 CMES, vol.43, no.1, pp.27-71, 2009

Derivation of a Bilayer Model for Shallow Water Equations with Viscosity. Numerical Validation

G. Narbona-Reina[1], J.D.D. Zabsonré [2], E.D. Fernández-Nieto[1] and D. Bresch [3]

Abstract: In this work we present a new two-dimensional bilayer Shallow-Water model including viscosity and friction effects on the bottom and interface level. It is obtained following [Gerbeau and Perthame (2001)] from an asymptotic analysis of non-dimensional and incompressible Navier-Stokes equations with hydrostatic approximation. In order to obtain the viscosity effects into the model we must have into account a second order approximation. To evaluate this model we perform two numerical tests consisting of an internal dam-break problem for both, one and two dimensional cases. In the first one we make a comparison between the model obtained and the Navier-Stokes simulation.

Keywords: Shallow Water equations, bilayer models, viscosity, friction, capillarity, Finite Volume methods.

1 Introduction

The goal of this paper is the derivation of a new viscous bilayer Shallow Water model. We also present some numerical test with the aim of checking its validity. So we shall compare the solution obtained by this new model with the Navier-Stokes solution.

The Shallow Water (SW) equations are usually used to simulate a large number of geophysical and engineering applications as ocean circulation, coastal areas, rivers, etc. But sometimes these equations are not sufficient to model specific situations as, for instance, the flow involved in the Strait of Gibraltar. In this physical domain two layers of water with different properties are founded, the denser Mediterranean

[1] Dpto. Matemática Aplicada I, E.T.S. Arquitectura. Universidad de Sevilla. Avda. Reina Mercedes 2. 41012 Sevilla, Spain, (gnarbona@us.es, edofer@us.es)
[2] Université Polytechnique de Bobo-Dioulasso, 01BP 1091 Bobo 01, jzabsonre@univ-ouaga.bf
[3] LAMA, UMR5127 CNRS, Université de Savoie, 73376 Le Bourget du Lac, France, Didier.Bresch@univ-savoie.fr

and the Atlantic water. So in this case we must consider a model of at least two layers. For this purpose we can find several derivations of one and two layers SW models to tackle these kind of situations in one and two dimensional case.

Usually there are two effects that are neglected in Shallow Water models: the viscosity and the Coriolis force. On the one hand it is important to remark that the viscous effects are essential to obtain an accurate approximation in dam-breaks problems or hydraulic jumps situations, as we can see in [Gerbeau and Perthame (2001); Marche (2005)]. On the other hand the Coriolis force plays an important role in geophysical fluid dynamics applications. To include this force in equations does not add any meaningful difficulty for the deduction of the model. Nevertheless its effects are of major importance in these kind of applications, see for example [Lucas (2007); Wacher and Givoli (2006)]. The objective is to find a bilayer model as complete as possible that takes the Coriolis force into consideration, includes the viscosity effect, friction -at the bottom and at the interface- and tension effect -on the surface and the interface-.

The pioneer work performed in [Gerbeau and Perthame (2001)] has been considered as a basis to develop the deduction of our model. In this work a viscous Shallow Water model is obtained for the one-dimensional one-layer case by performing the asymptotic analysis of the Navier-Stokes equations where friction effect at the bottom has been taken into account. When only first order approximation is considered the viscous terms doesn't appear in the equations, so moving on to the second order is needed to get the viscosity effects. The authors also put in evidence the difference between two approximations through an application to a dam-break problem.

In [Marche (2005)], a viscous one layer 2D Shallow-Water system is derived. The originality of that work is the introduction of a surface-tension term through the capillary effects at the free surface and quadratic friction term at the bottom. These terms have been useful to establish the existence of global weak solutions in [Bresch and Desjardins (2003)].

With regard to bilayer models we must mention the work performed in [Peybernes (2006)] and those developed in [Audusse (2005)].

In [Peybernes (2006)], it is deduced a bilayer viscous Shallow-Water model which takes into account the friction at the interface. But instead of asymptotic analysis development, several simplifications are used in the boundary conditions to deduce the final system. The energy of the system is also obtained under restrictive hypothesis on the data.

In [Audusse (2005)], a derivation of a multi-layer Shallow-Water model is performed to extend the case of one layer established in [Gerbeau and Perthame (2001)].

In this work, using the hydrostatic pressure and the kinematic boundary conditions, the author derives momentum equations of the form:

$$\partial_t \int_{H_{\alpha-1}}^{H_\alpha} u dz + \partial_x \int_{H_{\alpha-1}}^{H_\alpha} u^2 dz + g h_\alpha \partial_x h = \\ = \frac{\nu_0}{\varepsilon} \partial_z u(H_\alpha(t,x)) - \frac{\nu_0}{\varepsilon} \partial_z u(H_{\alpha-1}(t,x))$$

and use at the leading order a finite difference method with respect to the vertical variable when the equation is an interface equation to deduce the friction term:

$$\mu \partial_z u(H_\alpha) = \mu \frac{U_{\alpha+1} - U_\alpha}{h_{\alpha+1} + h_\alpha}. \tag{1}$$

Another works related to the derivation of 2D Shallow-Water model can be find in [Ferrari and Saleri (2004)] and in [García Rodríguez (2005)].

In [Ferrari and Saleri (2004)] the authors include the atmospheric pressure in the derivation. In [García Rodríguez (2005)] a non-viscous two-layer Shallow Water system is deduced following [Gerbeau and Perthame (2001)]. Linear friction conditions have been taken into account on the interface and on the bottom.

The deduction of the bilayer model developed in the present work has been obtained by integrating the three dimensional Navier-Stokes equations with Coriolis force and by using the asymptotic analysis to get the viscosity effects. We have also considered the friction effects on the surface and at the bottom and the tension effect on the surface and at the interface.

We would like to remark that the friction term usually carries some difficulties in its treatment (both in the model deduction and for the proof of the existence of solution). So we can find several definitions to avoid these troubles. Often one takes non linear friction terms, for example in [Song, McFarland, Bergman, and Vakakis (2005)] the tangential displacement has been considered in the definition of the friction traction or a more complex expression can be found in [Ozaki, Hashiguchi, Okayasu, and Chen (2007)] where a microscopic study has been tackled to set it. In [Marche (2005)] a non linear friction under the form $-\gamma|v|^\alpha v$ is considered at the bottom. So, in a equivalent way we could define the interface friction as $-\gamma|v_1 - v_2|(v_1 - v_2)$.

In [Zabsonré and Narbona-Reina (2009)] we have performed the theoretical analysis of the model presented in this work, obtaining the existence of weak solutions. In order to get it we have not been able to control the interface friction of quadratic form, so we have defined

$$fric(v_1, v_2) = -\xi B(h_1, h_2)(v_1 - v_2), \quad B(h_1, h_2) = \frac{h_1 h_2}{\frac{\rho_1}{\rho_2} h_1 + \frac{\rho_2}{\rho_1} h_2}, \tag{2}$$

with ρ_i the density of each layer and ξ a positive constant. So to be in agreement with these results we have considered in this paper the linear definition of the friction term (2).

The drag coefficient B is also used in [Ishii and Hibiki (2006)] and allows us to control the friction term. Note that in [Chueshov, Raugel, and Rekalo (2005)] the authors study a system of 3D Navier-Stokes equations in a two-layer thin domain with an interface condition

$$(\nu_i \partial_3 u^i_j - k(u^1_j - u^2_j))|_{x_3=0} = 0 \qquad i,j = 1,2. \tag{3}$$

This condition is the same kind condition appearing in the Primitive Equations of the Coupled Atmosphere and Ocean which describes the atmosphere/ocean interaction.

The paper is organized as follows: In Section 2 we develop the derivation of the model. First we write the equations in non-dimensional variables, later we perform the hydrostatic approximation to obtain the Shallow Water equations and finally we study the asymptotic analysis of two layers. In the last part of the section we state the final systems found and we include some remarks about them.

Section 3 is devoted to show some numerical test in which we notice the improvement obtained in the solution when considering viscous Shallow Water model in front of considering the first order one. In Test 1 we compare them with a solution of the Navier-Stokes equations for an internal dam-break problem in one-dimensional case. In the second test we present a 2D circular dam-break problem comparing the solutions of the models deduced in this work.

2 Derivation of the bilayer viscous Shallow Water model

In this section we perform the derivation of the model proposed in this paper. We start from the Navier-Stokes equations in a periodic domain $\mathscr{D}(t) \in \mathbb{R}^3$.

We consider a two layer environment of inmiscible fluids including three boundary regions. We assume that the bottom is defined by the function $b(x)$, and we denote by $\eta_{1,2}(t,x)$ the interface and the free surface is given by $\eta(t,x)$. The vertical direction is denoted by z and by x we denote a point in a domain $\chi \subset \mathbb{R}^2$. So the global domain is $\mathscr{D}(t) = \mathscr{D}_1(t) \cup \mathscr{D}_2(t) \cup \Gamma_b \cup \Gamma_{1,2}(t) \cup \Gamma_s(t)$, being:

$$\begin{aligned}
&\mathscr{D}_1(t) = \{(x,z) \in \mathbb{R}^3 / x \in \chi,\, b(x) < z < \eta_{1,2}(t,x)\};\\
&\mathscr{D}_2(t) = \{(x,z) \in \mathbb{R}^3 / x \in \chi,\, \eta_{1,2}(t,x) < z < \eta(t,x)\};\\
&\Gamma_b = \{(x,z) \in \mathbb{R}^3 / x \in \chi,\, z = b(x)\};\\
&\Gamma_{1,2} = \{(x,z) \in \mathbb{R}^3 / x \in \chi,\, z = \eta_{1,2}(t,x)\};\\
&\Gamma_s = \{(x,z) \in \mathbb{R}^3 / x \in \chi,\, z = \eta(t,x)\};
\end{aligned} \tag{4}$$

From now on, subscript 1 will correspond to the layer located below and subscript 2 to those located on the top. We denote by $h_1(t,x) = \eta_{1,2}(t,x) - b(x)$ the thickness of the layer 1 and by $h_2(t,x) = \eta(t,x) - \eta_{1,2}(t,x)$ the thickness of the second one. See Fig. 1.

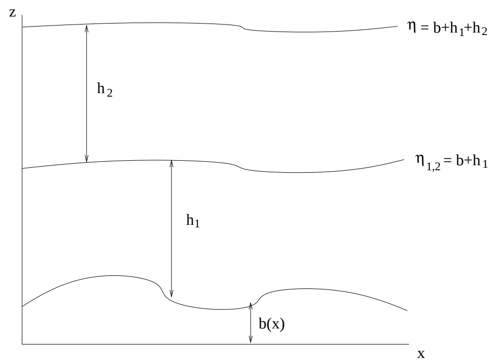

Figure 1: Domain.

We consider $u_i = (v_i, w_i)$ the velocity of each layer, ρ_i the density, μ_i denote the dynamic viscosity and p_i is the pressure, for $i = 1,2$. With this notation, the Navier-Stokes equations for each layer $i = 1,2$ state as:

$$\begin{cases} \rho_i \partial_t u_i + (\rho_i u_i \nabla) u_i - \text{div } (\sigma_i) + 2\rho_i \overrightarrow{\Omega} \times u_i = -\rho_i g e_z; \\ \text{div } (u_i) = 0. \end{cases} \tag{5}$$

We have included the Coriolis force given by the term $2\rho_i \overrightarrow{\Omega} \times u_i$ where $\overrightarrow{\Omega} = \Omega(0, \cos\theta, \sin\theta)$, being θ the latitude. The stress tensor is defined as $\sigma_i = 2\mu_i D(u_i) - p_i Id$ where $D(u) = \frac{\nabla u + \nabla^\perp u}{2}$ is the strain tensor. Finally g is the gravity constant. We denote by subscript n the normal component and by τ the tangent component, that is, $d = d_n n + d_\tau \tau$ for all $d \in \mathbb{R}^2$. In order to obtain a well-posed system we impose the following conditions on the boundaries:

- On the free surface $z = \eta(t,x)$:

  We assume the pressure to be constant. One usually neglect the atmospheric pressure effect but here we have considered it. If we denote by n_s the unit normal vector to $\eta(t,x)$ towards the increasing z and by κ the mean curvature of the surface ($\kappa = \text{div } (n_s)$), the tension effect on the surface is given by:

$$\sigma_2 \cdot n_s = \alpha_2 \kappa \cdot n_s, \tag{6}$$

being α_2 constant.

Finally we impose the kinematic condition for the surface:

$$\partial_t \eta + v_2 \cdot \nabla_x \eta = w_2. \tag{7}$$

- At the interface $z = \eta_{1,2}(t,x)$:

  First we consider the conditions related to the movement of the interface that is advected by the two flows, i.e., the kinematic condition that we write as:

  $$\partial_t \eta_{1,2} + v_i \cdot \nabla_x \eta_{1,2} = w_i; \quad i = 1,2. \tag{8}$$

  At the interface we consider the friction effects between the two layers and the continuity of the tension force. These conditions concern the tangent and normal components of the stress term respectively in the following sense. We consider the friction term between the two layers with coefficient γ done by $-\gamma(u_1 - u_2)$, so we impose:

  $$(\sigma_i \cdot n_{1,2})_\tau = -\gamma(u_1 - u_2)_\tau; \quad i = 1,2. \tag{9}$$

  We now consider $\kappa_{1,2} = \text{div}\ (n_{1,2})$ the mean curvature at the interface being $n_{1,2}$ the unit normal vector to $\Gamma_{1,2}$ pointing from layer 1 to layer 2, so we have:

  $$(\sigma_1 \cdot n_{1,2})_n = (\sigma_2 \cdot n_{1,2})_n + ((\alpha_1 - \alpha_2)\kappa_{1,2} \cdot n_{1,2})_n, \tag{10}$$

  being α_1 constant.

- At the bottom $z = b(x)$:

  We consider a Navier condition with a friction coefficient α:

  $$(\sigma_1 \cdot n_b)_\tau = \alpha(u_1)_\tau, \tag{11}$$

  and a no-penetration condition:

  $$u_1 \cdot n_b = 0; \tag{12}$$

  being n_b the unit normal vector to Γ_b pointing to the increasing z.

To obtain the model, first we shall write these equations under a non-dimensional form, secondly we shall develop the vertical integration in each layer to obtain the Shallow Water system. Finally we shall perform the asymptotic analysis studding both first and second order approximations. Therefore, two models are proposed depending if the viscous and friction terms are included or not.

