\mathbf{x})] = \psi_i(\mathbf{x}), \quad \forall \mathbf{x} \in \partial V_p \tag{2.2b}$$

The differential operator $P_{ij} : u_j \to p_i$ is called stress operator. It transforms the displacements into the tractions. For homogeneous isotropic elastic medium it has the forms

$$P_{ij} = \lambda n_i \partial_k + \mu\left(\delta_{ij}\partial_n + n_k\partial_i\right) \tag{2.3}$$

Here n_i are components of the outward unit normal vector, $\partial_n = n_i\partial_i$ is a derivative in direction of the vector $\mathbf{n}(\mathbf{x})$ normal to the surface ∂V.

If the equation (2.1) is defined in an infinite region, then its solution must satisfy additional conditions at the infinity

$$u_i(\mathbf{x}) = O(r^{-1}), \quad \sigma_{ij}(\mathbf{x}) = O(r^{-2}) \text{ for } r \to \infty \tag{2.4}$$

Here $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$is the distance in the 3-D Euclidian space.

The contact forces $\mathbf{q}(\mathbf{x}) = q_i(\mathbf{x})\mathbf{e}_i$ which arise on the cracks edges during the interaction are denoted by

$$\mathbf{q}(x) = -\boldsymbol{\sigma}(x) \cdot \mathbf{n}(x) \tag{2.5}$$

where $\boldsymbol{\sigma}(x) = \sigma_{ij}(x)\mathbf{e}_i \otimes \mathbf{e}_j$ is the strain tensor; $\mathbf{n}(\mathbf{x}) = n_i(\mathbf{x})\mathbf{e}_i$, $n_i(x) = n_i^+(x) = -n_i^-(x)$; $n_i^+(x)$ and $n_i^-(x)$ are the normal unit vectors directed to the positive side of the opposite cracks edges.

The displacement discontinuity vector characterizes mutual displacements of the cracks edges

$$\Delta u_i(x) = u_i^+(x) - u_i^-(x), \tag{2.6}$$

where $u_i^+(x)$ and $u_i^-(x)$ are displacements of opposite cracks edges.

Furthermore, we impose the following Signorini constraints

$$\Delta u_n \geq h_o,\ q_n \geq 0,\ (\Delta u_n - h_o)q_n = 0, \quad \forall \mathbf{x} \in \Omega \tag{2.7}$$

and Coulomb's friction law:

$$|\mathbf{q}_\tau| \leq k_\tau q_n \to \Delta\mathbf{u}_\tau = 0,\ |\mathbf{q}_\tau| = k_\tau q_n \to \Delta\mathbf{u}_\tau = -\lambda_t \mathbf{q},\ \forall \mathbf{x} \in \Omega \tag{2.8}$$

with $\lambda_\tau = |\Delta u_\tau(x)|/|q_\tau|$ for $x \in \Omega$; the Coulomb's friction coefficient $k_\tau > 0$ is here assumed to be constant.

Here the normal and tangential components of the displacement discontinuity on Ω are denoted by

$$u_n(x) = u_i(x)n_i(x),$$

$$\Delta\mathbf{u}_\tau(\mathbf{x}) = \Delta\mathbf{u}(\mathbf{x}) - \Delta u_n(\mathbf{x})\mathbf{n}(\mathbf{x}), \tag{2.9}$$

and the normal and tangential components of the contact forces are denoted by

$$q_n(x) = q_i(x)n_i(x),$$

$$\mathbf{q}_\tau(\mathbf{x}) = \mathbf{q}(\mathbf{x}) - q_n(\mathbf{x})\mathbf{n}(\mathbf{x}). \tag{2.10}$$

Classical formulation of the elastostatic problem for body with crack with considering opposite crack sides interaction consists in solution of the boundary value problem (2.1), (2.2) with considering Signorini contact conditions (2.7) with friction (2.8).

In classical elastostatics the equations of motion (2.1) must be satisfied exactly (see Antes and Panagiotopoulos 1992; Kikuchi and Oden 1987; Panagiotopoulos 1985). This means that the components of the displacement vector should be functions of the class $\mathbf{C}^2(V)$. Here $\mathbf{C}^k(V)$ is a functional space of functions, with k smooth derivatives with respect to the space coordinates. In order to satisfy all the equations of elastostatics in the classical sense, the components of the stress-strain state should belong to the following functional spaces

$$u_i \in \mathbf{C}^2(V) \cap \mathbf{C}^0(\partial V),\ \sigma_{ij}, \quad \varepsilon_{ij} \in \mathbf{C}^1(V), \quad p_i \in \mathbf{C}^0(\partial V). \tag{2.11}$$

These requirements of classical elastostatics are very stringent. Therefore many important physics and engineering problems, in particular problems with unilateral restrictions and friction, have no classical solution. For this reason it is necessary to consider "weakened" formulations to elastostatics problems. With such an approach it is not necessary to fulfill all the elastostatics equations in the classical sense.

3 Variational formulation of the problem without contact conditions

In order to formulate an elastostatic contact problem for body with crack in week form we will consider the principles of virtual displacements and of virtual stresses, and boundary variational principle.

3.1 Principles of Virtual Displacements

Consider the functional of Lagrange

$$\Phi_L(\mathbf{u}) = E(\mathbf{u}) + \langle u_i, b_i \rangle_V - \langle u_i, \psi_i \rangle_{\partial V_p}, \tag{3.1}$$

$$\langle u_i, b_i \rangle_V = \int_V b_i u_i dV, \quad \langle u_i, \psi_i \rangle_{\partial V} = \int_{\partial V} \psi_i u_i dS, \tag{3.2}$$

where $E(\mathbf{u})$ is the total potential energy of a deformed body,

$$E(\mathbf{u}) = \int_V 1/2 c_{ijkl} \varepsilon_{ij}(\mathbf{u}) \varepsilon_{kl}(\mathbf{u}) dV = \int_V [1/2 \lambda \varepsilon_{ii}^2(\mathbf{u}) + \mu \varepsilon_{ij}(\mathbf{u}) \varepsilon_{ij}(\mathbf{u})] dV, \tag{3.3}$$

and ε_{ij} and c_{ijkl} are the strain and elasticity modules tensors, respectively.

Functional (3.1) is considered on a set of admissible displacements:

$$\mathbf{K}_L(\mathbf{u}) = \{\mathbf{u} \in \mathbf{H}^1(V) \cap \mathbf{H}^{1/2}(\partial V);\ \varepsilon_{ij} = 1/2(\partial_i u_j + \partial_j u_i);\ \sigma_{ij} = c_{ijkl}\varepsilon_{kl};\ u_i(\mathbf{x}) = \varphi_i(\mathbf{x}),\, \forall \mathbf{x} \in \partial V_u\}. \quad (3.4)$$

Here $\mathbf{H}^l(V)$ is the Sobolev space with norm

$$\|f\|_{H^l} = \sqrt{\sum_{|\alpha| \le l} \int_V |\partial^\alpha f(\mathbf{x})|^2 d\mathbf{x} + \sum_{|\alpha| \le [l]} \int_V \int_V \frac{|\partial^\alpha f(\mathbf{x}) - \partial^\alpha f(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{n+2l-2[l]}} d\mathbf{x} d\mathbf{y}}, \quad (3.5)$$

where n is a dimension of the Euclidean space, $[l]$ - the integer part of l.

The variational formulation of an elastostatic problem for cracked body without unilateral constraints (2.7) and friction (2.8) is as follows:

$$\boxed{\begin{array}{l}\text{Find } \mathbf{u} \in \mathbf{K}_L(\mathbf{u}) \text{ such that} \\ \quad \Phi_L(\mathbf{u}) = \min\limits_{\mathbf{u}^* \in \mathbf{K}_L(\mathbf{u})} \{\Phi_L[\mathbf{u}^*] - \langle p_i, \Delta u_i \rangle_\Omega\}\end{array}} \quad (3.6)$$

3.2 Principles of Virtual Stresses

Consider the functional of Kastiliano

$$\Phi_K(\boldsymbol{\sigma}) = E^*(\sigma_{ij}) - \left\langle \sigma_{ij} n_j, \varphi_i \right\rangle_{\partial V_u}, \quad (3.7)$$

$$\left\langle \sigma_{ij} n_j, \varphi_i \right\rangle_{\partial V_u} = \int_{\partial V_u} \sigma_{ij} n_j \varphi_i dS \quad (3.8)$$

where $E^*(\boldsymbol{\sigma})$ is the total complementary potential energy of a deformed body,

$$E^*(\boldsymbol{\sigma}) = \int_V \frac{1}{2} C_{ijkl} \sigma_{ij} \sigma_{kl} dV = \int_V \frac{1}{\mu} [\sigma_{ij}\sigma_{ij} - \frac{\lambda}{3\mu + 2\lambda} \sigma_{ii}^2] dV, \quad (3.9)$$

On the set of admissible stresses

$$K_K(\boldsymbol{\sigma}) = \{\boldsymbol{\sigma} \in \mathbf{H}^0(V) \cap \mathbf{H}^{-1/2}(\partial V);\ \partial_j \sigma_{ij} = b_i;\ \varepsilon_{ij} = C_{ijkl}\sigma_{kl};\ \sigma_{ij}(\mathbf{x}) n_i(\mathbf{x}) = \psi_i(\mathbf{x}),\, \forall \mathbf{x} \in \partial V_p\}. \quad (3.10)$$

Here $\mathbf{H}^{-l}(V)$ is the functional space of linear functionals on $\mathbf{H}^l(V)$ with norm

$$\|v\|_{H^{-l}} = \sup_{f \in H_l} \frac{|\langle v, f \rangle|}{\|f\|_{H_l}}, \quad (3.11)$$

where $\langle v, f\rangle = \int\limits_V f(\mathbf{x})v(\mathbf{x})dV$ is a duality pairing between functional spaces $H^{-l}(V)$ and $H^l(V)$.