2.1 Dimensionless equations

Before changing equations to non-dimensional variables, we shall write the Navier-Stokes equations and the boundary conditions in a explicit form. First we look at the Coriolis term that reads:

$$\vec{\Omega} \times u_i = \Omega(w_i \cos\theta - (v_i \sin\theta)e_2, v_i \sin\theta e_1, -v_i \cos\theta e_1). \tag{13}$$

If we write the equations for each component of the velocity, we have:

$$\begin{aligned} &\rho_i \partial_t v_i + \rho_i v_i \nabla_x v_i + \rho_i w_i \partial_z v_i + 2\rho_i \Omega(w_i \cos\theta e_1 + (v_i)^{\perp} \sin\theta) - \\ &-2\mu_i \text{div}_{\,x}(D_x(v_i)) - \mu_i \partial_z^2 v_i - \mu_i \nabla_x(\partial_z w_i) + \nabla_x p_i = 0, \qquad i = 1,2; \end{aligned} \tag{14}$$

$$\begin{aligned} &\rho_i \partial_t w_i + \rho_i v_i \nabla_x w_i + \rho_i w_i \partial_z w_i - 2\rho_i \Omega v_i \cos\theta e_1 - \mu_i \Delta_x w_i - \\ &-\mu_i \partial_z(\text{div}_{\,x} v_i) - 2\mu_i \partial_z^2 w_i + \partial_z p_i = \rho_i g, \qquad i = 1,2; \end{aligned} \tag{15}$$

and

$$\text{div}_{\,x} v_i + \partial_z w_i = 0 \qquad i = 1,2. \tag{16}$$

Now, we explicitly write the boundary conditions concerning tension and friction terms using the definition of the stress tensor when needed.

- Free surface. We take the normal vector $n_s = \frac{1}{\sqrt{1+(\nabla_x \eta)^2}} \begin{pmatrix} -\nabla_x \eta \\ 1 \end{pmatrix}$, so the tension condition state at follows:

$$\begin{cases} (-2\mu_2 D_x(v_2) + p_2 + \alpha_2 \kappa)\nabla_x \eta + \mu_2(\nabla_x w_2 + \partial_z v_2) = 0; \\ -\mu_2(\nabla_x w_2 + \partial_z v_2)\nabla_x \eta + 2\mu_2 \partial_z w_2 - p_2 - \alpha_2 \kappa = 0. \end{cases} \tag{17}$$

- Interface. In this case the normal vector is

$$n_{1,2} = \frac{1}{\sqrt{1+(\nabla_x \eta_{1,2})^2}} \begin{pmatrix} -\nabla_x \eta_{1,2} \\ 1 \end{pmatrix},$$

  obtaining for the tension condition:

$$\begin{aligned} &(2\mu_1 D_x(v_1) - p_1)|\nabla_x \eta_{1,2}|^2 - 2\mu_1(\nabla_x w_1 + \partial_z v_1)\nabla_x \eta_{1,2} + \\ &+2\mu_1 \partial_z w_1 - p_1 = (2\mu_2 D_x(v_2) - p_2 + (\alpha_1 - \alpha_2)\kappa_{1,2})|\nabla_x \eta_{1,2}|^2 - \\ &-\mu_2(\nabla_x w_2 + \partial_z v_2)\nabla_x \eta_{1,2} + 2\mu_2 \partial_z w_2 - p_2 + (\alpha_1 - \alpha_2)\kappa_{1,2}. \end{aligned} \tag{18}$$

For the friction condition we must consider the tangent vector $\vec{\tau} = (\tau_1, \tau_2)$, that we define as follows:

$$\tau_1 = \frac{1}{|\nabla_x \eta_{1,2}|} \begin{pmatrix} \nabla_x^{\perp} \eta_{1,2} \\ 0 \end{pmatrix}, \tau_2 = \frac{1}{\sqrt{|\nabla_x \eta_{1,2}|^2 + |\nabla_x \eta_{1,2}|^4}} \begin{pmatrix} -\nabla_x \eta_{1,2} \\ -|\nabla_x \eta_{1,2}|^2 \end{pmatrix}. \tag{19}$$

So, the conditions for $i = 1,2$ are:

$$\begin{aligned} &\frac{1}{\sqrt{1+|\nabla_x \eta_{1,2}|^2}} \mu_i (\nabla_x w_i + \partial_z v_i) = -\gamma(v_1 - v_2); \\ &\frac{1}{\sqrt{1+|\nabla_x \eta_{1,2}|^2}} \big(-2\mu_i D_x(v_i) \nabla_x \eta_{1,2} + \mu_i (\nabla_x w_i + \partial_z v_i)(1 - |\nabla_x \eta_{1,2}|^2) + \\ &+2\mu_i \partial_z w_i \nabla_x \eta_{1,2} \big) = \gamma((v_1 - v_2) + (w_1 - w_2) \nabla_x \eta_{1,2}). \end{aligned} \tag{20}$$

- Bottom. In the same way, we consider the normal vector $n_b = \frac{1}{\sqrt{1+|\nabla_x b|^2}} \begin{pmatrix} -\nabla_x b \\ 1 \end{pmatrix}$. So for the no-penetration condition we obtain:

$$-v_1 \nabla_x b + w_1 = 0. \tag{21}$$

For the other one we take $\vec{\tau} = (\tau_1, \tau_2)$ being

$$\tau_1 = \frac{1}{|\nabla_x b|} \begin{pmatrix} \nabla_x^{\perp} b \\ 0 \end{pmatrix}, \quad \tau_2 = \frac{1}{\sqrt{|\nabla_x b|^2 + |\nabla_x b|^4}} \begin{pmatrix} -\nabla_x b \\ -|\nabla_x b|^2 \end{pmatrix}, \tag{22}$$

and the condition state as:

$$\frac{1}{\sqrt{1+|\nabla_x b|^2}} \mu_1 (\nabla_x w_1 + \partial_z v_1) = \alpha v_1; \tag{23}$$

$$\begin{aligned} &\frac{1}{\sqrt{1+|\nabla_x b|^2}} \big(2\mu_1 D_x(v_1) \nabla_x b - \mu_1 (\nabla_x w_1 + \partial_z v_1)(1 - |\nabla_x b|^2) - \\ &-2\mu_1 \partial_z w_1 \nabla_x b\big) = -\alpha(v_1 + w_1 \nabla_x b). \end{aligned} \tag{24}$$

We introduce now a small parameter $\varepsilon = \frac{H}{L}$ where H and L are two characteristics dimensions along the edges OZ and OX respectively. We also introduce some others characteristic dimensions: V for the horizontal velocity, $W = \varepsilon V$ for the vertical component of the velocity and $P = V^2$ for the pressure. Next, we consider the following dimensionless variables, we use the overline notation to denote them:

$$\begin{array}{lll} x = L\bar{x} & z = H\bar{z} & \\ v_i = V\overline{v_i} & w_i = \varepsilon V\overline{w_i} & \\ t = \dfrac{L}{V}\bar{t} & p_i = V^2\overline{p_i} & \\ Re_i = \dfrac{VL}{\mu_i} & Ro = \dfrac{V}{2L\Omega} & Fr = \dfrac{V}{\sqrt{gH}} \end{array} \tag{25}$$

$$\begin{array}{ll} \gamma = V\overline{\gamma} & \alpha = V\overline{\alpha} \\ \kappa = \dfrac{\varepsilon}{L}\overline{\kappa} & \kappa_{1,2} = \dfrac{\varepsilon}{L}\overline{\kappa_{1,2}} \\ \alpha_i = \dfrac{V^2 L}{\varepsilon}\overline{\alpha_i}; & i = 1,2 \\ b = H\bar{b}. & \end{array} \tag{26}$$

where we have denoted by *Re* the *Reynolds* number, *Ro* the *Rossby* number and *Fr* the *Froude* number.

Thus, the equations get as follows (we drop the "overline" notation for the sake of clarity):

$$\begin{aligned} &\rho_i \partial_t v_i + \rho_i v_i \nabla_x v_i + \rho_i w_i \partial_z v_i + \rho_i \frac{1}{Ro} \varepsilon w_i \cos\theta e_1 + \rho_i \frac{1}{Ro} (v_i)^{\perp} \sin\theta - \\ &- \frac{2}{Re_i} \text{div}_{\,x}(D_x(v_i)) - \frac{1}{Re_i}\frac{1}{\varepsilon^2} \partial_z^2 v_i - \frac{1}{Re_i} \nabla_x(\partial_z w_i) + \nabla_x p_i = 0; \end{aligned} \tag{27}$$

$$\begin{aligned} &\rho_i \partial_t w_i + \rho_i v_i \nabla_x w_i + \rho_i w_i \partial_z w_i - \rho_i \frac{1}{\varepsilon}\frac{1}{Ro} v_i \cos\theta e_1 - \frac{1}{Re_i} \Delta_x w_i - \\ &- \frac{1}{\varepsilon^2}\frac{1}{Re_i} \partial_z(\text{div}_{\,x} v_i) - 2\frac{1}{\varepsilon^2}\frac{1}{Re_i} \partial_z^2 w_i + \frac{1}{\varepsilon^2} \partial_z p_i = -\rho_i \frac{1}{\varepsilon^2}\frac{1}{Fr^2}; \end{aligned} \tag{28}$$

$$\text{div}_{\,x} v_i + \partial_z w_i = 0. \tag{29}$$

In the same way the boundary conditions must be modified, we specify them next.

- Conditions at the free surface:

$$\partial_t \eta + v_2 \cdot \nabla_x \eta = w_2; \tag{30}$$

$$\left(-2\frac{1}{Re_2} D_x(v_2) + p_2 + \alpha_2 \kappa\right) \nabla_x \eta + \frac{1}{Re_2}\nabla_x w_2 + \frac{1}{\varepsilon^2}\frac{1}{Re_2}\partial_z v_2 = 0; \tag{31}$$

$$\left(-\varepsilon^2 \frac{1}{Re_2}\nabla_x w_2 - \frac{1}{Re_2}\partial_z v_2\right)\nabla_x \eta + 2\frac{1}{Re_2}\partial_z w_2 - p_2 - \alpha_2 \kappa = 0. \tag{32}$$

- Conditions at the interface:

$$\partial_t \eta_{1,2} + v_2 \cdot \nabla_x \eta_{1,2} = w_2; \tag{33}$$

$$\partial_t \eta_{1,2} + v_1 \cdot \nabla_x \eta_{1,2} = w_1; \tag{34}$$

$$\begin{aligned} &-\left(2\varepsilon^2 \frac{1}{Re_1}\nabla_x w_1 + 2\frac{1}{Re_1}\partial_z v_1\right)\nabla_x \eta_{1,2} + 2\frac{1}{Re_1}\partial_z w_1 - p_1 = \\ &= -\left(2\varepsilon^2 \frac{1}{Re_2}\nabla_x w_2 + 2\frac{1}{Re_2}\partial_z v_2\right)\nabla_x \eta_{1,2} + 2\frac{1}{Re_2}\partial_z w_2 - p_2 + \\ &+(\alpha_1 - \alpha_2)\kappa_{1,2}; \end{aligned} \tag{35}$$

$$\frac{1}{Re_i}(\nabla_x w_i + \frac{1}{\varepsilon^2}\partial_z v_i) = -\frac{1}{\varepsilon}\gamma(v_1 - v_2)\sqrt{1+\varepsilon^2|\nabla_x \eta_{1,2}|^2}; \tag{36}$$

$$\begin{aligned} &-2\frac{1}{Re_i} D_x(v_i)\nabla_x \eta_{1,2} + \frac{1}{Re_i}(\nabla_x w_i + \frac{1}{\varepsilon^2}\partial_z v_i)(1 - \varepsilon^2|\nabla_x \eta_{1,2}|^2) + \\ &+2\frac{1}{Re_i}\partial_z w_i \nabla_x \eta_{1,2} = \\ &= \frac{1}{\varepsilon}\gamma((v_1 - v_2) + \varepsilon^2(w_1 - w_2)\nabla_x \eta_{1,2})\sqrt{1+\varepsilon^2|\nabla_x \eta_{1,2}|^2}. \end{aligned} \tag{37}$$

- Conditions at the bottom:

$$-v_1\nabla_x b + w_1 = 0; \tag{38}$$

$$\frac{1}{Re_1}(\nabla_x w_1 + \frac{1}{\varepsilon^2}\partial_z v_1) = \frac{1}{\varepsilon}\alpha v_1\sqrt{1+\varepsilon^2|\nabla_x b|^2}; \tag{39}$$

$$\begin{aligned} &2\frac{1}{Re_1}D_x(v_1)\nabla_x b - \frac{1}{Re_1}(\nabla_x w_1 + \frac{1}{\varepsilon^2}\partial_z v_1)(1-\varepsilon^2|\nabla_x b|^2) - \\ &-2\frac{1}{Re_1}\partial_z w_1 \nabla_x b = -\alpha\frac{1}{\varepsilon}(v_1 + \varepsilon^2 w_1 \nabla_x b)\sqrt{1+\varepsilon^2|\nabla_x b|^2}. \end{aligned} \tag{40}$$

2.2 Hydrostatic approximation

We assume ε to be small and we take the formal expression of the system at $\mathcal{O}(\varepsilon^2)$, keeping the terms of order zero and one. Thus, the hydrostatic system state as follows:

$$\begin{aligned} &\rho_i\partial_t v_i + \rho_i \mathrm{div}_x(v_i\otimes v_i) + \rho_i\partial_z(v_i w_i) + \varepsilon\rho_i\frac{1}{Ro}w_i\cos\theta e_1 + \rho_i\frac{1}{Ro}(v_i)^{\perp}\sin\theta \\ &-\frac{2}{Re_i}\mathrm{div}_x(D_x(v_i)) - \frac{1}{\varepsilon^2}\frac{1}{Re_i}\partial_z^2 v_i - \frac{1}{Re_i}\nabla_x(\partial_z w_i) + \nabla_x p_i = 0; \end{aligned} \tag{41}$$

$$-\varepsilon\rho_i\frac{1}{Ro}v_i\cos\theta e_1 - \frac{1}{Re_i}\partial_z(\mathrm{div}_x v_i) - 2\frac{1}{Re_i}\partial_z^2 w_i + \partial_z p_i = -\rho_i\frac{1}{Fr^2}; \tag{42}$$

$$\mathrm{div}_x v_i + \partial_z w_i = 0. \tag{43}$$

And boundary conditions:

- Conditions at the free surface:

$$\partial_t\eta + v_2\nabla_x\eta = w_2; \tag{44}$$

$$\left(-2\frac{1}{Re_2}D_x(v_2) + p_2 + \alpha_2\kappa\right)\nabla_x\eta + \frac{1}{Re_2}\nabla_x w_2 = 0; \tag{45}$$

$$\partial_z v_2 = \mathscr{O}(\varepsilon); \tag{46}$$

$$-\frac{1}{Re_2}\partial_z v_2 \nabla_x \eta + 2\frac{1}{Re_2}\partial_z w_2 - p_2 - \alpha_2 \kappa = 0. \tag{47}$$

- Conditions at the interface:

$$\partial_t \eta_{1,2} + v_2 \cdot \nabla_x \eta_{1,2} = w_2; \tag{48}$$

$$\partial_t \eta_{1,2} + v_1 \cdot \nabla_x \eta_{1,2} = w_1; \tag{49}$$

$$\begin{aligned} &-2\frac{1}{Re_1}\partial_z v_1 \nabla_x \eta_{1,2} + 2\frac{1}{Re_1}\partial_z w_1 - p_1 = \\ &= -2\frac{1}{Re_2}\partial_z v_2 \nabla_x \eta_{1,2} + 2\frac{1}{Re_2}\partial_z w_2 - p_2 + (\alpha_1 - \alpha_2)\kappa_{1,2}; \end{aligned} \tag{50}$$

$$\frac{1}{Re_i}\partial_z v_i = \varepsilon fric(v_1, v_2); \tag{51}$$

$$\begin{aligned} &-2\frac{1}{Re_i}D_x(v_i)\nabla_x \eta_{1,2} + \frac{1}{Re_i}(\nabla_x w_i + \frac{1}{\varepsilon^2}\partial_z v_i) + 2\frac{1}{Re_i}\partial_z w_i \nabla_x \eta_{1,2} - \\ &-\frac{1}{Re_i}\partial_z v_i |\nabla_x \eta_{1,2}|^2 = -\frac{1}{\varepsilon} fric(v_1, v_2). \end{aligned} \tag{52}$$

Where we have denoted by $fric(v_1, v_2) = -\gamma(v_1 - v_2)$ the friction term between the two layers.

- Conditions at the bottom:

$$-v_1 \nabla_x b + w_1 = 0; \tag{53}$$

$$\frac{1}{Re_1}\partial_z v_1 = \varepsilon\alpha v_1; \tag{54}$$

$$\begin{aligned}&2\frac{1}{Re_1}D_x(v_1)\nabla_x b - \frac{1}{Re_1}(\nabla_x w_1 + \frac{1}{\varepsilon^2}\partial_z v_1) + \frac{1}{Re_1}\partial_z v_1|\nabla_x b|^2 - \\ &-2\frac{1}{Re_1}\partial_z w_1 \nabla_x b = -\frac{1}{\varepsilon}\alpha v_1.\end{aligned} \tag{55}$$

2.3 Shallow Water system and asymptotic analysis

To obtain the Shallow Water equations, we assume that the height is small with respect to the length of the domain, that is, $\varepsilon \ll 1$. We first integrate the equations for each layer. Then we shall perform the asymptotic analysis of the system by introducing an asymptotic regime hypotheses over the physical data.

We shall obtain the system at first order, but we will must analyze the second order to obtain a system with viscosity.

We first perform the integration of the layer 1.

We want to obtain the pressure value, so from equation (42) for $i = 1$:

$$\partial_z p_1 = -\rho_1\frac{1}{Fr^2} + \varepsilon\rho_1\frac{1}{Ro}\cos\theta v_1 e_1 + \frac{1}{Re_1}\partial_z(\mathrm{div}\,_x v_1) + 2\frac{1}{Re_1}\partial_z^2 w_1. \tag{56}$$

To get p_1, we integre this equations from z to $\eta_{1,2}$, with $z \in (b, \eta_{1,2})$,

$$\begin{aligned}&p_1(z) - p_1(\eta_{1,2}) = -\rho_1\frac{1}{Fr^2}(z-\eta_{1,2}) + \varepsilon\rho_1\frac{1}{Ro}\cos\theta e_1\int_{\eta_{1,2}}^{z} v_1 dz + \\ &+\frac{1}{Re_1}(\mathrm{div}\,_x v_1 - \mathrm{div}\,_x v_{1|z=\eta_{1,2}}) + 2\frac{1}{Re_1}(\partial_z w_1 - \partial_z w_{1|z=\eta_{1,2}}).\end{aligned} \tag{57}$$

By using the divergence free condition we obtain the following expression for p_1:

$$\begin{aligned}&p_1(z) = p_1(\eta_{1,2}) - \rho_1\frac{1}{Fr^2}(z-\eta_{1,2}) + \varepsilon\rho_1\frac{1}{Ro}\cos\theta e_1\int_{\eta_{1,2}}^{z} v_1 dz - \\ &-\frac{1}{Re_1}(\mathrm{div}\,_x v_1 - \mathrm{div}\,_x v_{1|z=\eta_{1,2}}).\end{aligned} \tag{58}$$

Now, we integrate the system equations from b to $\eta_{1,2}$. For the equation (43) and using conditions (49) and (53), we have:

$$\partial_t \eta_{1,2} + \mathrm{div}\,_x \int_b^{\eta_{1,2}} v_1 dz = 0. \tag{59}$$

If we integrate the equation for the horizontal velocity (41), and we use the condition (49) it gives:

$$\begin{aligned}
&\rho_1 \partial_t \int_b^{\eta_{1,2}} v_1 dz + \rho_1 \mathrm{div}_x \int_b^{\eta_{1,2}} (v_1 \otimes v_1) dz + \varepsilon \rho_1 \frac{1}{Ro} \cos\theta e_1 \int_b^{\eta_{1,2}} w_1 dz + \\
&+\rho_1 \frac{1}{Ro} \sin\theta \int_b^{\eta_{1,2}} (v_1)^{\perp} dz - 2\frac{1}{Re_1} \mathrm{div}_x \int_b^{\eta_{1,2}} D_x(v_1) dz + \nabla_x \int_b^{\eta_{1,2}} p_1 dz + \\
&+\frac{1}{\varepsilon^2}\frac{1}{Re_1} \partial_z v_{1|z=b} + \frac{1}{Re_1} \nabla_x w_{1|z=b} - p_1(\eta_{1,2}) \nabla_x \eta_{1,2} + \\
&+2\frac{1}{Re_1} D_x(v_1)_{|z=\eta_{1,2}} \nabla_x \eta_{1,2} - \frac{1}{\varepsilon^2}\frac{1}{Re_1} \partial_z v_{1|z=\eta_{1,2}} - \frac{1}{Re_1} \nabla_x w_{1|z=\eta_{1,2}} - \\
&-2\frac{1}{Re_1} D_x(v_1)_{|z=b} \nabla_x b + p_1(b) \nabla_x b = 0.
\end{aligned} \tag{60}$$

Due to condition (52), we can write

$$\begin{aligned}
&2\frac{1}{Re_1} D_x(v_1)_{|z=\eta_{1,2}} \nabla_x \eta_{1,2} - \frac{1}{\varepsilon^2}\frac{1}{Re_1} \partial_z v_{1|z=\eta_{1,2}} - \frac{1}{Re_1} \nabla_x w_{1|z=\eta_{1,2}} = \\
&= \frac{1}{\varepsilon} fric(v_1, v_2) + 2\frac{1}{Re_1} \partial_z w_{1|z=\eta_{1,2}} \nabla_x \eta_{1,2} - \frac{1}{Re_1} \partial_z v_{1|z=\eta_{1,2}} |\nabla_x \eta_{1,2}|^2.
\end{aligned} \tag{61}$$

And thanks to condition (55), we have

$$\begin{aligned}
&-2\frac{1}{Re_1} D_x(v_1)_{|z=b} \nabla_x b + \frac{1}{\varepsilon^2}\frac{1}{Re_1} \partial_z v_{1|z=b} + \frac{1}{Re_1} \nabla_x w_{1|z=b} = \\
&= \frac{1}{\varepsilon} \alpha v_{1|z=b} + \frac{1}{Re_1} \partial_z v_{1|z=b} |\nabla_x b|^2 - 2\frac{1}{Re_1} \partial_z w_{1|z=b} \nabla_x b.
\end{aligned} \tag{62}$$

So, we get for the first layer the following equation:

$$\begin{aligned}
&\rho_1 \partial_t \int_b^{\eta_{1,2}} v_1 dz + \rho_1 \mathrm{div}_x \int_b^{\eta_{1,2}} (v_1 \otimes v_1) dz + \varepsilon \rho_1 \frac{1}{Ro} \cos\theta e_1 \int_b^{\eta_{1,2}} w_1 dz + \\
&+\rho_1 \frac{1}{Ro} \sin\theta \int_b^{\eta_{1,2}} (v_1)^{\perp} dz - 2\frac{1}{Re_1} \mathrm{div}_x \int_b^{\eta_{1,2}} D_x(v_1) dz + \nabla_x \int_b^{\eta_{1,2}} p_1 dz - \\
&-p_1(\eta_{1,2}) \nabla_x \eta_{1,2} + p_1(b) \nabla_x b + \frac{1}{Re_1} (2\partial_z w_1 - \partial_z v_1 \nabla_x \eta_{1,2})_{|z=\eta_{1,2}} \nabla_x \eta_{1,2} - \\
&-\frac{1}{Re_1} (2\partial_z w_1 - \partial_z v_1 \nabla_x b)_{|z=b} \nabla_x b + \frac{1}{\varepsilon} fric(v_1, v_2) + \frac{1}{\varepsilon} \alpha v_{1|z=b} = 0.
\end{aligned} \tag{63}$$

Now, we calculate the integration for the upper layer in the same way that for layer one. From equation (42) for $i = 2$ we have:

$$\partial_z p_2 = -\rho_2 \frac{1}{Fr^2} + \varepsilon \rho_2 \frac{1}{Ro} \cos\theta e_1 v_2 + \frac{1}{Re_2} \partial_z (\mathrm{div}_x v_2) + 2\frac{1}{Re_2} \partial_z^2 w_2. \tag{64}$$

We integrate this equations from z to η with $z \in (\eta_{1,2}, \eta)$ to obtain the value of p_2, we have also used the condition (47):

$$p_2(z) = -\rho_2 \frac{1}{Fr^2}(z-\eta) - \varepsilon\rho_2 \frac{1}{Ro}\cos\theta e_1 \int_z^\eta v_2 - \frac{1}{Re_2}(\mathrm{div}_{\,x} v_{2|z=\eta} + \mathrm{div}_{\,x} v_2) - \\ - \varepsilon\alpha_2\kappa. \quad (65)$$

Integrating from $\eta_{1,2}$ to η the divergence equation, we obtain:

$$\partial_t(\eta - \eta_{1,2}) + \mathrm{div}_{\,x} \int_{\eta_{1,2}}^{\eta} v_2 = 0. \quad (66)$$

Now, we integrate equation (41):

$$\begin{aligned} &\rho_2 \partial_t \int_{\eta_{1,2}}^{\eta} v_2 dz + \rho_2 \mathrm{div}_{\,x} \int_{\eta_{1,2}}^{\eta} (v_2 \otimes v_2) dz + \varepsilon\rho_2 \frac{1}{Ro}\cos\theta e_1 \int_{\eta_{1,2}}^{\eta} w_2 dz + \\ &+\rho_2 \frac{1}{Ro}\sin\theta \int_{\eta_{1,2}}^{\eta} (v_2)^{\perp} dz - 2\frac{1}{Re_2}\mathrm{div}_{\,x} \int_{\eta_{1,2}}^{\eta} D_x(v_2) + \nabla_x \int_{\eta_{1,2}}^{\eta} p_2 dz + \\ &+2\frac{1}{Re_2} D_x(v_2)_{|z=\eta} \nabla_x \eta - \frac{1}{Re_2}\nabla_x w_{2|z=\eta} - p_2(\eta)\nabla_x\eta - \\ &-2\frac{1}{Re_2} D_x(v_2)_{|z=\eta_{1,2}} \nabla_x \eta_{1,2} + \frac{1}{\varepsilon^2}\frac{1}{Re_2}\partial_z v_{2|z=\eta_{1,2}} + \frac{1}{Re_2}\nabla_x w_{2|z=\eta_{1,2}} + \\ &+p_2(\eta_{1,2})\nabla_x \eta_{1,2} = 0. \end{aligned} \quad (67)$$

We use conditions (52) and (45) to get:

$$\begin{aligned} &-2\frac{1}{Re_2} D_x(v_2)_{|z=\eta_{1,2}} \nabla_x \eta_{1,2} + \frac{1}{\varepsilon^2}\frac{1}{Re_2}\partial_z v_{2|z=\eta_{1,2}} + \frac{1}{Re_2}\nabla_x w_{2|z=\eta_{1,2}} = \\ &= -\frac{1}{\varepsilon} fric(v_1, v_2) - 2\frac{1}{Re_2}\partial_z w_{2|z=\eta_{1,2}} \nabla_x \eta_{1,2} + \frac{1}{Re_2}\partial_z v_{2|z=\eta_{1,2}} |\nabla_x \eta_{1,2}|^2 \end{aligned} \quad (68)$$

and

$$2\frac{1}{Re_2} D_x(v_2)_{|z=\eta} \nabla_x \eta - \frac{1}{Re_2}\nabla_x w_{2|z=\eta} - p_2(\eta)\nabla_x \eta = \alpha_2 \kappa \nabla_x \eta. \quad (69)$$

So the equation for the second layer state as follows

$$\begin{aligned}
&\rho_2\partial_t \int_{\eta_{1,2}}^{\eta} v_2 dz + \rho_2 \mathrm{div}\,_x \int_{\eta_{1,2}}^{\eta} (v_2\otimes v_2)dz + \varepsilon\rho_2 \frac{1}{Ro}\cos\theta e_1 \int_{\eta_{1,2}}^{\eta} w_2 dz+\\
&+\rho_2\frac{1}{Ro}\sin\theta \int_{\eta_{1,2}}^{\eta} (v_2)^{\perp}dz - 2\frac{1}{Re_2}\mathrm{div}\,_x \int_{\eta_{1,2}}^{\eta} D_x(v_2) + \nabla_x \int_{\eta_{1,2}}^{\eta} p_2 dz-\\
&-\frac{1}{Re_2}(2\partial_z w_2 + \partial_z v_2 \nabla_x \eta_{1,2})_{|z=\eta_{1,2}} \nabla_x \eta_{1,2} + p_2(\eta_{1,2})\nabla_x\eta_{1,2}-\\
&-\frac{1}{\varepsilon} fric(v_1,v_2) + \alpha_2 \kappa \nabla_x \eta = 0.
\end{aligned} \tag{70}$$

2.3.1 Asymptotic analysis

We assume the problem to be in an asymptotic regime by supposing the following hypothesis over the data:

$$\frac{1}{Re_i} = \varepsilon\mu_{0i}, \quad \alpha = \varepsilon\alpha_0, \quad \alpha_i = \varepsilon\alpha_{0i}, \quad \gamma = \varepsilon\gamma_0. \tag{71}$$

We make the development of the unknowns up to order 2:

$$\begin{array}{ll}
v_1 = v_1^0 + \varepsilon v_1^1 + \mathscr{O}(\varepsilon^2); & v_2 = v_2^0 + \varepsilon v_2^1 + \mathscr{O}(\varepsilon^2);\\
w_1 = w_1^0 + \varepsilon w_1^1 + \mathscr{O}(\varepsilon^2); & w_2 = w_2^0 + \varepsilon w_2^1 + \mathscr{O}(\varepsilon^2);\\
p_1 = p_1^0 + \varepsilon p_1^1 + \mathscr{O}(\varepsilon^2); & p_2 = p_2^0 + \varepsilon p_2^1 + \mathscr{O}(\varepsilon^2);\\
\eta = \eta^0 + \varepsilon \eta^1 + \mathscr{O}(\varepsilon^2); & \eta_{1,2} = \eta_{1,2}^0 + \varepsilon \eta_{1,2}^1 + \mathscr{O}(\varepsilon^2).
\end{array} \tag{72}$$

So from now on we denote $fric_0(v_1,v_2) = -\gamma_0(v_1^0 - v_2^0)$. For the development of h_1, we have into account that $\eta_{1,2} = h_1 + b$, so we can write

$$h_1 = h_1^0 + \varepsilon h_1^1 + \mathscr{O}(\varepsilon^2), \tag{73}$$

where $h_1^0 = \eta_{1,2}^0 - b$ and $h_1^1 = \eta_{1,2}^1 - b$. In the same way we can write

$$h_2 = h_2^0 + \varepsilon h_2^1 + \mathscr{O}(\varepsilon^2), \tag{74}$$

being $h_2^0 = \eta^0 - \eta_{1,2}^0$ and $h_2^1 = \eta^1 - \eta_{1,2}^1$, remember that $\eta = \eta_{1,2} + h_2$.