The variational formulation of an elastostatic problem without constraints is as follows:

$$\boxed{\begin{array}{l}\text{Find } \boldsymbol{\sigma} \in \mathbf{K}_K(\boldsymbol{\sigma}) \text{such that} \\ \quad \Phi_K(\boldsymbol{\sigma}) = \max_{\boldsymbol{\sigma}^* \in K_K(\boldsymbol{\sigma})}\{\Phi_K[\boldsymbol{\sigma}^*] - \langle p_i, \Delta u_i\rangle_\Omega\}\end{array}} \quad (3.12)$$

3.3 *Boundary Variational Principle*

Let us consider a variational formulation of the elastostatic contact problem in the form of boundary variational principles. Consider the functional of Lagrange in the form

$$\Phi_L(\mathbf{u}) = \int\limits_V 1/2\sigma_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{u})dV - \int\limits_V b_i u_i dV - \int\limits_{\partial V_p} \psi_i u_i dV - \int\limits_\Omega p_i \Delta u_i dS \quad (3.13)$$

on the set of permissible displacements $\mathbf{K}_L(\mathbf{u})$.

In order to eliminate from $\mathbf{K}_L(\mathbf{u})$ boundary condition on ∂V_u, methods of Lagrange multipliers will be used. As result we obtain the following functional

$$\Phi_L(\mathbf{u},\boldsymbol{\lambda}) = \int\limits_V 1/2\sigma_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{u})dV - \int\limits_V b_i u_i dV - \int\limits_{\partial V_p} \psi_i u_i dV - \int\limits_{\partial V_u} \lambda_i(u_i - \varphi_i)dV - \int\limits_\Omega p_i \Delta u_i dS \quad (3.14)$$

will be considered on the set of permissible displacements

$$\mathbf{K}_B(\mathbf{u}) = \{u_i \in \mathbf{C}^2(V),\ \sigma_{ij}(\mathbf{u}) = c_{ijkl}\varepsilon_{kl}(\mathbf{u}),\ \varepsilon_{ij}(\mathbf{u}) = 1/2(\partial_i u_j + \partial_j u_i), \partial_j\sigma_{ij} + b_i = 0\} \quad (3.15)$$

Let us consider variation of the functional (3.13) on the set of permissible displacements (3.15)

$$\delta\Phi_L(\mathbf{u},\boldsymbol{\lambda}) = \int\limits_V \sigma_{ij}(\mathbf{u})\delta\varepsilon_{ij}(\mathbf{u})dV - \int\limits_V b_i\delta u_i dV - \int\limits_{\partial V_p} \psi_i\delta u_i dV - \int\limits_{\partial V_u} \delta\lambda_i(u_i - \varphi_i)dV - \int\limits_{\partial V_u} \lambda_i\delta u_i dV - \int\limits_\Omega p_i\delta\Delta u_i dS \quad (3.16)$$

Taking into account that

$$\int_V \sigma_{ij}(\mathbf{u})\delta\varepsilon_{ij}(\mathbf{u})dV = \int_{\partial V} p_i\delta u_i dS - \int_V \partial_j\sigma_{ij}(\mathbf{u})\delta u_i(\mathbf{u})dV \tag{3.17}$$

we obtain

$$\delta\Phi_L(\mathbf{u},\boldsymbol{\lambda}) = \int_{\partial V} p_i\delta u_i dS - \int_V (\partial_j\sigma_{ij} + b_i)\delta u_i dV - \int_{\partial V_p} \psi_i\delta u_i dV - \int_{\partial V_u} \delta\lambda_i(u_i - \varphi_i)dV$$
$$- \int_{\partial V_u} \lambda_i\delta u_i dV - \int_{\Omega} p_i\delta\Delta u_i dS \tag{3.18}$$

In Guz and Zozulya 2002 have been shown that under such assumption the boundary variational functional may be presented in the form

$$\Phi_B(\mathbf{u},\mathbf{p}) = \int_{\partial V_u} (1/2p_i - \psi_i)u_i dS - \int_{\partial V_u} (1/2u_i - \varphi_i)p_i dS - \int_{\Omega} p_i\Delta u_i dS \tag{3.19}$$

Taking into account that displacements and traction may be represented in the form

$$u_i(\mathbf{y}) = \int_{\partial V_u} p_i(\mathbf{x})U_{ji}(\mathbf{x}-\mathbf{y})dS - \int_{\partial V_p} u_j(\mathbf{x})W_{ji}(\mathbf{x},\mathbf{y}) - \int_{\Omega} \Delta u_j(\mathbf{x})W_{ji}(\mathbf{x},\mathbf{y})$$
$$+ \int_{\partial V_p} \psi_i(\mathbf{x})U_{ji}(\mathbf{x}-\mathbf{y})dS - \int_{\partial V_u} \varphi_j(\mathbf{x})W_{ji}(\mathbf{x},\mathbf{y}) + \int_V b_i(\mathbf{x})U_{ji}(\mathbf{x}-\mathbf{y})dV$$

$$p_i(\mathbf{y}) = \int_{\partial V_u} p_i(\mathbf{x})K_{ji}(\mathbf{x},\mathbf{y})dS - \int_{\partial V_p} u_j(\mathbf{x})F_{ji}(\mathbf{x},\mathbf{y})dS - \int_{\Omega} \Delta u_j(\mathbf{x})F_{ji}(\mathbf{x},\mathbf{y})dS$$
$$+ \int_{\partial V_p} \psi_i(\mathbf{x})K_{ji}(\mathbf{x},\mathbf{y})dS - \int_{\partial V_u} \varphi_j(\mathbf{x})F_{ji}(\mathbf{x},\mathbf{y})dS + \int_V b_i(\mathbf{x})K_{ji}(\mathbf{x},\mathbf{y})dV \tag{3.20}$$

the boundary variational functional (3.16) may be represented in the form

$$\Phi_B(\mathbf{u},\mathbf{p}) = a(p_i,u_i) + \langle(p_i^* - \psi_i)u_i\rangle_{\partial V_p} - \langle(u_i^* - \varphi_i)p_i\rangle_{\partial V_u} - \langle p_i^*\Delta u_i\rangle_{\Omega} \tag{3.21}$$

where

$$
\begin{aligned}
1/2a(\mathbf{u},\mathbf{p}) = &-\int\limits_{\partial V_u} p_j(\mathbf{y}) \int\limits_{\partial V_u} p_i(\mathbf{x}) U_{ji}(\mathbf{x}-\mathbf{y}) dSdS + \int\limits_{\partial V_u} p_j(\mathbf{y}) \int\limits_{\partial V_p} u_j(\mathbf{x}) W_{ji}(\mathbf{x},\mathbf{y}) dSdS \\
&+ \int\limits_{\partial V_u} p_j(\mathbf{y}) \int\limits_{\Omega} \Delta u_j(\mathbf{x}) W_{ji}(\mathbf{x},\mathbf{y}) dSdS + \int\limits_{\partial V_p} u_j(\mathbf{y}) \int\limits_{\partial V_u} p_i(\mathbf{x}) K_{ji}(\mathbf{x},\mathbf{y}) dSdS \\
&+ \int\limits_{\Omega} \Delta u_j(\mathbf{y}) \int\limits_{\partial V_u} p_i(\mathbf{x}) K_{ji}(\mathbf{x},\mathbf{y}) dSdS - \int\limits_{\partial V_p} u_j(\mathbf{y}) \int\limits_{\partial V_p} u_j(\mathbf{x}) F_{ji}(\mathbf{x},\mathbf{y}) dSdS \\
&- \int\limits_{\Omega} \Delta u_j(\mathbf{y}) \int\limits_{\Omega} \Delta u_j(\mathbf{x}) F_{ji}(\mathbf{x},\mathbf{y}) dSdS
\end{aligned}
\tag{3.22}
$$

and

$$
u_i^*(\mathbf{x}) = \int\limits_{\partial V_p} \psi_i(\mathbf{x}) U_{ji}(\mathbf{x}-\mathbf{y}) dS - \int\limits_{\partial V_u} \varphi_j(\mathbf{x}) W_{ji}(\mathbf{x},\mathbf{y}) + \int\limits_{V} b_i(\mathbf{x}) U_{ji}(\mathbf{x}-\mathbf{y}) dV \tag{3.23}
$$

$$
p_i^*(\mathbf{x}) = \int\limits_{\partial V_p} \psi_i(\mathbf{x}) K_{ji}(\mathbf{x},\mathbf{y}) dS - \int\limits_{\partial V_u} \varphi_j(\mathbf{x}) F_{ji}(\mathbf{x},\mathbf{y}) dS + \int\limits_{V} b_i(\mathbf{x}) K_{ji}(\mathbf{x},\mathbf{y}) dV
$$

The variational formulation of an elastostatic problem for cracked body without unilateral constraints (2.7) and friction (2.8) is as follows:

$$
\boxed{\begin{array}{l}\text{Find } \mathbf{u},\mathbf{p} \in \mathbf{K}_B(\mathbf{u},\mathbf{p}) \text{ such that} \\ \quad \Phi_B(\mathbf{u},\mathbf{p}) = \max\limits_{\mathbf{p}^* \in \mathbf{K}_B(\mathbf{u},\mathbf{p})} \min\limits_{\mathbf{u}^* \in \mathbf{K}_B(\mathbf{u},\mathbf{p})} \{\Phi_B(\mathbf{u}^*,\mathbf{p}^*)\}\end{array}} \tag{3.24}
$$

where

$$
\mathbf{K}_B(\mathbf{u},\mathbf{p}) = \{\mathbf{u} \in \mathbf{H}^{1/2}(\partial V),\ \mathbf{p} \in \mathbf{H}^{-1/2}(\partial V)\}. \tag{3.25}
$$

4 Nonsmooth functionals for unilateral contact conditions with friction

In order to formulate boundary conditions in form of inequalities (2.7) and (2.8) in week form let us consider a maximal monotone operators $\beta_i : u_i \to p_i$. For each maximal monotone operator β_i may be defined with accuracy up to a constant component convex semi-continuous from below functional j_i such, that $\beta_i = \partial j_i$. Here ∂ is denoted the subdifferential of the nonsmooth functional (see. Antes and Panagiotopoulos1992; Panagiotopoulos1985 for details).