We can approximate $\kappa_{1,2} = \Delta_x \eta_{1,2} + \mathscr{O}(\varepsilon^2)$ and $\kappa = \Delta_x \eta + \mathscr{O}(\varepsilon^2)$.

Now we perform the asymptotic analysis for the two layers. Firstly we study the first order approximation where the viscosity terms do not appear. The second order approximation is derived to obtain a viscous system. Due to the bilayer situation we must consider the influence of two friction terms, one on the bottom and other one on the interface. In [Gerbeau and Perthame (2001)] one can find how to make the correction of the velocity in order to obtain a modified coefficient for the friction at the bottom. But in our case there is a friction term at the interface to be taken into account, so we have performed a correction in both velocities to derive model.

1. First order approximation.

Layer $\mathscr{D}_1$.

If we consider the terms of the principal order (ε^0), we obtain from (41), (51) and (54) that:

$$\begin{aligned} &\partial_z^2 v_1 = \mathscr{O}(\varepsilon); \\ &\partial_z v_{1|z=\eta_{1,2}} = \mathscr{O}(\varepsilon); \\ &\partial_z v_{1|z=b} = \mathscr{O}(\varepsilon). \end{aligned} \tag{75}$$

From here, we deduce that v_1 does not depend on z at first order so we can write:

$$v_1^0(t,x,z) = v_1^0(t,x). \tag{76}$$

Under this hypothesis, we can rewrite the expressions above up to order one to obtain the final equation for layer 1 at first order. First, we write (59) as:

$$\partial_t \eta_{1,2}^0 + \mathrm{div}\,_x(h_1^0 v_1^0) = 0, \tag{77}$$

and from (58) we obtain the pressure:

$$p_1^0(z) = -\rho_1 \frac{1}{Fr^2}(z-\eta_{1,2}^0) + p_1^0(\eta_{1,2}^0). \tag{78}$$

But the term appearing in (63) involves the integral of the pressure, so we calculate it from (78):

$$\nabla_x \int_b^{\eta_{1,2}} p_1^0 dz = h_1^0 \nabla_x(p_1^0(\eta_{1,2}^0)) + p_1^0(\eta_{1,2}^0)\nabla_x(\eta_{1,2}^0 - b) + \frac{1}{2}\rho_1 \frac{1}{Fr^2}\nabla_x(h_1^0)^2. \tag{79}$$

If we take these values into equation (63), considering only principal order terms, we obtain:

$$\begin{aligned} &\rho_1 \partial_t((\eta_{1,2}^0 - b)v_1^0) + \rho_1 \mathrm{div}\,_x((\eta_{1,2}^0 - b)v_1^0 \otimes v_1^0) + \rho_1 \frac{1}{Ro}\sin\theta(\eta_{1,2}^0 - b)(v_1^0)^{\perp} + \\ &+ (\eta_{1,2}^0 - b)\nabla_x(p_1^0(\eta_{1,2}^0)) + p_1^0(\eta_{1,2}^0)\nabla_x(\eta_{1,2}^0 - b) + \frac{1}{2}\rho_1 \frac{1}{Fr^2}\nabla_x(b-\eta_{1,2}^0)^2 - \\ &- p_1^0(\eta_{1,2}^0)\nabla_x \eta_{1,2}^0 + p_1^0(b)\nabla_x b + fric_0(v_1,v_2) + \alpha_0 v_1^0 = 0. \end{aligned} \tag{80}$$

Now we simplify the pressure terms. We use the definition of p_1^0 (78) to write

$$-p_1^0(\eta_{1,2}^0) + p_1^0(b) = -\rho_1 \frac{1}{Fr^2}(b - \eta_{1,2}^0). \tag{81}$$

From (50) up to first order we have $p_1^0 = p_2^0 + \mathscr{O}(\varepsilon)$.
So the final equation reads:

$$\begin{aligned}&\rho_1 \partial_t (h_1^0 v_1^0) + \rho_1 \text{div}\,_x (h_1^0 v_1^0 \otimes v_1^0) + \rho_1 \frac{1}{Ro} \sin\theta h_1^0 (v_1^0)^{\perp} + h_1^0 \nabla_x (p_2^0(\eta_{1,2}^0)) \\ &+ \frac{1}{2}\rho_1 \frac{1}{Fr^2} \nabla_x (h_1^0)^2 + \rho_1 \frac{1}{Fr^2} h_1^0 \nabla_x b + fric_0(v_1, v_2) + \alpha_0 v_1^0 = 0.\end{aligned} \tag{82}$$

Layer $\mathscr{D}_2$.

Following the same way, we obtain the equations for the second layer.
Thus, from equations (41), (51) and (45) we can write:

$$\begin{aligned}&\partial_z^2 v_2 = \mathscr{O}(\varepsilon); \\ &\partial_z v_{2|z=\eta_{1,2}} = \mathscr{O}(\varepsilon); \\ &\partial_z v_{2|z=\eta} = \mathscr{O}(\varepsilon).\end{aligned} \tag{83}$$

So we can deduce that v_2 does not depend on z at order zero:

$$v_2^0(t,x,z) = v_2^0(t,x), \tag{84}$$

That allows us to write (66) as

$$\partial_t h_2^0 + \text{div}\,_x (h_2^0 v_2^0) = 0 \tag{85}$$

and from the pressure (65),

$$p_2^0(z) = -\rho_2 \frac{1}{Fr^2}(z - \eta^0) + \mathscr{O}(\varepsilon). \tag{86}$$

We integrate (86) to obtain the pressure term in equation (70):

$$\nabla_x \int_{\eta_{1,2}^0}^{\eta^0} p_2^0 = \frac{1}{2}\rho_2 \frac{1}{Fr^2} \nabla_x (h_2^0)^2. \tag{87}$$

So, the equation for the layer 2 at first order is:

$$\begin{aligned}&\rho_2 \partial_t (h_2^0 v_2^0) + \rho_2 \text{div}\,_x (h_2^0 v_2^0 \otimes v_2^0) + \rho_2 \frac{1}{Ro} \sin\theta h_2^0 (v_2^0)^{\perp} + \frac{1}{2}\rho_2 \frac{1}{Fr^2} \nabla_x (h_2^0)^2 + \\ &+ \rho_2 \frac{1}{Fr^2} h_2^0 \nabla_x \eta_{1,2}^0 - fric_0(v_1, v_2) = 0.\end{aligned} \tag{88}$$

Remark 2.1 *The equation for layer 1 given by (82) includes the value of the pressure of the second layer. Now, using (86) we can write the equation for $\mathscr{D}_1$ as follows:*

$$\rho_1\partial_t(h_1^0v_1^0)+\rho_1 div\,_x(h_1^0v_1^0\otimes v_1^0)+\rho_1\frac{1}{Ro}\sin\theta h_1^0(v_1^0)^{\perp}+\rho_2\frac{1}{Fr^2}h_1^0\nabla_x h_2^0+$$
$$+\frac{1}{2}\rho_1\frac{1}{Fr^2}\nabla_x(h_1^0)^2+\rho_1\frac{1}{Fr^2}h_1^0\nabla_x b-fric_0(v_1,v_2)+\alpha_0 v_1^0=0. \tag{89}$$

2. Second order approximation: Correcting the friction terms.

As we can see, there are no viscous terms involved in the equations above. To obtain a viscous system we must take into account the second order approximation. We perform the equivalent correction for the bottom friction presented in [Gerbeau and Perthame (2001)] but for the bilayer case.

We purpose an additional correction for the friction term at the interface in the equations obtained. For this aim first we are going to develop the second order approximation for each term in both layers equations, later we shall perform the velocities correction and we shall state the final model.

Layer $\mathscr{D}_1$.

Now we consider the approximation up to order 2 for the unknows:

$$\begin{aligned}
\widetilde{v_1}&=v_1^0+\varepsilon v_1^1,\\
\widetilde{p_1}&=p_1^0+\varepsilon p_1^1,\\
\widetilde{\eta_{1,2}}&=\eta_{1,2}^0+\varepsilon\eta_{1,2}^1,\\
\widetilde{h_1}&=h_1^0+\varepsilon h_1^1,
\end{aligned} \tag{90}$$

and we back to the equations writing them up to second order. First, for (43):

$$\partial_t\widetilde{h_1}+\text{div}\,_x(\widetilde{h_1}\widetilde{v_1})=\mathscr{O}(\varepsilon^2). \tag{91}$$

Now, we want to get an expression for v_1 with the aim of determine its average. We take equation (41) to principal order:

$$\rho_1\partial_t v_1+\rho_1\text{div}\,_x(v_1\otimes v_1)+\rho_1\partial_z(v_1w_1)+\rho_1\frac{1}{Ro}(v_1)^{\perp}\sin\theta-$$
$$-\frac{1}{\varepsilon}\mu_{01}\partial_z^2 v_1+\nabla_x p_1=0. \tag{92}$$

Thus,

$$\frac{1}{\varepsilon}\mu_{01}\partial_z^2 v_1 = \rho_1\partial_t v_1^0 + \rho_1 v_1^0\nabla_x v_1^0 + \rho_1\frac{1}{Ro}(v_1^0)^\perp \sin\theta + \nabla_x(p_1^0(\eta_{1,2}^0)) + \\ + \rho_1\frac{1}{Fr^2}\nabla_x\eta_{1,2}^0 + \mathcal{O}(\varepsilon). \quad (93)$$

From (82), we can write

$$h_1^0(\rho_1\partial_t v_1^0 + \rho_1 v_1^0\nabla_x v_1^0 + \rho_1\frac{1}{Ro}\sin\theta(v_1^0)^\perp + \nabla_x(p_2^0(\eta_{1,2}^0)) + \rho_1\frac{1}{Fr^2}\nabla_x\eta_{1,2}^0) = \\ = fric_0(v_1,v_2) - \alpha_0 v_1^0 \quad (94)$$

So we can simplify the expression of $\partial_z^2 v_1$:

$$\frac{1}{\varepsilon}\mu_{01}\partial_z^2 v_1 = \frac{1}{h_1^0}fric_0(v_1,v_2) - \frac{\alpha_0}{h_1^0}v_1^0 + \mathcal{O}(\varepsilon). \quad (95)$$

If we integrate this equation from b to z and we use condition (54) we obtain:

$$v_1 = \widetilde{v_1}_{|z=b} + \frac{\varepsilon}{\mu_{01}}\alpha_0 v_1^0(z-b) + \frac{\varepsilon}{\mu_{01}}(fric_0(v_1,v_2) - \alpha_0 v_1^0)\frac{(z-b)^2}{2h_1^0} + \mathcal{O}(\varepsilon^2). \quad (96)$$

From here, we can obtain the average of the velocity v_1:

$$\overline{v_1} = \frac{1}{h_1}\int_b^{\eta_{1,2}} v_1 = \left(1 + \varepsilon\frac{h_1^0\alpha_0}{3\mu_{01}}\right)\widetilde{v_1}_{|z=b} + \varepsilon\frac{h_1^0}{6\mu_{01}}fric_0(v_1,v_2) + \mathcal{O}(\varepsilon^2). \quad (97)$$

Note that $\overline{v_1\otimes v_1} = \overline{v_1}\otimes\overline{v_1} + \mathcal{O}(\varepsilon^2)$. For the sake of brevity we do not include the proof, (Cf. [Zabsonré (2008)] for details).
We must calculate the term for the pressure at first order, we back to equation (58):

$$p_1(z) = \widetilde{p_1}(\widetilde{\eta_{1,2}}) - \rho_1\frac{1}{Fr^2}(z - \widetilde{\eta_{1,2}}) + \varepsilon\rho_1\frac{1}{Ro}\cos\theta v_1^0 e_1(z - \eta_{1,2}^0) + \mathcal{O}(\varepsilon^2). \quad (98)$$

Thus the integral of the pressure reads:

$$\nabla_x\int_b^{\eta_{1,2}} p_1 = \frac{1}{2}\rho_1\frac{1}{Fr^2}\nabla_x(\widetilde{h_1})^2 - \frac{1}{2}\varepsilon\rho_1\frac{1}{Ro}\cos\theta e_1\nabla_x((h_1^0)^2 v_1^0) + \nabla_x(\widetilde{h_1}\widetilde{p_1}(\widetilde{\eta_{1,2}})). \quad (99)$$

Looking at (63), we need to obtain the integral of w_1^0, so we integrate the equation of free divergence

$$\int_b^{\eta_{1,2}} w_1^0 dz = h_1^0 v_1^0\nabla_x b - \frac{(h_1^0)^2}{2}\mathrm{div}\,_x v_1^0. \quad (100)$$

And finally, from equation (97) we obtain the velocity at the bottom:

$$\widetilde{v_1}_{|z=b} = \beta \overline{v_1} - \varepsilon \frac{h_1^0}{6\mu_{01}} \beta fric_0(v_1, v_2) + \mathscr{O}(\varepsilon^2), \tag{101}$$

where $\beta = \beta(h_1) = \left(1 + \varepsilon \frac{h_1^0 \alpha_0}{3\mu_{01}}\right)^{-1}$.

Layer $\mathscr{D}_2$.