Boundary condition for the displacements (2.2a) using subdifferential or superpotential may be written in the form

$$p_i \in \beta_i(u_i) \text{ or } p_i \in \partial j_i(u_i) \tag{4.1}$$

where

$$\beta_i(u_i) = \begin{cases} (-\infty, \infty) & \text{if } u_i = \varphi_i \\ \emptyset & \text{othervise} \end{cases}, \quad j_i(u_i) = \begin{cases} 0 & \text{if } u_i = \varphi_i \\ \infty & \text{othervise} \end{cases} \tag{4.2}$$

Boundary condition for the traction (2.2b) using subdifferentials and superpotentials may be written in the form

$$u_i \in \beta_i^c(p_i) \text{ or } u_i \in \partial j_i^c(p_i) \tag{4.3}$$

where

$$\beta_i^c(p_i) = \begin{cases} (-\infty, \infty) & \text{if } p_i = \psi_i \\ \emptyset & \text{othervise} \end{cases}, \quad j_i^c(p_i) = \begin{cases} 0 & \text{if } p_i = \psi_i \\ \infty & \text{othervise} \end{cases} \tag{4.4}$$

Subdifferential boundary conditions (4.1) and (4.3) should be considered point by point. The functionals $j_i(u_i)$ and $j_i^c(p_i)$ are referred as superpotentials. They express potential and complementary energy of the local constraint.

Let us consider the maximal monotonic operators and the superpotentials for boundary conditions, which are used in the formulation of contact problems with the unilateral constrains in elastostatics.

4.1 Signorini boundary conditions

Usually Signorini boundary conditions are presented in the form (2.7). They are equivalent to the following conditions (fig. 1.)

$$\text{if } \Delta u_n \geq h_0 \text{ then } q_n = 0 \tag{4.5}$$

$$\text{if } \Delta u_n = h_0 \text{ then } q_n > 0$$

The maximal monotone operator in this case has the form

$$\beta_n(\Delta u_n) = \begin{cases} 0, & \text{if } \Delta u_n > h_0 \\ [0, \infty), & \text{if } \Delta u_n = h_0 \\ \emptyset, & \text{if } \Delta u_n < h_0 \end{cases} \tag{4.6}$$

and the corresponding superpotential

$$j_n(\Delta u_n) = \begin{cases} 0, & \text{if } \Delta u_n \geq h_0 \\ \infty, & \text{otherwise} \end{cases} \tag{4.7}$$

The conjugate maximal monotone operator has the form

$$\beta_n^c(q_n) = \begin{cases} [h_0, \infty), & \text{if } q_n = 0 \\ h_0, & \text{if } q_n < 0 \\ \emptyset, & \text{if } q_n > 0 \end{cases} \tag{4.8}$$

and the corresponding superpotential

$$j_n^c(q_n) = \begin{cases} 0, & \text{if } q_n > 0 \\ \infty, & \text{if } q_n \leq 0 \end{cases} \tag{4.9}$$

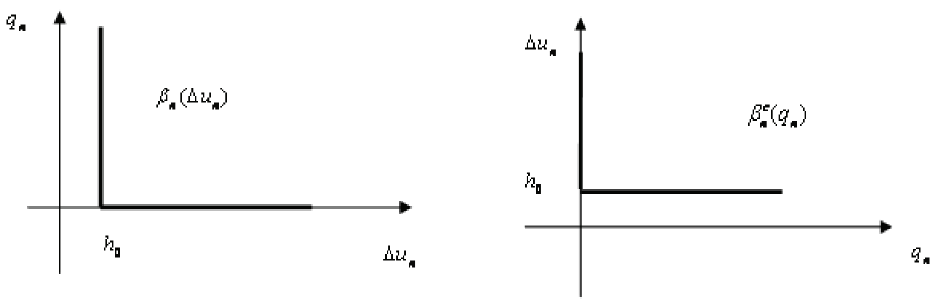

Fig. 1

4.2 Boundary conditions with Coulomb friction

Usually boundary conditions with Coulomb friction are presented in the form (2.8). They are equivalent to the following conditions (fig. 2.)

$$\text{if } |\mathbf{q}_\tau| < k_\tau p_n \text{ then } \Delta \mathbf{u}_\tau = 0 \tag{4.10}$$

$$\text{if } |\mathbf{q}_\tau| = k_\tau p_n \text{ then } \Delta \mathbf{u}_\tau = -\lambda_\tau \mathbf{q}_\tau$$

The maximal monotone operator in this case has the form

$$\beta_\tau(\Delta \mathbf{u}_\tau) = \begin{cases} [-k_\tau |q_n| + k_\tau |q_n|], & \text{if } \Delta \mathbf{u}_\tau = 0 \\ k_\tau |q_n|, & \text{if } \Delta \mathbf{u}_\tau > 0 \\ -k_\tau |q_n|, & \text{if } \Delta \mathbf{u}_\tau < 0 \end{cases} \tag{4.11}$$

and the corresponding superpotential

$$j_\tau(\Delta\mathbf{u}_\tau) = k_\tau\,|q_n|\,|\Delta\mathbf{u}_\tau| \tag{4.12}$$

The conjugate maximal monotone operator has the form

$$\beta_\tau^c(\mathbf{q}_\tau) = \begin{cases} \emptyset, & \text{if } |\mathbf{q}_\tau| < -k_\tau\,|q_n| \\ (-\infty, 0], & \text{if } |\mathbf{q}_\tau| = -k_\tau\,|q_n| \\ 0, & \text{if } [-k_\tau\,|q_n|\ ,\ +k_\tau\,|q_n|] \\ [0, \infty), & \text{if } |\mathbf{q}_\tau| = k_\tau\,|q_n| \\ \emptyset, & \text{if } |\mathbf{q}_\tau| > k_\tau\,|q_n| \end{cases} \tag{4.13}$$

and the corresponding superpotential

$$j_\tau^c(\mathbf{q}_\tau) = \begin{cases} \emptyset, & \text{if } |\mathbf{q}_\tau| \leq -k_\tau\,|q_n| \\ \infty, & \text{if } |\mathbf{q}_\tau| > k_\tau\,|q_n| \end{cases} \tag{4.14}$$

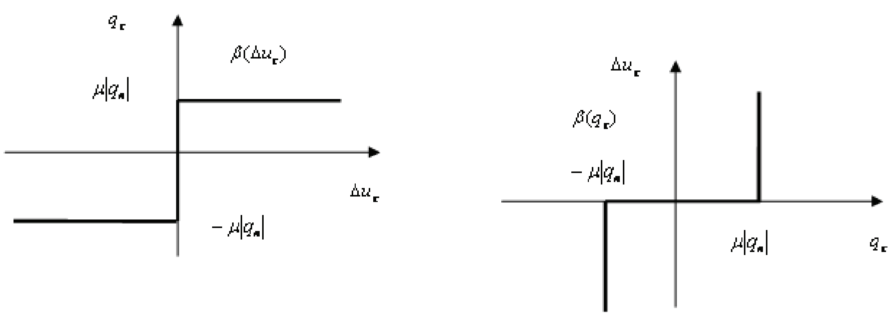

Fig. 2

In the maximal monotone operators and the superpotentials defined in (4.10) – (4.14) the normal traction q_n can be considered as known before function of unknown before one. In last case we have Signorini contact problem with friction.

In fact the maximal monotone operators and the superpotentials define relations between q_n and Δu_n and also between $\mathbf{q}_\tau$ and $\Delta\mathbf{u}_\tau$ at the points of the close contact area. The unilateral constrains (2.7) and (2.8) may be written using subdifferential boundary conditions (4.7), (4.9 and (4.12, (4.14) in the form

$$q_n \in \partial j_n(\Delta u_n), \quad \mathbf{q}_\tau \in \partial j_\tau(\Delta\mathbf{u}_\tau) \tag{4.15}$$

or in the form

$$\Delta u_n \in \partial j_n^c(q_n), \quad \Delta \mathbf{u}_\tau \in \partial j_\tau^c(\mathbf{q}_\tau) \tag{4.16}$$

We have now considered subdifferential boundary conditions in the pointwice sense. This means that maximal monotone operators and the superpotentials in (4.7) – (4.9) and (4.11)-(4.14) define unilateral contact conditions with friction point by point. But for variational formulation of the problem however it is necessary to extend these boundary conditions to the functional spaces. It means that we have to consider functionals of the form

$$\Phi(\Delta \mathbf{u}) = \int_\Omega j(\Delta \mathbf{u}(\mathbf{x}))dS \text{ and } \Phi^c(\mathbf{q}) = \int_\Omega j(\mathbf{q}(\mathbf{x}))dS \tag{4.17}$$

In Antes and Panagiotopoulos1992; Panagiotopoulos1985 it was shown that the corresponding functionals have the following form.