As in the case of layer 1, we look for the second order to obtain the viscosity terms in the equation. We define the approximations of second order:

$$\begin{aligned} \widetilde{v_2} &= v_2^0 + \varepsilon v_1^2; \\ \widetilde{p_2} &= p_2^0 + \varepsilon p_2^1; \\ \widetilde{\eta} &= \eta^0 + \varepsilon \eta^1; \\ \widetilde{h_2} &= h_2^0 + \varepsilon h_2^1. \end{aligned} \tag{102}$$

So we obtain for the first equation:

$$\partial_t \widetilde{h_2} + \text{div}\,_x(\widetilde{h_2}\widetilde{v_2}) = \mathscr{O}(\varepsilon^2). \tag{103}$$

We write the equation (41) at principal order to obtain v_2,

$$\begin{aligned} \tfrac{1}{\varepsilon}\mu_{02}\partial_z^2 v_2 &= \rho_2 \partial_t v_2^0 + \rho_2 \text{div}\,_x(v_2^0 \otimes v_2^0) + \rho_2 v_2^0 \partial_z w_2^0 + \rho_2 \tfrac{1}{Ro} \sin\theta (v_2^0)^\perp \\ &+ \rho_2 \frac{1}{Fr^2} \nabla_x \eta^0 + \mathscr{O}(\varepsilon), \end{aligned} \tag{104}$$

and using (88),

$$h_2^0 \left(\rho_2 \partial_t v_2^0 + \rho_2 v_2^0 \nabla_x v_2^0 + \rho_2 \frac{1}{Ro} \sin\theta (v_2^0)^\perp + \rho_2 \frac{1}{Fr^2} \nabla_x \eta^0 \right) = fric_0(v_1, v_2), \tag{105}$$

from where we get

$$\frac{1}{\varepsilon}\mu_{02}\partial_z^2 v_2 = \frac{1}{h_2^0} fric_0(v_1, v_2). \tag{106}$$

Integrating this expression from $\eta_{1,2}^0$ to z and using (51), we find the expression for v_2:

$$v_2 = \widetilde{v_2}_{|z=\eta_{1,2}} + \frac{\varepsilon}{\mu_{02}} fric_0(v_1, v_2)(z - \eta_{1,2}^0) \left(1 - \frac{z - \eta_{1,2}^0}{2h_2^0}\right) + \mathscr{O}(\varepsilon^2), \tag{107}$$

that we integrate to obtain the average for the velocity of the second layer:

$$\overline{v_2} = \frac{1}{\eta - \eta_{1,2}} \int_{\eta_{1,2}}^{\eta} v_2 = \widetilde{v_2}_{|z=\eta_{1,2}} + \varepsilon \frac{1}{3\mu_{02}} h_2^0 fric_0(v_1, v_2) + \mathscr{O}(\varepsilon^2). \tag{108}$$

Again, $\overline{v_2 \otimes v_2} = \overline{v_2} \otimes \overline{v_2} + \mathscr{O}(\varepsilon^2)$.
The divergence condition allows us to know the integral of w_2^0:

$$\int_{\eta_{1,2}^0}^{\eta^0} w_2^0 = w_{2x=|\eta_{1,2}^0}^0 h_2^0 - \frac{1}{2}(h_2^0)^2 \text{div}\,_x v_2^0, \tag{109}$$

and from (65) we get the integral of the pressure:

$$\begin{aligned}\nabla_x \int_{\eta_{1,2}^0}^{\eta^0} p_2 = \frac{1}{2}\rho_2 \frac{1}{Fr^2} \nabla_x (h_2^0)^2 - \varepsilon \rho_2 \frac{1}{2} \frac{1}{Ro} \cos\theta e_1 \nabla_x ((h_2^0)^2 v_2^0) - \\ -2\varepsilon \mu_{02} \nabla_x (h_2^0 \text{div}\,_x v_2^0) - \varepsilon \alpha_{02} \nabla_x (h_2^0 \Delta_x \eta^0).\end{aligned} \tag{110}$$

Now we perform the correction for the specifics friction terms. This correction is based in the same idea that we have developed for the layer 1 to obtain the value of the velocity $v_{1_{|z=b}}$ in function of the average $\overline{v_1}$ in equation (101). This will provide us the correction of the friction coefficient at the bottom.

For this aim we take the definition of the friction term at the interface:

$$fric(v_1, v_2) = -\gamma(v_1 - v_2), \quad \text{with } \gamma > 0. \tag{111}$$

Remember that for the asymptotic assumption we have taken $\gamma = \varepsilon\gamma_0$, and

$$fric_0(v_1, v_2) = -\gamma_0(v_1^0 - v_2^0).$$

On the contrary to the first case we make the correction at the same time for both layers.

Now we want to get a modified friction coefficient at the interface, the idea is to find a value of the difference of velocities $(v_1 - v_2)_{|z=\eta_{1,2}}$ in function of the averages $\overline{v_1}$ and $\overline{v_2}$ because this is the term appearing in the friction term (111). First we give the expression of $\overline{v_1}$ in function of $v_{1_{|z=\eta_{1,2}}}$. So we return to (95) that we write now as:

$$\frac{1}{\varepsilon}\mu_{01}\partial_z^2 v_1 = -\frac{1}{h_1^0}\gamma_0(v_1^0 - v_2^0) - \frac{\alpha_0}{h_1^0} v_1^0 + \mathscr{O}(\varepsilon). \tag{112}$$

We integrate this equation from z to $\eta_{1,2}$ getting:

$$v_1 = \widetilde{v_1}_{|z=\eta_{1,2}} - \frac{\varepsilon}{\mu_{01}}\gamma_0(v_1^0 - v_2^0)(z-\eta_{1,2}) - \\ - \frac{\varepsilon}{\mu_{01}}\left(\gamma_0(v_1^0 - v_2^0) + \alpha_0 v_1^0\right)\frac{(z-\eta_{1,2})^2}{2h_1^0} + \mathscr{O}(\varepsilon^2) \tag{113}$$

and we compute the average of v_1:

$$\overline{v_1} = -\frac{1}{6}\frac{\varepsilon}{\mu_{01}}h_1^0\alpha_0\widetilde{v_1}_{|z=b} + \left(1 + \frac{\varepsilon}{3\mu_{01}}\gamma_0 h_1^0\right)\widetilde{v_1}_{|z=\eta_{1,2}} - \frac{\varepsilon}{3\mu_{01}}h_1^0\gamma_0\widetilde{v_2}_{|z=\eta_{1,2}} + \mathscr{O}(\varepsilon^2). \tag{114}$$

Remember that thanks to (101) we have

$$\overline{v_1} = \left(1 + \frac{\varepsilon}{3\mu_{01}}\alpha_0 h_1^0\right)\widetilde{v_1}_{|z=b} - \frac{1}{6}\frac{\varepsilon}{\mu_{01}}h_1^0\gamma_0(\widetilde{v_1} - \widetilde{v_2})_{|z=\eta_{1,2}} + \mathscr{O}(\varepsilon^2). \tag{115}$$

We had too the value for the average of velocity v_2 in equation (108):

$$\overline{v_2} = -\frac{\varepsilon}{3\mu_{02}}\gamma_0 h_2^0\widetilde{v_1}_{|z=\eta_{1,2}} + \left(1 + \frac{\varepsilon}{3\mu_{02}}h_2^0\gamma_0\right)\widetilde{v_2}_{|z=\eta_{1,2}} + \mathscr{O}(\varepsilon^2). \tag{116}$$

Subtracting the last two expressions we get:

$$\overline{v_1} - \overline{v_2} = -\frac{\varepsilon}{6\mu_{01}}h_1^0\alpha_0\widetilde{v_1}_{|z=b} + \left(1 + \frac{\varepsilon}{3}\gamma_0\left(\frac{h_1^0}{\mu_{01}} + \frac{h_2^0}{\mu_{02}}\right)\right)(\widetilde{v_1} - \widetilde{v_2})_{|z=\eta_{1,2}} + \\ + \mathscr{O}(\varepsilon^2). \tag{117}$$

We solve the following system for obtaining $\widetilde{v_1}_{|z=b}$ and $(\widetilde{v_1} - \widetilde{v_2})_{|z=\eta_{1,2}}$:

$$\begin{cases} \overline{v_1} = \left(1 + \frac{\varepsilon}{3\mu_{01}}\alpha_0 h_1^0\right)\widetilde{v_1}_{|z=b} - \frac{\varepsilon}{6\mu_{01}}h_1^0\gamma_0(\widetilde{v_1} - \widetilde{v_2})_{|z=\eta_{1,2}}; \\ \overline{v_1} - \overline{v_2} = -\frac{\varepsilon}{6\mu_{01}}h_1^0\alpha_0\widetilde{v_1}_{|z=b} + \left(1 + \frac{\varepsilon}{3}\gamma_0\left(\frac{h_1^0}{\mu_{01}} + \frac{h_2^0}{\mu_{02}}\right)\right)(\widetilde{v_1} - \widetilde{v_2})_{|z=\eta_{1,2}}. \end{cases} \tag{118}$$

The results are:

$$\widetilde{v_1}_{|z=b} = D\delta\overline{v_1} + D\frac{\varepsilon\gamma_0 h_1^0}{6\mu_{01}}(\overline{v_1} - \overline{v_2}), \tag{119}$$

and

$$(\widetilde{v_1}-\widetilde{v_2})_{|z=\eta_{1,2}} = D\frac{\varepsilon\alpha_0 h_1^0}{6\mu_{01}}\overline{v_1} + D\beta^{-1}(\overline{v_1}-\overline{v_2}), \tag{120}$$

being

$$\beta = \left(1+\frac{\varepsilon}{3\mu_{01}}\alpha_0 h_1^0\right)^{-1}, \quad \delta = 1+\frac{\varepsilon}{3}\gamma_0\left(\frac{h_1^0}{\mu_{01}}+\frac{h_2^0}{\mu_{02}}\right) \tag{121}$$

and

$$D = \left(\beta^{-1}\delta - \frac{\varepsilon^2}{36\mu_{01}^2}(h_1^0)^2\alpha_0\gamma_0\right)^{-1}. \tag{122}$$

As we can check that $\beta^{-1}\delta > 1$ and having into account that we have a second order approximation, we set:

$$D = \beta\delta^{-1}. \tag{123}$$

So we can rewrite the solutions as:

$$\widetilde{v_1}_{|z=b} = \beta\overline{v_1} + \beta\delta^{-1}\frac{\varepsilon\gamma_0 h_1^0}{6\mu_{01}}(\overline{v_1}-\overline{v_2}), \tag{124}$$

and

$$(\widetilde{v_1}-\widetilde{v_2})_{|z=\eta_{1,2}} = \beta\delta^{-1}\frac{\varepsilon\alpha_0 h_1^0}{6\mu_{01}}\overline{v_1} + \delta^{-1}(\overline{v_1}-\overline{v_2}). \tag{125}$$

We take these values to equations (63) and (70):

$$\begin{aligned}
&\rho_1\partial_t(\widetilde{h_1}\overline{v_1}) + \rho_1 \text{div}\,_x(\widetilde{h_1}\overline{v_1}\otimes\overline{v_1}) - \frac{\varepsilon}{2}\rho_1\frac{1}{Ro}\cos\theta e_1(\widetilde{h_1})^2\text{div}\,_x\overline{v_1} + \\
&+\rho_1\frac{1}{Ro}\sin\theta\widetilde{h_1}(\overline{v_1})^{\perp} - 2\varepsilon\mu_{01}\text{div}\,_x(\widetilde{h_1}D_x(\overline{v_1})) + \frac{1}{2}\rho_1\frac{1}{Fr^2}\nabla_x(\widetilde{h_1})^2 - \\
&-\frac{\varepsilon}{2}\rho_1\frac{1}{Ro}\cos\theta e_1\nabla_x((\widetilde{h_1})^2\overline{v_1}) + \rho_1\frac{1}{Fr^2}\widetilde{h_1}\nabla_x\widetilde{b} + \\
&+\varepsilon\rho_1\frac{1}{Ro}\cos\theta\widetilde{h_1}\left[(\overline{v_1}\cdot\nabla_x\widetilde{b})e_1 - (\overline{v_1}\cdot e_1)\nabla_x\widetilde{b}\right] + \\
&+\delta^{-1}\gamma_0\left(\beta\frac{\varepsilon\alpha_0 h_1^0}{6\mu_{01}}\overline{v_1} + (\overline{v_1}-\overline{v_2})\right) + \beta\alpha_0\left(\overline{v_1} + \delta^{-1}\frac{\varepsilon\gamma_0 h_1^0}{6\mu_{01}}(\overline{v_1}-\overline{v_2})\right) - \\
&-2\varepsilon\mu_{01}\nabla_x\widetilde{h_1}\text{div}\,_x\overline{v_1} + \widetilde{h_1}\nabla_x(\widetilde{p_1}(\widetilde{\eta_{1,2}})) = 0.
\end{aligned} \tag{126}$$

$$
\begin{aligned}
&\rho_2\partial_t(\widetilde{h_2}\overline{v_2})+\rho_2\mathrm{div}\,_x(\widetilde{h_2}\overline{v_2}\otimes\overline{v_2})-\frac{\varepsilon}{2}\rho_2\frac{1}{Ro}\cos\theta e_1(\widetilde{h_2})^2\mathrm{div}\,_x\overline{v_2}+\\
&+\rho_2\frac{1}{Ro}\sin\theta\widetilde{h_2}(\overline{v_2})^\perp-2\varepsilon\mu_{02}\mathrm{div}\,_x(\widetilde{h_2}D_x(\overline{v_2}))+\frac{1}{2}\rho_2\frac{1}{Fr^2}\nabla_x(\widetilde{h_2})^2-\\
&-\varepsilon\frac{1}{2}\rho_2\frac{1}{Ro}\cos\theta e_1\nabla_x((\widetilde{h_2})^2\overline{v_2})-2\varepsilon\mu_{02}\nabla_x(\widetilde{h_2}\mathrm{div}\,_x\overline{v_2})+\\
&+\varepsilon\rho_2\frac{1}{Ro}\cos\theta e_1 h_2^0 w^0_{2|z=\eta_{1,2}}-\delta^{-1}\gamma_0\left(\beta\frac{\varepsilon\alpha_0 h_1^0}{6\mu_{01}}\overline{v_1}+(\overline{v_1}-\overline{v_2})\right)+\\
&+\nabla_x\widetilde{\eta_{1,2}}(\widetilde{p_2}(\widetilde{\eta_{1,2}})+2\varepsilon\mu_{02}\mathrm{div}\,_x\overline{v_2})-\\
&-\varepsilon\alpha_{02}\nabla_x(h_2^0\Delta_x\eta^0)+\varepsilon\alpha_{02}\nabla_x\eta^0\Delta_x\eta^0=0,
\end{aligned}
\tag{127}
$$

where we can check now how the coefficients for both friction terms have been corrected for two layers.

In order to simplify the last two terms in (126), we use the interfaz condition (35) to write:

$$
\widetilde{p_1}(\widetilde{\eta_{1,2}})+2\varepsilon\mu_{01}\mathrm{div}\,_x\overline{v_1}=\widetilde{p_2}(\widetilde{\eta_{1,2}})+2\varepsilon\mu_{02}\mathrm{div}\,_x\overline{v_2}-\varepsilon(\alpha_{01}-\alpha_{02})\kappa_{1,2}. \tag{128}
$$

So if we also include the expression of $\kappa_{1,2}$ we get:

$$
\begin{aligned}
&-2\varepsilon\mu_{01}\nabla_x\widetilde{h_1}\mathrm{div}\,_x\overline{v_1}+\widetilde{h_1}\nabla_x(\widetilde{p_1}(\widetilde{\eta_{1,2}}))=-2\varepsilon\mu_{01}\nabla_x\widetilde{h_1}\mathrm{div}\,_x\overline{v_1}+\\
&+\widetilde{h_1}\nabla_x(-2\varepsilon\mu_{01}\mathrm{div}\,_x\overline{v_1}+\widetilde{p_2}(\widetilde{\eta_{1,2}})+2\varepsilon\mu_{02}\mathrm{div}\,_x\overline{v_2}-\varepsilon(\alpha_{01}-\alpha_{02})\Delta_x\widetilde{\eta_{1,2}})\\
&=-2\varepsilon\mu_{01}\nabla_x(\widetilde{h_1}\mathrm{div}\,_x\overline{v_1})+\widetilde{h_1}\nabla_x(\widetilde{p_2}(\widetilde{\eta_{1,2}})+2\varepsilon\mu_{02}\mathrm{div}\,_x\overline{v_2})-\\
&-\varepsilon(\alpha_{01}-\alpha_{02})\widetilde{h_1}\nabla_x(\Delta_x\widetilde{\eta_{1,2}}).
\end{aligned}
\tag{129}
$$

Finally the equation for layer 1 reads as:

$$\begin{aligned}
&\rho_1\partial_t(\widetilde{h_1}\overline{v_1})+\rho_1\mathrm{div}\,_x(\widetilde{h_1}\overline{v_1}\otimes\overline{v_1})-\frac{\varepsilon}{2}\rho_1\frac{1}{Ro}\cos\theta e_1(\widetilde{h_1})^2\mathrm{div}\,_x\overline{v_1}+\\
&+\rho_1\frac{1}{Ro}\sin\theta\widetilde{h_1}(\overline{v_1})^{\perp}-2\varepsilon\mu_{01}\mathrm{div}\,_x(\widetilde{h_1}D_x(\overline{v_1}))+\frac{1}{2}\rho_1\frac{1}{Fr^2}\nabla_x(\widetilde{h_1})^2-\\
&-\frac{\varepsilon}{2}\rho_1\frac{1}{Ro}\cos\theta e_1\nabla_x((\widetilde{h_1})^2\overline{v_1})+\rho_1\frac{1}{Fr^2}\widetilde{h_1}\nabla_x\widetilde{b}-2\varepsilon\mu_{01}\nabla_x(\widetilde{h_1}\mathrm{div}\,_x\overline{v_1})+\\
&+\varepsilon\rho_1\frac{1}{Ro}\cos\theta\widetilde{h_1}\left[(\overline{v_1}\cdot\nabla_x\widetilde{b})e_1-(\overline{v_1}\cdot e_1)\nabla_x\widetilde{b}\right]+\\
&+\delta^{-1}\gamma_0\left(\beta\frac{\varepsilon\alpha_0h_1^0}{6\mu_{01}}\overline{v_1}+(\overline{v_1}-\overline{v_2})\right)+\beta\alpha_0\left(\overline{v_1}+\delta^{-1}\frac{\varepsilon\gamma_0h_1^0}{6\mu_{01}}(\overline{v_1}-\overline{v_2})\right)+\\
&+\widetilde{h_1}\nabla_x(\widetilde{p_2}(\widetilde{\eta_{1,2}})+2\varepsilon\mu_{02}\mathrm{div}\,_x\overline{v_2})-\varepsilon(\alpha_{01}-\alpha_{02})\widetilde{h_1}\nabla_x(\Delta_x\widetilde{\eta_{1,2}})=0.
\end{aligned}\tag{130}$$

In the same way we work on the last terms in (127) by using (65) to rewrite them as:

$$\widetilde{p_2}(\widetilde{\eta_{1,2}})+2\varepsilon\mu_{02}\mathrm{div}\,_x\overline{v_2}=\rho_2\frac{1}{Fr^2}\widetilde{h_2}-\varepsilon\rho_2\frac{1}{Ro}\cos\theta e_1\widetilde{h_2}\overline{v_2}-\varepsilon\alpha_{02}\kappa.\tag{131}$$

So:

$$\begin{aligned}
&\nabla\widetilde{\eta_{1,2}}(\widetilde{p_2}(\widetilde{\eta_{1,2}})+2\varepsilon\mu_{02}\mathrm{div}\,_x\overline{v_2})=\\
&=\nabla\widetilde{\eta_{1,2}}(\rho_2\frac{1}{Fr^2}\widetilde{h_2})-\varepsilon\rho_2\frac{1}{Ro}\cos\theta e_1\nabla_x(\widetilde{\eta_{1,2}})\widetilde{h_2}\overline{v_2}-\varepsilon\alpha_{02}\nabla_x\eta_{1,2}^0\Delta_x\eta^0.
\end{aligned}\tag{132}$$

We can finally write the equation for layer 2 as

$$\begin{aligned}
&\rho_2\partial_t(\widetilde{h_2}\overline{v_2})+\rho_2\mathrm{div}\,_x(\widetilde{h_2}\overline{v_2}\otimes\overline{v_2})-\frac{\varepsilon}{2}\rho_2\frac{1}{Ro}\cos\theta e_1(\widetilde{h_2})^2\mathrm{div}\,_x\overline{v_2}+\\
&+\rho_2\frac{1}{Ro}\sin\theta\widetilde{h_2}(\overline{v_2})^{\perp}-2\varepsilon\mu_{02}\mathrm{div}\,_x(\widetilde{h_2}D_x(\overline{v_2}))+\rho_2\frac{1}{Fr^2}(\widetilde{h_2}\nabla_x\widetilde{h_2}+\widetilde{h_2}\nabla_x\widetilde{\eta_{1,2}})-\\
&-\varepsilon\frac{1}{2}\rho_2\frac{1}{Ro}\cos\theta e_1\nabla_x((\widetilde{h_2})^2\overline{v_2})-2\varepsilon\mu_{02}\nabla_x(\widetilde{h_2}\mathrm{div}\,_x\overline{v_2})+\\
&+\varepsilon\rho_2\frac{1}{Ro}\cos\theta e_1h_2^0w^0_{2|z=\eta_{1,2}}-\delta^{-1}\gamma_0\left(\beta\frac{\varepsilon\alpha_0h_1^0}{6\mu_{01}}\overline{v_1}+(\overline{v_1}-\overline{v_2})\right)-\\
&-\varepsilon\rho_2\frac{1}{Ro}\cos\theta e_1\widetilde{h_2}\overline{v_2}\nabla_x\widetilde{\eta_{1,2}}-\varepsilon\alpha_{02}h_2^0\nabla_x(\Delta_x\eta^0)=0.
\end{aligned}$$

(133)

Remark 2.2 *We can use the equation (131) to write (130) as:*

$$\rho_1 \partial_t(\widetilde{h_1}\overline{v_1}) + \rho_1 div\,_x(\widetilde{h_1}\overline{v_1} \otimes \overline{v_1}) - \frac{\varepsilon}{2}\rho_1 \frac{1}{Ro}\cos\theta e_1(\widetilde{h_1})^2 div\,_x\overline{v_1} +$$
$$+\rho_1 \frac{1}{Ro}\sin\theta \widetilde{h_1}(\overline{v_1})^{\perp} - 2\varepsilon\mu_{01} div\,_x(\widetilde{h_1}D_x(\overline{v_1})) + \frac{1}{2}\rho_1 \frac{1}{Fr^2}\nabla_x(\widetilde{h_1})^2 -$$
$$-\frac{\varepsilon}{2}\rho_1 \frac{1}{Ro}\cos\theta e_1 \nabla_x((\widetilde{h_1})^2\overline{v_1}) + \rho_1 \frac{1}{Fr^2}\widetilde{h_1}\nabla_x\widetilde{b} - 2\varepsilon\mu_{01}\nabla_x(\widetilde{h_1} div\,_x\overline{v_1}) +$$
$$+\varepsilon\rho_1 \frac{1}{Ro}\cos\theta\widetilde{h_1}\left[(\overline{v_1}\cdot\nabla_x\widetilde{b})e_1 - (\overline{v_1}\cdot e_1)\nabla_x\widetilde{b}\right] + \tag{134}$$
$$+\delta^{-1}\gamma_0\left(\beta\frac{\varepsilon\alpha_0 h_1^0}{6\mu_{01}}\overline{v_1} + (\overline{v_1} - \overline{v_2})\right) + \beta\alpha_0\left(\overline{v_1} + \delta^{-1}\frac{\varepsilon\gamma_0 h_1^0}{6\mu_{01}}(\overline{v_1} - \overline{v_2})\right) -$$
$$-\varepsilon\alpha_{01}\widetilde{h_1}\nabla_x\Delta_x\widetilde{h_1} - \varepsilon\alpha_{01}\widetilde{h_1}\nabla_x\Delta_x b + \rho_2\frac{1}{Fr^2}\widetilde{h_1}\nabla_x\widetilde{h_2} -$$
$$-2\varepsilon\rho_2\frac{1}{Ro}\cos\theta e_1\widetilde{h_1}\nabla_x(\widetilde{h_2}\overline{v_2}) - \varepsilon\alpha_{02}\widetilde{h_1}\nabla_x\Delta_x\widetilde{h_2} = 0.$$

2.4 Final models

In this section we write the final equations for the two models obtained with dimension and dropping the cosines terms, having into account that

$$\frac{1}{Re_i} = \varepsilon\mu_{0i}, \quad \alpha = \varepsilon\alpha_0, \quad \alpha_i = \varepsilon\alpha_{0i}, \quad \gamma = \varepsilon\gamma_0. \tag{135}$$

We also divide the second and fourth equations in the systems by ρ_1 and ρ_2 respectively. For a good writing of equations we introduce some notation about the coefficients involved in the system. First we define the density relation by $r = \frac{\rho_2}{\rho_1}$, we explicit the dynamic viscosity as $\mu_i = \rho_i\nu_i$ for $i = 1,2$, being ν_i the kinematic viscosity and finally we take the following definition for the friction and tension coefficients: $\gamma = \tilde{\gamma}\rho_2$, $\alpha = \tilde{\alpha}\rho_1$, $\alpha_i = \tilde{\alpha}_i\rho_i$ with $\tilde{\gamma}$, $\tilde{\alpha}$ and $\tilde{\alpha}_i$ being positive constants.

Next we introduce some remarks about the approximations obtained mainly related to the friction terms.

First, we state the system without viscosity, from equations (77), (89), (85) and

(88), we found the following system that we denoted by (BL1):

$$
(BL1)\begin{cases}
\partial_t h_2 + \text{div}\,_x(h_2 v_2) = 0;\\
\partial_t(h_2 v_2) + \text{div}\,_x(h_2 v_2 \otimes v_2) + 2\Omega\sin\theta h_2 (v_2)^{\perp} +\\
+\dfrac{1}{2} g\nabla_x h_2^2 + g h_2 \nabla_x \eta_{1,2} = \tilde{\gamma}(v_1 - v_2);\\
\partial_t h_1 + \text{div}\,_x(h_1 v_1) = 0;\\
\partial_t(h_1 v_1) + \text{div}\,_x(h_1 v_1 \otimes v_1) + 2\Omega\sin\theta h_1 (v_1)^{\perp} + \dfrac{1}{2} g\nabla_x h_1^2 +\\
+ g h_1 \nabla_x b + r g h_1 \nabla_x h_2 = -r\tilde{\gamma}(v_1 - v_2) - \tilde{\alpha} v_1.
\end{cases} \tag{136}
$$

In the same way, we consider equations (91), (134), (103) and (133) to write the viscous model with correction on the bottom and interface friction, (BL2):

$$
(BL2)\begin{cases}
\partial_t h_2 + \text{div}\,_x(h_2 v_2) = 0;\\
\partial_t(h_2 v_2) + \text{div}\,_x(h_2 v_2 \otimes v_2) + 2\Omega\sin\theta h_2 (v_2)^{\perp} + \dfrac{1}{2} g\nabla_x h_2^2 +\\
+ g h_2 \nabla_x \eta_{1,2} = \delta^{-1}\tilde{\gamma}\left(\beta \dfrac{\tilde{\alpha} h_1}{6\nu_1} v_1 + (v_1 - v_2)\right) + \tilde{\alpha}_2 h_2 \nabla_x \Delta_x h_2 +\\
+ \tilde{\alpha}_2 h_2 \nabla_x \Delta_x \eta_{1,2} + 2\nu_2 \text{div}\,_x(h_2 D_x(v_2)) + 2\nu_2 \nabla_x (h_2 \text{div}\,_x v_2);\\
\partial_t h_1 + \text{div}\,_x(h_1 v_1) = 0;\\
\partial_t(h_1 v_1) + \text{div}\,_x(h_1 v_1 \otimes v_1) + 2\Omega\sin\theta h_1 (v_1)^{\perp} + \dfrac{1}{2} g\nabla_x h_1^2 +\\
+ g h_1 \nabla_x b + r g h_1 \nabla_x h_2 = -\delta^{-1}\tilde{\gamma} r\left(\beta \dfrac{\tilde{\alpha} h_1}{6\nu_1} v_1 + (v_1 - v_2)\right) -\\
-\beta\tilde{\alpha}\left(v_1 + \delta^{-1} r \dfrac{\tilde{\gamma} h_1}{6\nu_1}(v_1 - v_2)\right) + \tilde{\alpha}_1 h_1 \nabla_x \Delta_x h_1 +\\
+ \tilde{\alpha}_1 h_1 \nabla_x \Delta_x b + 2\nu_1 \text{div}\,_x(h_1 D_x(v_1)) + 2\nu_1 \nabla_x (h_1 \text{div}\,_x v_1),
\end{cases} \tag{137}
$$

being

$$\beta = \left(1 + \frac{\tilde{\alpha}}{3\nu_1} h_1\right)^{-1}, \quad \delta = 1 + \frac{\tilde{\gamma}}{3}\left(r\frac{h_1}{\nu_1} + \frac{h_2}{\nu_2}\right). \tag{138}$$

Remark 2.3 *We want to remark that friction terms that we have obtained in (137) have the same order of the friction and viscosity coefficients. That is to say, if we suppose that the coefficients $\nu_i, \tilde{\alpha}_i, \tilde{\alpha}$ and $\tilde{\gamma}$ are of order ε, with $\varepsilon \sim 10^{-3}$, then we next prove that the terms*

$$\delta^{-1}\tilde{\gamma} r\left(\beta\frac{\tilde{\alpha} h_1}{6\nu_1} v_1 + (v_1 - v_2)\right), \quad \beta\tilde{\alpha}\left(v_1 + \delta^{-1} r\frac{\tilde{\gamma} h_1}{6\nu_1}(v_1 - v_2)\right) \tag{139}$$

have order ε too. We write them as follows:

$$\delta^{-1}\tilde{\gamma} r\beta\frac{\tilde{\alpha}}{6\nu_1} h_1 v_1 + \delta^{-1}\tilde{\gamma} r(v_1 - v_2), \quad \beta\tilde{\alpha} v_1 + \delta^{-1}\tilde{\gamma} r\beta\frac{\tilde{\alpha}}{6\nu_1} h_1(v_1 - v_2). \tag{140}$$

So, it is enough to prove that the coefficients given by

$$\underbrace{\delta^{-1}\tilde{\gamma}\beta\frac{\tilde{\alpha}}{\nu_1}}_{[1]}, \quad \underbrace{\delta^{-1}\tilde{\gamma}}_{[2]}, \text{ and } \underbrace{\beta\tilde{\alpha}}_{[3]} \tag{141}$$

have order ε.