4.3 Signorini boundary conditions in functional space

Lets $\Delta u_n \in \mathbf{H}^{1/2}(\Omega)$ and $q_n \in \mathbf{H}^{-1/2}(\Omega)$ satisfy following conditions $\Delta u_n \geq h_0$, $q_n \geq 0$. $\langle q_n, (\Delta u_n - h_0)\rangle_\Omega = 0$, Here $\langle \cdot,\cdot\rangle$ denotes the duality pairing between the functional spaces $\mathbf{H}^{1/2}(\Omega)$ and $\mathbf{H}^{-1/2}(\partial\Omega)$. Then corresponding functional has the form

$$\Phi_n(\Delta u_n) = \begin{cases} 0, & \text{if } \Delta u_n \geq h_0 \\ \infty, & \text{otherwise} \end{cases} \tag{4.18}$$

The conjugate functional has the form

$$\Phi_n^c(q_n) = \begin{cases} 0, & \text{if } q_n \geq 0 \\ \infty, & \text{otherwise} \end{cases} \tag{4.19}$$

4.4 Boundary conditions with Coulomb friction

Lets $\Delta \mathbf{u}_\tau \in (\mathbf{H}^{1/2}(\Omega))^2$ and $\mathbf{q}_\tau \in (\mathbf{H}^{-1/2,0}(\Omega))^2$ satisfy following conditions if $|\mathbf{q}_\tau| \leq kq_n$ then $\Delta \mathbf{u}_\tau = 0$, if $|\mathbf{q}_\tau| = kq_n$ then $\Delta \mathbf{u}_\tau = -\lambda_\tau \mathbf{q}_\tau$ and also $\langle (kq_n - |\mathbf{q}_\tau|), \Delta \mathbf{u}_\tau\rangle_\Omega = 0$. Here $\langle \cdot,\cdot\rangle$ denotes the duality pairing between the functional spaces $(\mathbf{H}^{1/2}(\Omega))^2$ and $(\mathbf{H}^{-1/2}(\Omega))^2$. Then corresponding functional has the form

$$\Phi_\tau(\Delta \mathbf{u}_\tau) = \langle \mathbf{q}_\tau, \Delta \mathbf{u}_\tau\rangle_\Omega \tag{4.20}$$

The conjugate functional has the form

$$\Phi_\tau^c(\mathbf{q}_\tau) = \begin{cases} 0, & \text{if } |\mathbf{q}_\tau| = kq_n \\ \infty, & \text{otherwise} \end{cases} \tag{4.21}$$

4.5 Signorini boundary conditions with friction

These boundary conditions may be considered as combination of the previously considered boundary conditions. Really lets $\Delta u_n \in \mathbf{H}^{1/2}(\Omega)$ and $q_n \in \mathbf{H}^{-1/2}(\Omega)$ satisfy following conditions $\Delta u_n \geq h_0$, $q_n \geq 0$, $\langle q_n, (\Delta u_n - h_0)\rangle_\Omega = 0$ and also $\Delta \mathbf{u}_\tau \in (\mathbf{H}^{1/2}(\Omega))^2$ and $\mathbf{q}_\tau \in (\mathbf{H}^{-1/2}(\Omega))^2$ satisfy following conditions if $|\mathbf{q}_\tau| \leq kq_n$ then $\Delta \mathbf{u}_\tau = 0$, if $|\mathbf{q}_\tau| = kq_n$ then $\Delta \mathbf{u}_\tau = -\lambda_\tau \mathbf{q}_\tau$ and also $\langle (kq_n - |\mathbf{q}_\tau|)\Delta \mathbf{u}_\tau\rangle_\Omega = 0$. We consider functionals such that

$$\Phi_{n,\tau}(\Delta \mathbf{u}) = \Phi_n(\Delta u_n) + \Phi_\tau(\Delta \mathbf{u}_\tau) \text{ and } \Phi^c_{n,\tau}(\mathbf{q}) = \Phi^c_n(q_n) + \Phi^c_\tau(\mathbf{q}_\tau) \tag{4.22}$$

These functionals have the form

$$\Phi_{n,\tau}(\Delta \mathbf{u}) = \begin{cases} \langle \mathbf{q}_\tau, \Delta \mathbf{u}_\tau \rangle, & \text{if } \Delta u_n \geq h_0 \\ \infty, & \text{otherwise} \end{cases} \tag{4.23}$$

$$\Phi^c_{n,\tau}(\mathbf{q}) = \begin{cases} 0, & \text{if } q_n \geq 0,\ |\mathbf{q}_\tau| = kq_n \\ \infty, & \text{otherwise} \end{cases} \tag{4.24}$$

5 Variational formulation of the problem with contact and friction

In order to formulate an elastostatic contact problem for body with crack in week form we will consider the principles of virtual displacements and of virtual stresses and boundary variational principle.

5.1 Principles of Virtual Displacements

Consider the functional of Lagrange (3.1) on a set of admissible displacements

$$\mathbf{K}_{L,n,\tau}(\mathbf{u}) = \mathbf{K}_L(\mathbf{u}) \cap \mathbf{K}_n(\mathbf{u}) \cap \mathbf{K}_\tau(\mathbf{u}),$$

$$\mathbf{K}_n(\mathbf{u}) = \{\mathbf{u} \in \mathbf{H}^{1/2}(\Omega),\ \Delta u_n - h_0 \geq 0\},\ \forall \mathbf{x} \in \Omega\}, \tag{5.1}$$

$$\mathbf{K}_\tau(\mathbf{u}) = \{\mathbf{u} \in \mathbf{H}^{1/2}(\Omega),\ \Delta \mathbf{u}_\tau = 0 \text{ for } |\mathbf{q}_\tau| \leq k_\tau q_n \text{ and } \Delta \mathbf{u}_\tau = -\lambda_t \mathbf{q}_\tau \text{ for } |\mathbf{q}_\tau| = k_\tau q_n,\ \forall \mathbf{x} \in \Omega\}$$

The variational formulation of an elastostatic problem for cracked body with unilateral constraints (2.7) and (2.8) is as follows:

$$\boxed{\begin{array}{l}\text{Find } \mathbf{u} \in \mathbf{K}_{L,n,\tau}(\mathbf{u}) \text{ such that} \\ \quad \Phi_L[\mathbf{u}] = \inf\limits_{\mathbf{u}^* \in \mathbf{K}_{L,n,\tau}(\mathbf{u})} \{\Phi_L[\mathbf{u}^*]\}\end{array}} \tag{5.2}$$

Let's exclude unilateral restrictions (2.7) and (2.8) from the set (5.1). Using Lagrange multiplier functional of Lagrange (3.1) may be presented in the form

$$\Phi_{L,n,\tau}[\mathbf{u}] = \Phi_L[\mathbf{u}] + \Phi_{n,\tau}(\Delta\mathbf{u}), \tag{5.3}$$

Now the variational formulation of an elastostatic problem for cracked body with unilateral constraints (2.7) and (2.8) is as follows:

$$\boxed{\begin{array}{l}\text{Find } \mathbf{u} \in \mathbf{K}_L(\mathbf{u}) \text{ such that}\\ \quad \Phi_{L,n,\tau}[\mathbf{u}] = \inf\limits_{\mathbf{u}^* \in \mathbf{K}_L(\mathbf{u})} \{\Phi_{L,n,\tau}[\mathbf{u}^*]\}\end{array}} \tag{5.4}$$

Functional in (5.2) has a simple form, but the set of restrictions (5.1) is complicate, it contains unilateral constraints (2.7) and (2.8). Functional (5.3) is more complicate and nonsmooth, but the set of restrictions (3.4) is simple, it does not contain unilateral constraints (2.7) and (2.8). Sometime it is convenient for numerical solution of the problem to present it in form (5.2) or (5.4).

5.2 Principles of Virtual Stresses

Consider the functional of Kastiliano (3.7) on a set of admissible displacements

$$\mathbf{K}_{K,n,\tau}(\boldsymbol{\sigma}) = \mathbf{K}_K(\boldsymbol{\sigma}) \cap \mathbf{K}_n^c(\boldsymbol{\sigma}) \cap \mathbf{K}_\tau^c(\boldsymbol{\sigma}),$$

$$\mathbf{K}_n^c(\boldsymbol{\sigma}) = \{\boldsymbol{\sigma} \in \mathbf{H}^{-1/2}(\partial V \cup \Omega),\ q_n \geq 0,\ \forall \mathbf{x} \in \Omega\},$$
$$\mathbf{K}_\tau^c(\boldsymbol{\sigma}) = \{\boldsymbol{\sigma} \in \mathbf{H}^{-1/2}(\partial V \cup \Omega),\ |\mathbf{q}_\tau| \leq k_\tau p_n,\ \forall \mathbf{x} \in \Omega\} \tag{5.5}$$

The variational formulation of an elastostatic problem for cracked body with unilateral constraints (2.7) and (2.8) is as follows:

$$\boxed{\begin{array}{l}\text{Find } \boldsymbol{\sigma} \in \mathbf{K}_{K,n,\tau}(\boldsymbol{\sigma}) \text{ such that}\\ \quad \Phi_K[\boldsymbol{\sigma}] = \sup\limits_{\boldsymbol{\sigma}^* \in \mathbf{K}_{K,n,\tau}(\sigma)} \{\Phi_K[\boldsymbol{\sigma}^*]\}\end{array}} \tag{5.6}$$

Let's exclude unilateral restrictions (2.7) and (2.8) from the set (5.5). Using Lagrange multiplier functional Kastiliano (3.7) may be presented in the form

$$\Phi_{K,n}[\boldsymbol{\sigma}] = \Phi_K[\boldsymbol{\sigma}] + \Phi_{n,\tau}^c(\mathbf{q}), \tag{5.7}$$

Now the variational formulation of an elastostatic problem for cracked body with unilateral constraints (2.7) and (2.8) is as follows:

$$\boxed{\begin{array}{l}\text{Find } \boldsymbol{\sigma} \in \mathbf{K}_K(\boldsymbol{\sigma}) \text{ such that}\\ \quad \Phi_{K,n,\tau}[\boldsymbol{\sigma}] = \sup\limits_{\boldsymbol{\sigma}^* \in \mathbf{K}_K(\boldsymbol{\sigma})} \{\Phi_{K,n,\tau}[\boldsymbol{\sigma}^*]\}\end{array}} \tag{5.8}$$

Functional in (5.6) has a simple form, but the set of restrictions (5.5) is complicate, it contains unilateral constraints (2.7) and (2.8). Functional (5.7) is more complicate and nonsmooth, but the set of restrictions (3.7) is simple, it does not contain unilateral constraints (2.7) and (2.8). Sometime it is convenient for numerical solution of the problem to present it in form (5.6) or (5.8).

5.3 Boundary Variational Principles

Consider the boundary variational functional (3.18) on a set of admissible displacements

$$\mathbf{K}_{B,n,\tau}(\mathbf{u},\mathbf{p}) = \mathbf{K}_B(\mathbf{u},\mathbf{p}) \cap \mathbf{K}_n(\mathbf{u}) \cap \mathbf{K}_\tau(\mathbf{u}). \tag{5.9}$$

The variational formulation of an elastostatic problem for cracked body with unilateral constraints (2.7) and (2.8) is as follows:

Find $\mathbf{u}$ and $\mathbf{p}$ such that

$$\Phi_B(\mathbf{u},\mathbf{p}) = \sup_{\mathbf{p}^*\in\mathbf{K}_B(\mathbf{u},\mathbf{p})} \inf_{\mathbf{u}^*\in\mathbf{K}_{B,n,\tau}(\mathbf{u},\mathbf{p})} \{\Phi_B(\mathbf{u}^*,\mathbf{p}^*)\} \tag{5.10}$$

Let's exclude unilateral restrictions (2.7) and (2.8) from the set (5.9). Using Lagrange multiplier boundary variational functional (3.18) may be presented in the form

$$\Phi_{B,n,\tau}(\mathbf{u},\mathbf{p}) = \Phi_B(\mathbf{u},\mathbf{p}) + \Phi_{n,\tau}(\Delta\mathbf{u}), \tag{5.11}$$

The variational formulation of an elastostatic problem for cracked body with unilateral constraints (2.7) and (2.8) is as follows:

Find $\mathbf{u}$ and $\mathbf{p}$ such that

$$\Phi_{B,n,\tau}(\mathbf{u},\mathbf{p}) = \inf\sup_{\mathbf{u}^*,\mathbf{p}^*\in\mathbf{K}_B(\mathbf{u},\mathbf{p})} \{\Phi_{B,n,\tau}(\mathbf{u}^*,\mathbf{p}^*)\} \tag{5.12}$$

In the same way we can consider the functional

$$\Phi^c_{B,n,\tau}(\mathbf{u},\mathbf{p}) = \Phi_B(\mathbf{u},\mathbf{p}) + \Phi^c_{n,\tau}(\mathbf{q}), \tag{5.13}$$

The variational formulation of an elastostatic problem for cracked body with unilateral constraints (2.7) and (2.8) is as follows:

Find $\mathbf{u}$ and $\mathbf{p}$ such that

$$\Phi^c_{B,n,\tau}(\mathbf{u},\mathbf{p}) = \sup\inf_{\mathbf{u}^*,\mathbf{p}^*\in\mathbf{K}_B(\mathbf{u},\mathbf{p})} \{\Phi^c_{B,n,\tau}(\mathbf{u}^*,\mathbf{p}^*)\} \tag{5.14}$$

Functional in (5.10) has a simple form, but the set of restrictions (5.9) is complicate, it contains unilateral constraints (2.7) and (2.8). Functionals (5.11) and (5.13) are more complicate and nonsmooth, but the set of restrictions (3.22) is simple, it does not contain unilateral constraints (2.7) and (2.8). Sometime it is convenient for numerical solution of the problem to present it in form (5.10) or (5.12), (5,14).

It is necessary to mention that the principle of virtual displacements and virtual stress are usually used with FEM and the boundary variational principle with BEM.

6 Dual variational formulation and Uzawa's optimization algorithm

Variational formulations (5.3) and (5.8) also (5.12) and (5.14) are referred as complementary. We can reformulate above variational problems using duality feature. On these dual formulations are based Uzawa's nonsmooth optimization algorithms. Let us consider dual formulations and corresponding Uzawa's algorithms for the problems under consideration.

6.1 Principles of Virtual Displacements

Let us introduce functional

$$L(\mathbf{u},\mathbf{q}) = \Phi_L[\mathbf{u}] + \langle \mathbf{q}, \Phi(\Delta\mathbf{u})\rangle_\Omega \tag{6.1}$$

which is considered on the following sets of restrictions

$$\mathbf{u} \in \mathbf{K}_L(\mathbf{u}), \quad \mathbf{q} \in \mathbf{K}^c_{n,\tau}(\boldsymbol{\sigma}) \tag{6.2}$$

Dual to (5.3) variational formulation of the contact problem with friction for elastic body with crack has the form

$$L(\mathbf{u},\mathbf{q}) = \inf_{\mathbf{u}^*\in\mathbf{K}_L(\mathbf{u})} \sup_{\mathbf{q}^*\in\mathbf{K}^c_{n,\tau}(\boldsymbol{\sigma})} \{L(\mathbf{u}^*,\mathbf{q}^*)\} \tag{6.3}$$

The Uzawa's algorithm includes the following steps:

1. specify an initial value $\mathbf{q}^0 \in \mathbf{K}^c_{n,\tau}(\boldsymbol{\sigma})$,

2. solve the minimization problem for known $\mathbf{q}^n_\tau$ and determine the unknown quantity $\mathbf{u}^n \in \mathbf{K}_L(\mathbf{u})$

$$L(\mathbf{u}^n,\mathbf{q}^n) = \inf_{\mathbf{u}^*\in\mathbf{K}_L(\mathbf{u})} \{L(\mathbf{u},\mathbf{q}^n)\} = \inf_{\mathbf{u}^*\in\mathbf{K}_L(\mathbf{u})} \{\Phi_L[\mathbf{u}] + \langle \mathbf{q}^n, \Phi(\Delta\mathbf{u})\rangle_\Omega\} \tag{6.4}$$

3. correct the quantity $\mathbf{q}^n$ to satisfy the constraints

$$\mathbf{q}^{n+1} = \mathbf{P}_{\mathbf{K}^c_{n,\tau}(\boldsymbol{\sigma})}[\mathbf{q}^n + \rho\Phi(\Delta\mathbf{u}^n)] \tag{6.5}$$

where $\mathbf{P}_{\mathbf{K}^c_{n,\tau}(\boldsymbol{\sigma})}$ is the operator of projection in $\mathbf{H}^{-1/2}(\Omega)$ on $\mathbf{K}^c_{n,\tau}(\boldsymbol{\sigma})$ and coefficient ρ is selected so as to provide the best convergence of the algorithm,

4. proceed to the next step of iteration.

6.2 Principles of Virtual Stresses

Let us introduce functional

$$L(\boldsymbol{\sigma}, \Delta\mathbf{u}) = \Phi_K[\boldsymbol{\sigma}^*] + \langle \Delta\mathbf{u}^*, \Phi^c(\mathbf{q}^*)\rangle_\Omega \tag{6.6}$$

which is considered on the following sets of restrictions

$$\boldsymbol{\sigma} \in \mathbf{K}_K(\boldsymbol{\sigma}), \quad \Delta\mathbf{u} \in \mathbf{K}_{n,\tau}(\mathbf{u}) \tag{6.7}$$

Dual to (5.8) variational formulation of the contact problem with friction for elastic body with crack has the form

$$L(\boldsymbol{\sigma}, \Delta\mathbf{u}) = \sup_{\boldsymbol{\sigma}^* \in \mathbf{K}_K(\boldsymbol{\sigma})} \inf_{\Delta\mathbf{u}^* \in \mathbf{K}_{n,\tau}(\mathbf{u})} \{L(\boldsymbol{\sigma}^*, \Delta\mathbf{u}^*)\} \tag{6.8}$$

The Uzawa's algorithm includes the following steps:

1. specify an initial value $\Delta\mathbf{u}^0 \in \mathbf{K}^{(}_{n,\tau}\mathbf{u})$,

2. solve the minimization problem for known $\Delta\mathbf{u}^n$ and determine the unknown quantity $\boldsymbol{\sigma}^n \in \mathbf{K}_2(\boldsymbol{\sigma})$

$$L(\boldsymbol{\sigma}, \Delta\mathbf{u}^n) = \sup_{\boldsymbol{\sigma} \in \mathbf{K}_K(\boldsymbol{\sigma})} \{L(\boldsymbol{\sigma}, \Delta\mathbf{u}^n)\} = \sup_{\boldsymbol{\sigma} \in \mathbf{K}_K(\boldsymbol{\sigma})} \{\Phi_K[\boldsymbol{\sigma}] + \langle \Delta\mathbf{u}^n, \Phi^c(\mathbf{q})\rangle_\Omega\} \tag{6.9}$$

3. correct the quantity $\Delta\mathbf{u}^n$ to satisfy the constraints

$$\Delta\mathbf{u}^{n+1} = \mathbf{P}_{\mathbf{K}_{n,\tau}(\mathbf{u})}[\Delta\mathbf{u}^n + \rho\Phi^c(\mathbf{q}^n)] \tag{6.10}$$

where $\mathbf{P}_{\mathbf{K}_{n,\tau}(\mathbf{u})}$ is the operator of projection in $\mathbf{H}^{1/2}(\Omega)$ on $\mathbf{K}^{(}_{n,\tau}\mathbf{u})$ and coefficient ρ is selected so as to provide the best convergence of the algorithm,

4. proceed to the next step of iteration.