First we develop β and δ^{-1}:

$$\beta = \frac{3\nu_1}{3\nu_1 + \tilde{\alpha} h_1}, \quad \delta^{-1} = \frac{3\nu_1\nu_2}{3\nu_1\nu_2 + \tilde{\gamma} r\nu_2 h_1 + \tilde{\gamma}\nu_1 h_2} \tag{142}$$

and we observe that they have order ε^0 because $\nu_i, \tilde{\alpha}$ and $\tilde{\gamma}$ have the same order. Now we study each term in (141).

Term [1]*: Since β and δ^{-1} have order ε^0 and $\nu_1, \tilde{\alpha}$ and $\tilde{\gamma}$ have order ε we deduce that this first term have order ε.*

Term [2]*: this term has order ε because δ^{-1} has order ε^0 and $\tilde{\gamma}$ has order ε.*

Term [3]*: for the same reason that for the second one, we find the ε order for this term, because β has order ε^0 and $\tilde{\alpha}$ has order ε.*

Remark 2.4 *We have performed the deduction of a bilayer Shallow Water equations following the work developed in [Gerbeau and Perthame (2001)] for the one-layer case.*

Then, if we throw out the layer on top, we should get the model for one-layer obtained by them in that work. So, taking $h_2 = \rho_2 = 0$ *in the final system (137):*

$$\left\{\begin{array}{l} \partial_t h_1 + div\,_x(h_1 v_1) = 0; \\ \\ \partial_t(h_1 v_1) + div\,_x(h_1 v_1 \otimes v_1) + 2\Omega \sin\theta h_1 (v_1)^{\perp} + \dfrac{1}{2} g \nabla_x h_1^2 + g h_1 \nabla_x b + \\ \\ + r g h_1 \nabla_x h_2 = -\beta\tilde{\alpha} v_1 + 2\nu_1 div\,_x(h_1 D_x(v_1)) + 2\nu_1 \nabla_x (h_1 div\,_x v_1); \end{array}\right. \tag{143}$$

that is just the same correction for the friction that Gerbeau and Perthame have found but in 2d case.

Remark 2.5 *In [Zabsonré and Narbona-Reina (2009)] a theoretical study of a simplified (BL2) model is performed proving the existence of global weak solution for the system above but in the particular case when* $b = 0$. *To obtain this result, the following form for the friction coefficient is taken:*

$$\gamma = \frac{h_1 h_2}{\frac{\nu_1}{\nu_2} h_1 + \frac{\nu_2}{\nu_1} h_2}. \tag{144}$$

Remark 2.6 *The proof of the global weak solution for the (BL2) system is in course, it shall appear in a forthcoming paper.*

3 Numerical assessment

This section is devoted to check the validity of the new viscous bilayer model that we have derived in the previous section. In the first test we solve a 1D internal dam-break problem following the work [Gerbeau and Perthame (2001)] and we compare the numerical solution obtained by solving (BL1) and (BL2) with the Navier-Stokes equations. In Test 2 we make a comparison of the solution of the models for a 2D dam-break problem.

The results obtained show us that the new viscous model improves the no viscous one for both unknowns, height and discharge. But as it is already confirmed in precedent works (see [Gerbeau and Perthame (2001); Marche (2005); Audusse (2005); Peybernes (2006)]) we notice that the more significant difference relies on the discharge.

Test 1: An internal dam-break problem.

We present a test for which we compare the solutions obtained for Navier-Stokes equations with variable density with those given by systems (BL1) and (BL2). In the numerical discretization we have defined the friction term at the interface as follows:

$$\mathrm{fric}(v_1,v_2) = -\kappa B(h_1,h_2)(v_1-v_2), \quad \text{with } B(h_1,h_2) = \frac{h_1h_2}{\frac{\rho_1}{\rho_2}h_1 + \frac{\rho_2}{\rho_1}h_2} \text{ and } \kappa > 0. \tag{145}$$

In [Gerbeau and Perthame (2001)] a dam-break test is calculated to validate the viscous model obtained. In this work we present a similar case but solving an internal dam-break problem where two flows with different densities are involved, with the aim of emphasizing the importance of the friction term between layers.

To make the comparison between the approximated Shallow-Water systems and Navier-Stokes equations, we have computed the non-dimensional problem in each case without tension terms (i.e. $\tilde{\alpha}_i = 0$, for $i = 1,2$). For the sake of clarity we specify these problems below.

So for the first order approximation we have:

$$(BL1_{adim}) \left\{ \begin{array}{l} \partial_t h_2 + \mathrm{div}\,_x(h_2v_2) = 0; \\ \partial_t(h_2v_2) + \mathrm{div}\,_x(h_2v_2 \otimes v_2) + \frac{1}{2}\frac{1}{Fr^2}\nabla_x h_2^2 + \frac{1}{Fr^2}h_2\nabla_x\eta_{1,2} = \\ = \tilde{\gamma}_0(v_1-v_2); \\ \partial_t h_1 + \mathrm{div}\,_x(h_1v_1) = 0; \\ \partial_t(h_1v_1) + \mathrm{div}\,_x(h_1v_1 \otimes v_1) + \frac{1}{2}\frac{1}{Fr^2}\nabla_x h_1^2 + \\ + \frac{1}{Fr^2}h_1\nabla_x b + r\frac{1}{Fr^2}h_1\nabla_x h_2 = -r\tilde{\gamma}_0(v_1-v_2) - \tilde{\alpha}_0 v_1. \end{array} \right. \tag{146}$$

For the second order approximation we solve:

$$(BL2_{adim}) \left\{ \begin{array}{l} \partial_t h_2 + \mathrm{div}\,_x(h_2v_2) = 0; \\ \partial_t(h_2v_2) + \mathrm{div}\,_x(h_2v_2 \otimes v_2) + \frac{1}{2}\frac{1}{Fr^2}\nabla_x h_2^2 + \\ + \frac{1}{Fr^2}h_2\nabla_x\eta_{1,2} = \delta^{-1}\tilde{\gamma}_0\left(\beta\frac{\tilde{\alpha}_0h_1}{6}\varepsilon^2 Re_1\, v_1 + (v_1-v_2)\right) + \\ + 2\frac{1}{Re_2}\mathrm{div}\,_x(h_2D_x(v_2)) + 2\frac{1}{Re_2}\nabla_x(h_2\mathrm{div}\,_x v_2); \end{array} \right. \tag{147}$$

for the upper layer and for the lower one:

$$(BL2_{adim})\begin{cases}\partial_t h_1+\text{div}_{\,x}(h_1v_1)=0;\\[2ex] \partial_t(h_1v_1)+\text{div}_{\,x}(h_1v_1\otimes v_1)+\dfrac{1}{2}\dfrac{1}{Fr^2}\nabla_x h_1^2+\dfrac{1}{Fr^2}h_1\nabla_x b+\\[2ex] +r\dfrac{1}{Fr^2}h_1\nabla_x h_2=-\delta^{-1}\tilde{\gamma}_0\,r\left(\beta\dfrac{\tilde{\alpha}_0 h_1}{6}\varepsilon^2 Re_1\,v_1+(v_1-v_2)\right)-\\[2ex] -\beta\,\tilde{\alpha}_0\left(v_1+\delta^{-1}r\,\dfrac{\tilde{\gamma}_0 h_1}{6}\varepsilon^2 Re_1\,(v_1-v_2)\right)+\\[2ex] +2\dfrac{1}{Re_1}\text{div}_{\,x}(h_1D_x(v_1))+2\dfrac{1}{Re_1}\nabla_x(h_1\text{div}_{\,x}v_1),\end{cases}\tag{148}$$

being

$$\beta=\left(1+\frac{\tilde{\alpha}_0}{3}\varepsilon^2 Re_1\,h_1\right)^{-1},\quad \delta=1+\frac{\tilde{\gamma}_0}{3}\varepsilon^2\left(rh_1Re_1+h_2Re_2\right).\tag{149}$$

And finally for the Navier-Stokes equations we take the following problem:

$$(NS_{adim})\begin{cases}\rho\partial_t v+\rho v\nabla_x v+\rho w\partial_z v-\\ -\dfrac{2}{Re}\text{div}_{\,x}(D_x(v))-\dfrac{1}{Re}\dfrac{1}{\varepsilon^2}\partial_z^2 v-\dfrac{1}{Re}\nabla_x(\partial_z w)+\nabla_x p=\\ =-\gamma[v]\delta_\varepsilon(\rho);\\[2ex] \rho\partial_t w+\rho v\nabla_x w+\rho w\partial_z w-\\ -\frac{1}{Re}\Delta_x w--\dfrac{1}{\varepsilon^2}\dfrac{1}{Re}\partial_z(\text{div}_{\,x}v)-2\dfrac{1}{\varepsilon^2}\dfrac{1}{Re}\partial_z^2 w+\dfrac{1}{\varepsilon^2}\partial_z p=\\ =-\rho\frac{1}{\varepsilon^2}\frac{1}{Fr^2}-\gamma[w]\delta_\varepsilon(\rho);\\[2ex] \text{div}_{\,x}v+\partial_z w=0.\end{cases}\tag{150}$$

with boundary conditions at the bottom:

$$\begin{cases}\dfrac{1}{Re}(\nabla_x w+\dfrac{1}{\varepsilon^2}\partial_z v)=\dfrac{1}{\varepsilon}\alpha\,v;\\ w=0.\end{cases}\tag{151}$$

And δ_ε being an approximation of the Dirac mass on the interface, defined as follows:

$$\delta_\varepsilon=\frac{1}{\varepsilon}\xi\left(\frac{\rho}{\varepsilon}\right)|\nabla\rho|,\ \text{with}\ \xi(\vartheta)=\begin{cases}\frac{1}{2}(1+cos(\pi\vartheta)) & \text{if }|\vartheta|<1\\ 0 & \text{otherwise}\end{cases}\tag{152}$$

So the term involving δ_ε in (150) is an approximation of the friction condition at the interface that we have in the two layers case.

We take a domain $\mathscr{D}$ with length $L = 1$ meters and we set the friction coefficient as $\nu = \nu_0\,\zeta, \quad \gamma = \gamma_0\,\zeta, \quad \alpha = \alpha_0\,\zeta$ with ζ being a quantity depending on the jump height, given by $\zeta = \frac{h_{1_L} - h_{1_R}}{L}$. The height before the jump is taken as $h_{1_L} = 0.9$ and $h_{1_R} = 0.1$ after the jump.

If we denote by ρ_i, $i = 1,2$ the densities associated with each layer, we have defined $r = \frac{\rho_2}{\rho_1} = 0.98$ and the upper layer density, $\rho_2 = 1$.

We have also set the following constant data:

$\varepsilon = 0.04, \dfrac{1}{Re_i} = \dfrac{1}{Re} = \dfrac{\varepsilon}{10}$ (for $i = 1,2$), $\dfrac{1}{Fr^2} = 1$, and $\zeta = 10$ to put in evidence the influence of the friction term.

The resolution of problems $(BL1_{adim})$ and $(BL2_{adim})$ has been performed by using a Finite Volume method of Roe's type; see [Parés and Castro (2004)]. We take the following initial conditions:

$$h(t=0) = \begin{cases} h_{1_L} & x < 0; \\ h_{1_R} & x > 0, \end{cases} \qquad q(t=0) = 0. \tag{153}$$

In order to solve the Navier-Stokes equations for this problem, we have solved it in a two-dimensional domain $\mathscr{D} \times \Upsilon$, Υ with length 1, in the terms that we specify next.

To calculate the density ρ, we have solved the transport problem for the salinity S given by:

$$\partial_t S + \nabla(uS) - \varsigma \Delta S = 0, \tag{154}$$

ς being the molecular diffusion. Then, the density is updated by the state equation

$$\rho = \rho_0\,[1 + F(S)], \tag{155}$$

where ρ_0 is a reference density and $F(S)$ is a function of the salinity S.

For the time being we are not able to solve the bilayer problem using Navier-Stokes equations so in order to simulate this situation, we have taken the following constant piecewise function for the initial density value, (see Fig. 3) related to initial condition (153):

$$\rho(t=0) = \begin{cases} \rho_2 & \text{if } \{x < 0, y > h_{1_L}\} \text{ and } \{x > 0, y > h_{1_R}\} \\ \rho_1 & \text{otherwise,} \end{cases} \tag{156}$$

and the corresponding initial data for the salinity. At the initial time, we have taken $u = 0$. Regarding the constants involving the Navier-Stokes problem, we have fixed

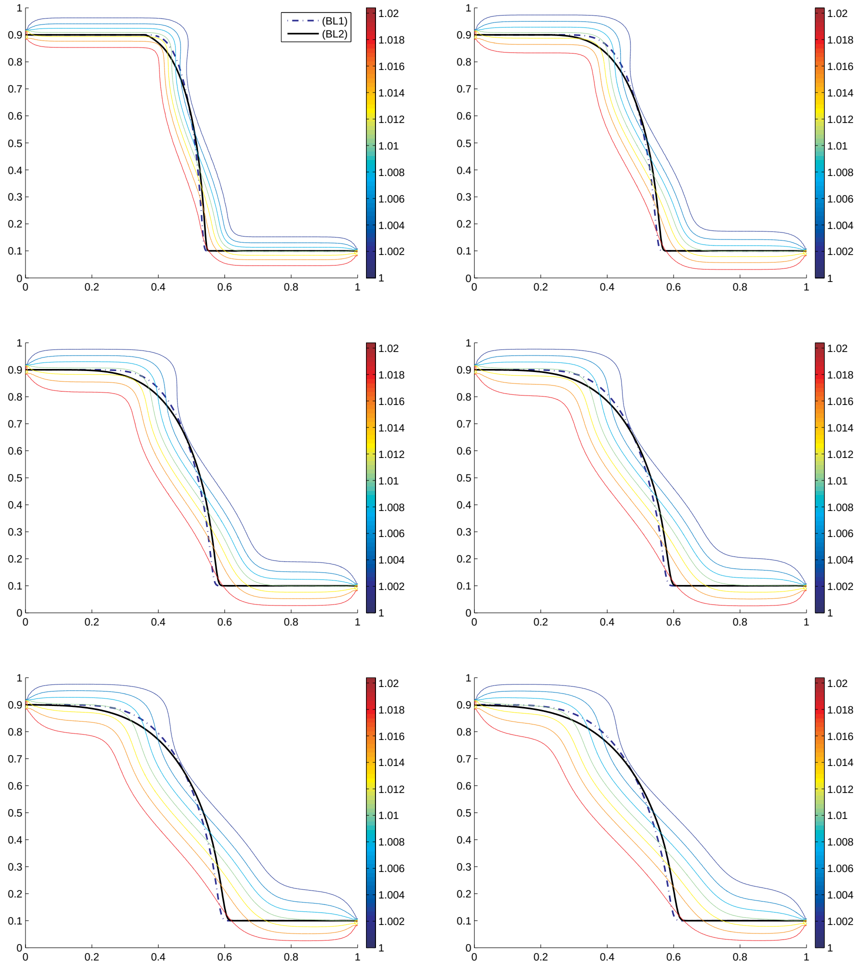

Figure 2: Evolution in time of the interface from t=1 to t=6.

the reference density $\rho_0 = 1$ and in our case, we have considered linear functions F, $F(S) = bS$ for a positive constant b, concretely we set $b = 1$. Finally, the molecular diffusion for the salinity problem is taken as $\varsigma = 10^{-5}$.