6.3 Bounadry Variational Principles I

Let us introduce functional

$$L(\mathbf{u},\mathbf{p},\mathbf{q}) = \Phi_B(\mathbf{u}^*,\mathbf{p}^*) + \langle \mathbf{q}^*, \Phi_\tau(\Delta\mathbf{u}^*)\rangle_\Omega \tag{6.11}$$

which is considered on the following sets of restrictions

$$\mathbf{u},\mathbf{p} \in \mathbf{K}_B(\mathbf{u}), \quad \mathbf{q} \in \mathbf{K}^c_{n,\tau}(\boldsymbol{\sigma}) \tag{6.12}$$

Dual to (5.12) variational formulation of the contact problem with friction for elastic body with crack has the form

$$L(\mathbf{u},\mathbf{p},\mathbf{q}) = \inf_{\mathbf{u}^*,\mathbf{p}^* \in \mathbf{K}_B(\mathbf{u},\mathbf{p})} \sup \sup_{\mathbf{q}^* \in \mathbf{K}^c_{n,\tau}(\boldsymbol{\sigma})} \{L(\mathbf{u}^*,\mathbf{p}^*,\mathbf{q}^*)\} \tag{6.13}$$

The Uzawa's algorithm includes the following steps:

1. specify an initial value $\mathbf{q}^0 \in \mathbf{K}^c_{n,\tau}(\boldsymbol{\sigma})$,
2. solve the minimization problem for known $\mathbf{q}^n_\tau$ and determine the unknown quantity $\mathbf{u}^n, \mathbf{p}^n \in \mathbf{K}_B(\mathbf{u},\mathbf{p})$

$$L(\mathbf{u}^n,\mathbf{p}^n,\mathbf{q}^n) = \inf_{\mathbf{u},\mathbf{p} \in \mathbf{K}_B(\mathbf{u},\mathbf{p})} \sup \{L(\mathbf{u},\mathbf{p},\mathbf{q}^n)\} = \inf_{\mathbf{u},\mathbf{p} \in \mathbf{K}_B(\mathbf{u},\mathbf{p})} \sup \{\Phi_B(\mathbf{u},\mathbf{p}) + \langle \mathbf{q}^n, \Phi_\tau(\Delta\mathbf{u})\rangle_\Omega\} \tag{6.14}$$

3. correct the quantity $\mathbf{q}^n$ to satisfy the constraints

$$\mathbf{q}^{n+1} = \mathbf{P}_{\mathbf{K}^c_{n,\tau}(\boldsymbol{\sigma})}[\mathbf{q}^n + \rho\Phi(\Delta\mathbf{u}^n)] \tag{6.15}$$

   where $\mathbf{P}_{\mathbf{K}^c_{n,\tau}(\boldsymbol{\sigma})}$ is the operator of projection in $\mathbf{H}^{-1/2}(\Omega)$ on $\mathbf{K}^c_{n,\tau}(\boldsymbol{\sigma})$ and coefficient ρ is selected so as to provide the best convergence of the algorithm,

4. proceed to the next step of iteration.

6.4 Bounadry Variational Principles II

Let us introduce functional

$$L(\mathbf{u},\mathbf{p},\Delta\mathbf{u}) = \Phi_B(\mathbf{u}^*,\mathbf{p}^*) + \langle \Delta\mathbf{u}^*, \Phi^c_\tau(\mathbf{q}^*)\rangle_\Omega \tag{6.16}$$

which is considered on the following sets of restrictions

$$\mathbf{u},\mathbf{p} \in \mathbf{K}_B(\mathbf{u}), \quad \Delta\mathbf{u} \in \mathbf{K}_{n,\tau}(\mathbf{u}) \tag{6.17}$$

Dual to (5.14) variational formulation of the contact problem with friction for elastic body with crack has the form

$$L(\mathbf{u},\mathbf{p},\Delta\mathbf{u}) = \inf_{\mathbf{u}^*,\mathbf{p}^*\in\mathbf{K}_B(\mathbf{u},\mathbf{p})}\sup\ \sup_{\mathbf{q}^*_\tau\in\mathbf{K}^c_{n,\tau}(\boldsymbol{\sigma})}\{L(\mathbf{u}^*,\mathbf{p}^*,\Delta\mathbf{u}^*)\} \tag{6.18}$$

The Uzawa's algorithm includes the following steps:

1. specify an initial value $\Delta\mathbf{u}^0 \in \mathbf{K}^(_{n,\tau}\mathbf{u})$,

2. solve the minimization problem for known $\Delta\mathbf{u}^n$ and determine the unknown quantity $\mathbf{u}^n,\mathbf{p}^n \in \mathbf{K}_B(\mathbf{u},\mathbf{p})$

$$L(\mathbf{u}^n,\mathbf{p}^n,\Delta\mathbf{u}^n) = \inf_{\mathbf{u},\mathbf{p}\in\mathbf{K}_B(\mathbf{u},\mathbf{p})}\sup\{L(\mathbf{u},\mathbf{p},\Delta\mathbf{u}^n)\} = \inf_{\mathbf{u},\mathbf{p}\in\mathbf{K}_B(\mathbf{u},\mathbf{p})}\sup\{\Phi_B(\mathbf{u},\mathbf{p}) + \langle\Delta\mathbf{u}^n,\Phi^c_\tau(\mathbf{q})\rangle_\Omega\} \tag{6.19}$$

3. correct the quantity $\Delta\mathbf{u}^n$ to satisfy the constraints

$$\Delta\mathbf{u}^{n+1} = \mathbf{P}_{\mathbf{K}_{n,\tau}(\mathbf{u})}[\Delta\mathbf{u}^n + \rho\Phi^c(\mathbf{q}^n)] \tag{6.20}$$

   where $\mathbf{P}_{\mathbf{K}^(_{n,\tau}\mathbf{u})}$ is the operator of projection in $\mathbf{H}^{1/2}(\Omega)$ on $\mathbf{K}^(_{n,\tau}\mathbf{u})$ and coefficient ρ is selected so as to provide the best convergence of the algorithm,

4. proceed to the next step of iteration.

It is important to mention that in the algorithms based on principle of virtual displacements and virtual stress the FEM and the algorithms based on the boundary variational principle the BEM are usually used.

7 Formulation of the problem for crack in infinite elastic region

In the case of crack in infinite body situation become simpler. Indeed in this case

$$a(\mathbf{u},\mathbf{p}) = -1/2\int_\Omega \Delta u_j(\mathbf{y})\int_\Omega \Delta u_j(\mathbf{x})F_{ji}(\mathbf{x},\mathbf{y})dSdS, \quad u_i^*(\mathbf{x}) = 0, \quad p_i^*(\mathbf{x}) = p_i(\mathbf{x}) \tag{7.1}$$

where $\mathbf{p}(\mathbf{x}) = p_i(\mathbf{x})\mathbf{e}_i = \mathbf{p}^0(\mathbf{x}) + \mathbf{q}(\mathbf{x})$, $\mathbf{p}^0(\mathbf{x})$ is a vector of given loading applied to the crack edges, $\mathbf{q}(\mathbf{x})$ is a vector of contact forces.

Boundary variational functional become of the form

$$\Phi_B(\Delta\mathbf{u},\mathbf{p}) = -1/2\int\limits_{\Omega}\Delta u_i(\mathbf{y})\int\limits_{\Omega}\Delta u_j(\mathbf{x})F_{ji}(\mathbf{x},\mathbf{y})dSdS - \int\limits_{\Omega}p_i^{(}\mathbf{y})\Delta u_i(\mathbf{y})dS \tag{7.2}$$

In the case of the crack faces contact interaction absence $\mathbf{q}(\mathbf{x}) = 0$ and boundary variational functional has the form

$$\Phi_B(\Delta\mathbf{u}) = -1/2\int\limits_{\Omega}\Delta u_i(\mathbf{y})\int\limits_{\Omega}\Delta u_j(\mathbf{x})F_{ji}(\mathbf{x},\mathbf{y})dSdS - \int\limits_{\Omega}p_i^0(\mathbf{y})\Delta u_i(\mathbf{y})dS \tag{7.3}$$

Then variational formulation of an elastostatic problem for cracked body without unilateral constraints (2.7) and friction (2.8) is as follows:

$$\boxed{\begin{array}{l}\text{Find } \Delta\mathbf{u},\mathbf{p}\in\mathbf{K}_B(\Delta\mathbf{u},\mathbf{p}) \text{ such that}\\ \quad \Phi_B(\Delta\mathbf{u},\mathbf{p}) = \min\limits_{\mathbf{u}^*,\mathbf{p}^*\in\mathbf{K}_B(\Delta\mathbf{u},\mathbf{p})}\{\Phi_B[\Delta\mathbf{u}^*,\mathbf{p}^*]\}\end{array}} \tag{7.4}$$

where

$$\mathbf{K}_B(\Delta\mathbf{u},\mathbf{p}) = \{\Delta\mathbf{u}\in\mathbf{H}^{1/2}(\Omega)\,,\ \mathbf{p}\in\mathbf{H}^{-1/2}(\Omega),\ \forall\mathbf{x}\in\Omega\} \tag{7.5}$$

The boundary variational functional (7.3) is smooth and Gateaux-differentiable, therefore from (7.4) follows the following condition of functional minima

$$\delta\Phi_B(\Delta\mathbf{u}) = 0 \tag{7.6}$$

and therefore the problem (7.4) is equivalent to the following integral equation

$$\int\limits_{\Omega}\Delta u_j(\mathbf{x})F_{ji}(\mathbf{x},\mathbf{y})dS = p_i(\mathbf{y}) \tag{7.7}$$

Now we can represent boundary variational functional (7.2) in the form

$$\Phi_B(\Delta\mathbf{u}) = -1/2\,\langle\mathbf{F}\cdot\Delta\mathbf{u},\Delta\mathbf{u}\rangle - \langle\mathbf{p},\Delta\mathbf{u}\rangle_\Omega \tag{7.8}$$

where $\mathbf{F}$ is matrix integral operator defined in (7.7),

Then boundary variational formulation of the elastostatic contact problem with friction for cracked body has the form

$$\boxed{\begin{array}{l}\text{Find } \Delta\mathbf{u},\mathbf{p}\in\mathbf{K}_{B,n,\tau}(\Delta\mathbf{u},\mathbf{p}) \text{ such that}\\ \quad \Phi_B(\Delta\mathbf{u}) = \inf\limits_{\Delta\mathbf{u}^*\in\mathbf{K}_{B,n,\tau}(\mathbf{u})}\{-\langle\mathbf{F}\cdot\Delta\mathbf{u}^*,\Delta\mathbf{u}^*\rangle - \langle\mathbf{p},\Delta\mathbf{u}^*\rangle\}_\Omega\end{array}} \tag{7.9}$$

The Uzawa's algorithm in this case is simplified. We refer to this algorithm as Algorithm 1.

Algorithm 1 includes the following steps:

1. specify an initial value $\Delta\mathbf{u}^0 \in \mathbf{K}_{n,\tau}(\Delta\mathbf{u})$,

2. calculate $\mathbf{q}^n$ substituting known $\Delta\mathbf{u}^n$ in the internals equation

$$\mathbf{q}^n = \mathbf{F}\cdot\Delta\mathbf{u}^n - \mathbf{p}^0 \tag{7.10}$$

3. correct the quantity $\Delta\mathbf{u}^n$ to satisfy the constraints

$$\Delta\mathbf{u}^{n+1} = \mathbf{P}_{\mathbf{K}_{n,\tau}(\Delta\mathbf{u})}[\Delta\mathbf{u}^n + \rho\mathbf{q}^n] \tag{7.11}$$

where $\mathbf{P}_{\mathbf{K}_{n,\tau}(\Delta\mathbf{u})}$ is the operator of projection in $\mathbf{H}^{1/2}(\Omega)$ on $\mathbf{K}_{n,\tau}(\Delta\mathbf{u})$ and coefficient ρ is selected so as to provide the best convergence of the algorithm,

4. proceed to the next step of iteration.

We can formally introduce complementary boundary variational functional in the form

$$\Phi_B^c(\mathbf{p}) = -1/2\left\langle\mathbf{p},\mathbf{F}^{-1}\cdot\mathbf{p}\right\rangle - \langle\mathbf{p},\Delta\mathbf{u}\rangle_\Omega \tag{7.12}$$

and $\mathbf{F}^{-1}$ is inverse operator.

Then boundary variational formulation of the elastostatic contact problem with friction for cracked body has the form

$$\boxed{\begin{array}{l}\text{Find } \Delta\mathbf{u},\mathbf{p} \in \mathbf{K}_{B,n,\tau}^c(\Delta\mathbf{u},\mathbf{p}) \text{ such that}\\ \quad\Phi_B^c(\mathbf{p}) = \sup\limits_{\mathbf{p}^*\in\mathbf{K}_{B,n,\tau}^c(\mathbf{u})}\left\{-1/2\left\langle\mathbf{p}^*,\mathbf{F}^{-1}\cdot\mathbf{p}^*\right\rangle - \langle\mathbf{p}^*,\Delta\mathbf{u}^*\rangle_\Omega\right\}\end{array}} \tag{7.13}$$

The Uzawa's algorithm in this case is simplified. We refer to this algorithm as Algorithm 2.

Algorithm 2 includes the following steps:

1. specify an initial value $\mathbf{q}^0 \in \mathbf{K}_{n,\tau}^c(\Delta\mathbf{q})$,

2. calculate $\Delta\mathbf{u}^n$ substituting known $\mathbf{q}^n$ in the internals equation

$$\Delta\mathbf{u}^n = \mathbf{F}^{-1}\cdot\mathbf{p}^n \tag{7.14}$$

3. correct the quantity $\mathbf{q}^n$ to satisfy the constraints

$$\mathbf{q}^{n+1} = \mathbf{P}_{\mathbf{K}_{n,\tau}^c(\mathbf{q})}[\mathbf{q}^n + \rho\Delta\mathbf{u}^n] \tag{7.15}$$

where $\mathbf{P}_{\mathbf{K}_{n,\tau}^c(\mathbf{q})}$ is the operator of projection in $\mathbf{H}^{-1/2}(\Omega)$ on $\mathbf{K}_{n,\tau}^c(\mathbf{q})$ and coefficient ρ is selected so as to provide the best convergence of the algorithm,

4. proceed to the next step of iteration.

Two slightly modified algorithms were proposed and investigated. We refer to these algorithms as Algorithm 3 and Algorithm 4.
Algorithm 3 includes the following steps:

1. specify an initial value $\Delta\mathbf{u}^0 \in \mathbf{K}^{(}_{n,\tau}\Delta\mathbf{u})$,
2. calculate $\mathbf{q}^n$ substituting known $\Delta\mathbf{u}^n$ in the internals equation

$$\mathbf{q}^n = \mathbf{F}\cdot\Delta\mathbf{u}^n - \mathbf{p}^0 \tag{7.16}$$

3. correct the quantity $\mathbf{q}^n$ to satisfy the constraints

$$\mathbf{q}^{n+1} = \mathbf{P}_{\mathbf{K}^c_{n,\tau}(\mathbf{q})}[\mathbf{q}^n] \tag{7.17}$$

where $\mathbf{P}_{\mathbf{K}^c_{n,\tau}(\mathbf{q})}$ is the operator of projection in $\mathbf{H}^{-1/2}(\Omega)$ on $\mathbf{K}^c_{n,\tau}(\mathbf{q})$ and coefficient ρ is selected so as to provide the best convergence of the algorithm,

4. calculate $\Delta\mathbf{u}^n$ substituting known $\mathbf{q}^n$ in the internals equation

$$\Delta\mathbf{u}^n = \mathbf{F}^{-1}\cdot\mathbf{p}^n \tag{7.18}$$

5. correct the quantity $\Delta\mathbf{u}^n$ to satisfy the constraints

$$\Delta\mathbf{u}^{n+1} = \mathbf{P}_{\mathbf{K}_{n,\tau}(\Delta\mathbf{u})}[\Delta\mathbf{u}^n] \tag{7.19}$$

where $\mathbf{P}_{\mathbf{K}_{n,\tau}(\Delta\mathbf{u})}$ is the operator of projection in $\mathbf{H}^{1/2}(\Omega)$ on $\mathbf{K}_{n,\tau}(\Delta\mathbf{u})$,

6. proceed to the next step of iteration.

Algorithm 4 includes the following steps:

1. specify an initial value $\mathbf{q}^0 \in \mathbf{K}^{(}_{n,\tau}\mathbf{q})$,
2. calculate $\Delta\mathbf{u}^n$ substituting known $\mathbf{q}^n$ in the internals equation

$$\Delta\mathbf{u}^n = \mathbf{F}^{-1}\cdot\mathbf{p}^n \tag{7.20}$$

3. correct the quantity $\Delta\mathbf{u}^n$ to satisfy the constraints

$$\Delta\mathbf{u}^{n+1} = \mathbf{P}_{\mathbf{K}_{n,\tau}(\Delta\mathbf{u})}[\Delta\mathbf{u}^n] \tag{7.21}$$

where $\mathbf{P}_{\mathbf{K}_{n,\tau}(\Delta\mathbf{u})}$ is the operator of projection in $\mathbf{H}^{1/2}(\Omega)$ on $\mathbf{K}_{n,\tau}(\Delta\mathbf{u})$,

4. calculate $\mathbf{q}^n$ substituting known $\Delta\mathbf{u}^n$ in the internals equation

$$\mathbf{q}^n = \mathbf{F} \cdot \Delta\mathbf{u}^n - \mathbf{p}^0 \tag{7.22}$$

5. correct the quantity $\mathbf{q}^n$ to satisfy the constraints

$$\mathbf{q}^{n+1} = \mathbf{P}_{\mathbf{K}^c_{n,\tau}(\mathbf{q})}[\mathbf{q}^n] \tag{7.23}$$

where $\mathbf{P}_{\mathbf{K}^c_{n,\tau}(\mathbf{q})}$ is the operator of projection in $\mathbf{H}^{-1/2}(\Omega)$ on $\mathbf{K}^c_{n,\tau}(\mathbf{q})$,

6. proceed to the next step of iteration.

Our calculations show that all four above algorithms are convergent not only in elastostatics but also elastodynamics contact problems for infinite cracked body. It is important to mention that Algorithm 3 and Algorithm 4 have significantly faster convergence.

8 Conclusions

This paper present various variational formulation of elastostatic problem for body with crack with considering possibility for unilateral crack faces contact interaction and friction in the for that permit easy extension to elastodynamic problems. Variational formutations based on principles of virtual displacements and virtual stresses and also boundary variational principle based on fundamental solutions are established. Nonsmooth functionals that correspond to unilareral frictional contact conditions are constructed. Iterative algorithms of the Uzawa's type that are based on projection on the set of unilateral restrictions and friction are proposed. It was shown that in the case if varational formulation is based on principles formulated for whole domain the FEM have to be used and in the case if varational formulation is based on principles formulated only for boundary the BEM have to be used. The case of the crack in infinite elastic media is considered in more details and four new algorithms are proposed.

References

Anderson T.L. (1995): *Fracture Mechanics. Fundamentals and Applications*, CRC Press Boca Raton, London, New York.

Antes H.; Panagiotopoulos P.D. (1992): *The boundary integral approach to static and dynamic contact problems*, Birkhauser, Basel-Boston-Berlin.

Atluri S.N.; Liu H.T.; Han Z.D. (2006a): Meshless Local Petrov-Galerkin (MLPG) Mixed Finite Difference Method for Solid Mechanics, *CMES:Computer Modeling in Engineering & Sciences*, vol. 15, no. 1, pp. 1-16.

Atluri S.N.; Liu H.T.; Han Z.D. (2006b): Meshless local Petrov-Galerkin (MLPG) mixed collocation method for elasticity problems, *CMES:Computer Modeling in Engineering & Sciences*, vol. 14, no. 3, pp. 141-152.

Balas J.; Sladek J.; Sladek V. (1989): *Stress analysis by boundary element methods,* Elsevier, Amsterdam.

Cea J. (1971): *Optimization. Teorie et algorithmes.* Dunod, Paris, (in French),

Eck C.; Jarusek J.; Krbec M. (2005): *Unilateral Contact Problems. Variational Methods and Existence Theorems*, CRC Press, Boca Raton London New York.

Ekeland I.; Temam R. (1975): *Convex analysis and variational problems*. North-Holland, Amsterdam.

Guz A.N.; Zozulya V.V. (1993): *Brittle fracture of constructive materials under dynamic loading*, Naukova Dumka, Kiev. (in Russian).

Guz A.N.; Zozulya V.V. (2001): Fracture dynamics with allowance for a crack edges contact interaction. *International Journal of Nonlinear Sciences and Numerical Simulation*, vol 2, pp. 173-233.

Guz A.N.; Zozulya V.V. (2002): Elastodynamic unilateral contact problems with friction for bodies with cracks. *International Applied Mechanics*, vol. 38, no. 8, pp. 895-932.

Guz A.N.; Menshykov O.V.; Zozulya V.V.; Guz I.A. (2007): Contact problem for the flat elliptical crack under normally incident shear Wave. CMES: Computer Modeling in Engineering & Science, vol. 17, no. 3, pp. 205-214.

Guz A.N.; Zozulya V.V. (2007): Investigation of the effect of frictional contact in III-mode crack under action of the SH-wave harmonic load. CMES: Computer Modeling in Engineering & Science, vol. 22, no. 2, pp. 119-128.

Han, Z.D.; Atluri, S.N. (2007): A Systematic Approach for the Development of Weakly-Singular BIEs. *CMES: Computer Modeling in Engineering & Sciences*, vol. 21, no. 1, pp. 41–52.

Karlis, G.; Tsinopoulos, S.; Polyzos, D.; Beskos, D. (2008): 2D and 3D boundary element analysis of mode-I cracks in gradient elasticity. *CMES: Computer Modeling in Engineering & Sciences*, vol. 26, no. 3, pp. 189-207.

Keppas L.K., Giannopoulos G.I. , Anifantis N.K. (2008): Transient Coupled Thermoelastic Contact Problems Incorporating Thermal Resistance: a BEM Approach, *CMES: Computer Modeling in Engineering & Sciences*, 25 (3), 181-196.

Khludnev A.M.; Kovtunenko V.A. (1999): *Analysis of cracks in solids*, WIT Press, Southampton, Boston.

Kikuchi, N.; Oden, J.T. (1987): *Contact problems in elasticity*, SIAM, Publications, Philadephia,.

Kravchuk A.S.; Neittaanmaki P.J. (2007): *Variational and Quasi-Variational Inequalities in Mechanics*, Springer, Netherlands.

Panagiotopoulos P.D. (1985): *Inequality problems in mechanics and applications. Convex and non convex energy functions*, Birkhauser, Stuttgart.

Polizzotto C. (1991): A boundary min – max principle as a tool for boundary element formulations, *Engineering Analysis with Boundary Elements*, vol. 8, no. 2, p. 89-93.

Polizzotto C. (1993): Variational boundary-integral-equation approach to unilateral contact problems in elasticity, *Computational Mechanics*, vol. 13, no. 1-2, p. 100-115.

Sanz, J.; Solis, M.; Dominguez, J. (2007): Hypersingular BEM for piezoelectric solids: formulation and applications for fracture mechanics. *CMES: Computer Modeling in Engineering & Sciences*, vol. 17, no. 3, pp. 215-229.

Shillor M.; Sofonea M.; Telega J.J. (2004): *Models and Analysis of Quasistatic Contact. Variational Methods*, Springer-Verlag Berlin Heidelberg.

Sladek, J.; Sladek, V.; Zhang, C. (2007): Fracture analyses in continuously nonhomogeneous piezoelectric solids by the MLPG. *CMES: Computer Modeling in Engineering & Sciences*, vol. 19, no. 3, pp. 247–262.

Shu Li; Atluri S.N. (2008a): Topology-optimization of structures based on the MLPG mixed collocation method. *CMES: Computer Modeling in Engineering & Science*, vol. 26, no. 1, pp. 61-74.

Shu Li; Atluri S.N. (2008b): The MLPG mixed collocation method for material orientation and topology optimization of anisotropic solids and structures. *CMES: Computer Modeling in Engineering & Science*, vol. 30, no. 1, pp. 37-56.

Tan C.L.; Shiah Y.C.; Lin C.W. (2009): Stress analysis of 3D generally anisotropic elastic solids using the boundary element method. *CMES: Computer Modeling in Engineering & Science*, vol. 41, no. 3, pp. 195-214.

Theocaris P.S.; Panagiotopoulos P.D. (1992): On the consideration of unilateral contact and friction in cracks. The boundary integral method, *International Journal for Numerical Method in Engineering*, vol. 35, 1697-1708

Washizu K. (1982): *Variational Methods in Elasticity and Plasticity*, Pergamon Press, New York.

Wriggers P. (2006): *Computational Contact Mechanics*, Springer, Berlin, Heidelberg, New York.

Young D.L.; Chen K.H.; Liu T.Y.; Shen L.H.; Wu C.S. (2009): Hypersingular meshless method for solving 3D potential problems with arbitrary domain. *CMES: Computer Modeling in Engineering & Science,* vol. 40, no. 3, pp. 225-269.

Yueting Zhou; Xing Li; Dehao Yu (2008): Integral method for contact problem of bonded plane material with arbitrary cracks. *CMES: Computer Modeling in Engineering & Science*, vol. 36, no. 2, pp. 147-172.

Zhang, Y.; Chen, L. (2008): A simplified meshless method for dynamic crack growth. *CMES: Computer Modeling in Engineering& Sciences*, vol. 31, no. 3, pp. 189-200.

Zhou Y.; Li X.; Dehao Yu D. (2008): Integral Method for Contact Problem of Bonded Plane Material with Arbitrary Cracks. *CMES: Computer Modeling in Engineering & Sciences*, vol. 36, no. 2, pp. 147-172.

Zozulya V.V. (1990): On solvability of the dynamic problems in theory of cracks with contact, friction and sliding domains, *Doklady Akademii Nauk Ukrainskoy SSR*, vol. 3, pp.53-55. (in Russian).

Zozulya V.V. (1992): Method of boundary functionals in contact problems of dynamics of bodies with cracks, *Doklady Akademii Nauk Ukraine*, vol. 2, pp. 38-44. (in Russian).

Zozulya V.V. (1992): Solution of the problems on dynamics of bodies with cracks by the method of boundary integral equations, *Doklady Akademii Nauk Ukraine*, vol.3, pp. 38-43. (in Russian).

Zozulya V.V. (2001): Variational principles and algorithms in contact problem with friction, in: *Advances in Scientific Computing, Computational Intelligence and Applications. Eds . N. Mastorakis, V. Mladenov, B. Suter, L.J. Wang, WSES Press, Danvers,* pp. 181-186.

Zozulya V.V. (2001): Variational principles and algorithms in elastodynamic contact problem with friction, in: *Advances in Computational and Engineering Sciences, Eds. S.N. Atluri, M., Nishioka, M. Kikuchi, Tech. Science Press, Puerto Vallarta, Mexico.*

Zozulya V.V. (2002): Fracture dynamics with allowance for crack edge contact interaction, in: *C. Constanda, P. Schiavone, and A. Mioduchowski, Integral Methods in Science and Engineering, Birkhauser, Boston*, pp. 257-262.

Zozulya V.V. (2003): Mathematical investigation of nonsmooth optimization algorithm in elastodynamic contact problems with friction for bodies with cracks, *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 4, no. 4, pp. 405-422.

Zozulya V.V. (2006): Regularization of the divergent integrals. I. General consideration. *Electronic Journal of Boundary Elements,* vol. 4, no 2, pp. 49-57.

Zozulya V.V. (2006): Regularization of the divergent integrals. II. Application in Fracture Mechanics. *Electronic Journal of Boundary Elements,* vol. 4, no. 2, pp. 58-56.

Zozulya V.V.; Gonzalez-Chi P. I. (2000): Dynamic fracture mechanics with crack edges contact interaction, *Engineering Analysis with Boundary Elements,* vol. 24, no. 9, pp. 643-659

Zozulya V.V.; Menshykov O.V. (2003): Use of the Constrained Optimization Algorithms in Some Problems of Fracture Mechanics, *Optimization and Engineering*, vol. 4, pp. 365-384.

Zozulya V.V.; Menshykova M.V. (2002): Study of iterative algorithms for solution of dynamic contact problems for elastic cracked bodies, *International Applied Mechanics*, vol. 38, no. 5, pp. 573-577.

Zozulya V.V.; Rivera P. (2000): Boundary integral equations and problem of existence in contact problems with friction, *Journal of the Chinese Institute of Engineers*, vol. 3, no. 3, pp. 313-320.