We numerically solve this problem using a Finite Element discretization in the stable pair of spaces $(\mathbb{P}_2, \mathbb{P}_1)$. The computational work has been performed by using the software *Freefem++* (`http://www.freefem.org`).

Computing the test at time $T = 6$ seconds for the three problems, we compare the solutions obtained for the interface and the velocity.

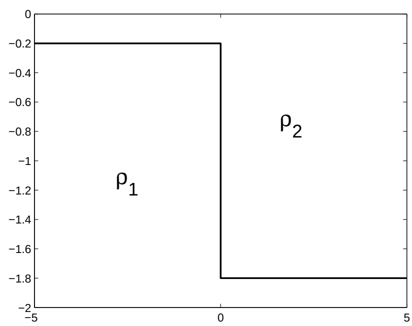

Figure 3: Initial density for Navier-Stokes problem.

In Fig. 3 we show the interface level obtained in each case for times from $t = 1$ to $t = 6$. For problems ($BL1_{adim}$) and ($BL2_{adim}$) we plot the height of the lower layer, h_1, but for Navier-Stokes equations, this value must be obtained as a function of the density profile, so we show the isolines of the density (colored lines).

As we can see in this figure the approximation for Navier-Stokes is not too meaningful to compare it with the models studied here. Anyway we can see that the solution given by the two systems keep on the profile of the Navier-Stokes solution.

Finally, in Fig. 3 we show the velocities of the lower layer obtained by solving problems ($BL1_{adim}$), ($BL2_{adim}$) and (NS_{adim}) at times $t = 1$ to 6. We notice that the solutions of the Shallow-Water systems are getting further when the time increases and that the second order approximation gives us a closer solution to the Navier-Stokes velocity.

Test 2: Circular dam-break problem in a 2D domain.

We consider a circular dam-break problem in both, surface and interface with a no constant bottom.

The domain is the square $\mathscr{D} = [0,2] \times [0,2]$, the bottom is given by the following function:

$$b(x,y) = \begin{cases} \frac{1}{8}(1+\cos(2\pi x))(1+\cos(2\pi y)) & (x-1)^2 + (y-1)^2 \leq 0.1^2; \\ 0 & \text{otherwise.} \end{cases} \tag{157}$$

The initial condition is given by:

$$h_1(t=0) + b(x,y) = \begin{cases} 1.1 & (x-0.9)^2 + (y-1)^2 \leq 0.2^2; \\ 0.6 & \text{otherwise.} \end{cases} \tag{158}$$

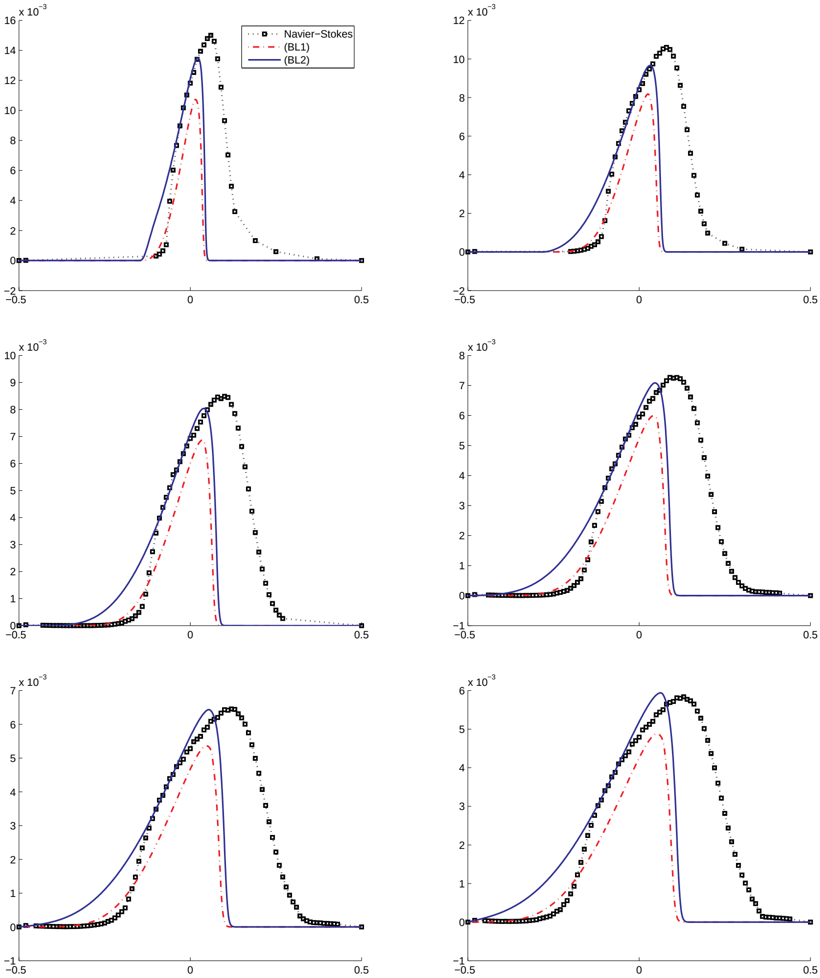

Figure 4: Evolution in time of the velocity of the layer below for times t=1 to t=6.

$$h_2(t=0)+h_1(t=0)+b(x,y)=\begin{cases} 1.7 & (x-1.6)^2+(y-1)^2 \leq 0.1^2; \\ 1.2 & \text{otherwise;} \end{cases} \tag{159}$$

and $q_1(t=0)=q_2(t=0)=0$. A longitudinal section in $y=1$ of the height initial condition is shown in Fig. 5.

The CFL is set to 0.7 and we consider a partition with $\Delta x=\Delta y=0.02$, the final time is $T=2$. The friction coefficients and the kinematic viscosity has been taken

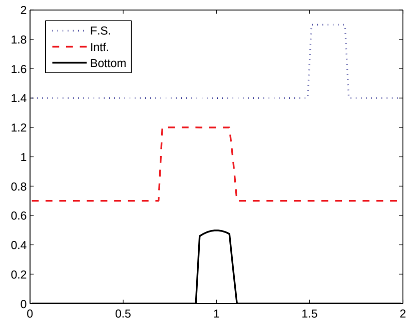

Figure 5: initial condition

as $\tilde{\gamma} = \tilde{\alpha} = \nu_1 = \nu_2 = 10^{-3}$ and the density ratio is set to 0.8.

In Fig. 6 we show the evolution in time, from $t = 0.2$ to $t = 2$ of the interface and the free surface.

In Figs. 7-10 a longitudinal section in $y = 1$ is drawn. The heights of layers are shown in Fig. 7 for the same times values. We can see that the difference between the solution of problems (BL1) and (BL2) is getting higher in time. This behavior can also been check for the discharges, see Fig. 8 for layer 1 and Fig. 9 for layer 2.

In order to weigh up the influence of the friction at the interface we show in Fig. 10 the difference between horizontal velocities $v_{1x} - v_{2x}$. Remember that $fric(v_1, v_2) = -\gamma(v_1 - v_2)$. We can see that for small times the difference is about 0.4 near the bottom bump and for final times this quantity is reduced to the half.

4 Conclusions

In this work we propose a bilayer Shallow Water 2D model, taking into account viscosity and tension effects on the surface and the interface. The model is obtained from the Navier-Stokes equations through a second order development in the asymptotic analysis and the integration process, following [Gerbeau and Perthame (2001)].

The main difficulty is related to the correction of the friction term at the interface. Usually to make this correction we must write the velocities at the interface in function of the average velocities. Due to that the friction term depends on the velocity difference of the two layers, we have a coupled problem and a second order friction correction cannot be performed layer by layer. To solve this problem we set out a linear system of equations where the unknowns are the velocities at the

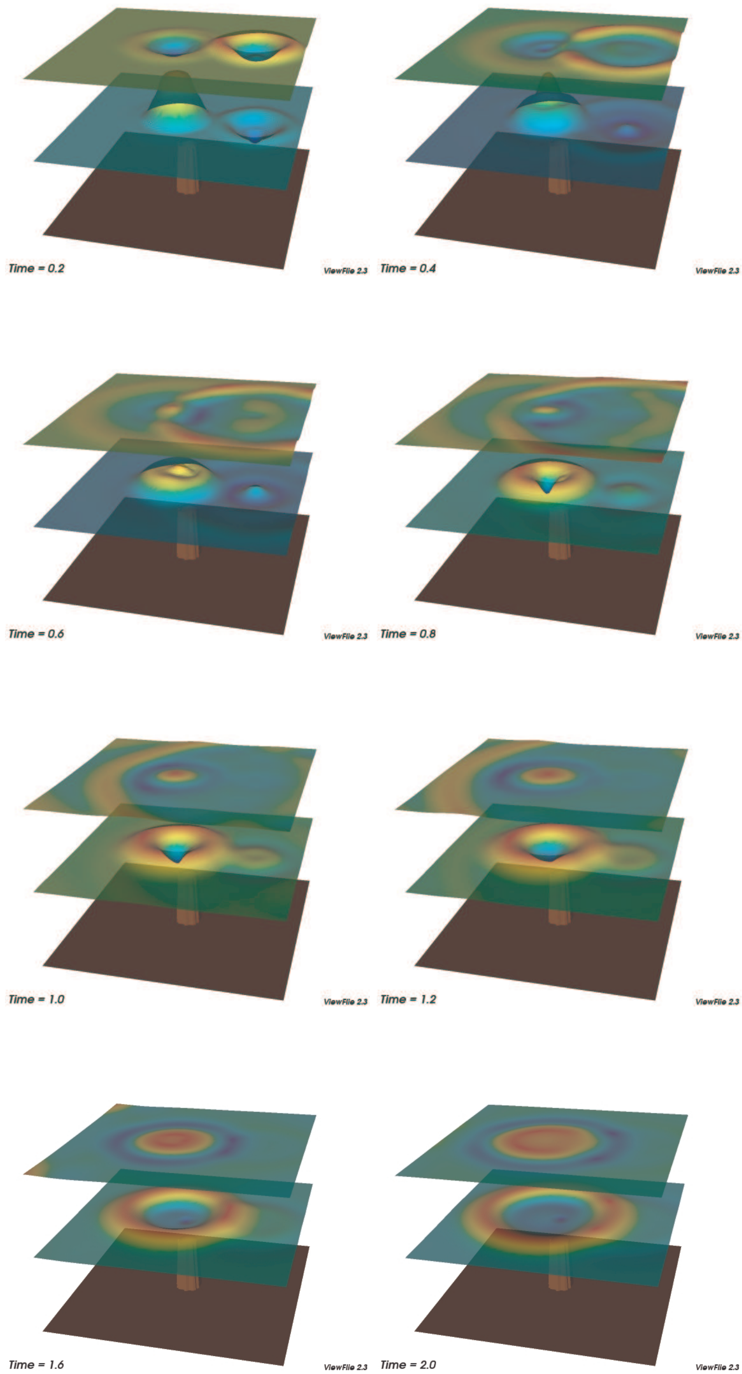

Figure 6: Free surface and interface.

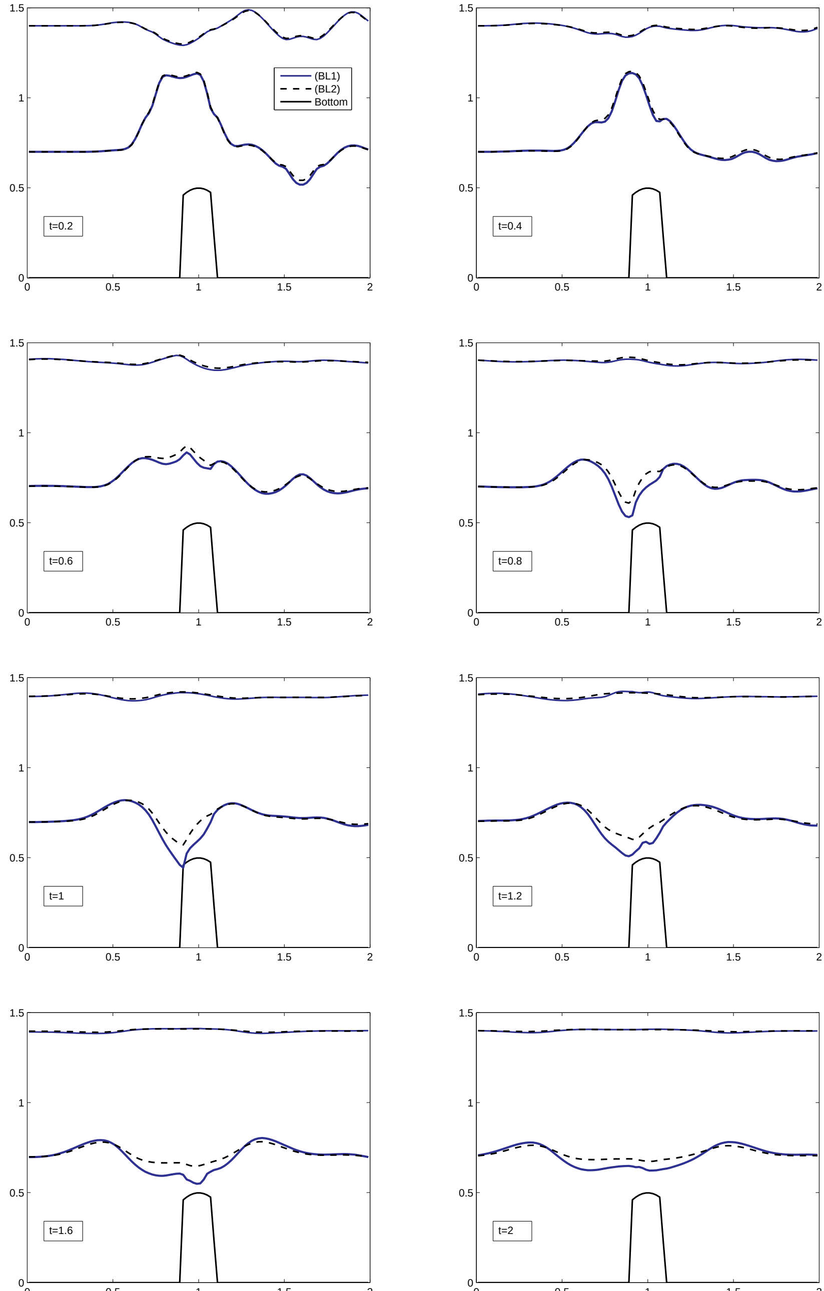

Figure 7: Longitudinal section of heights.

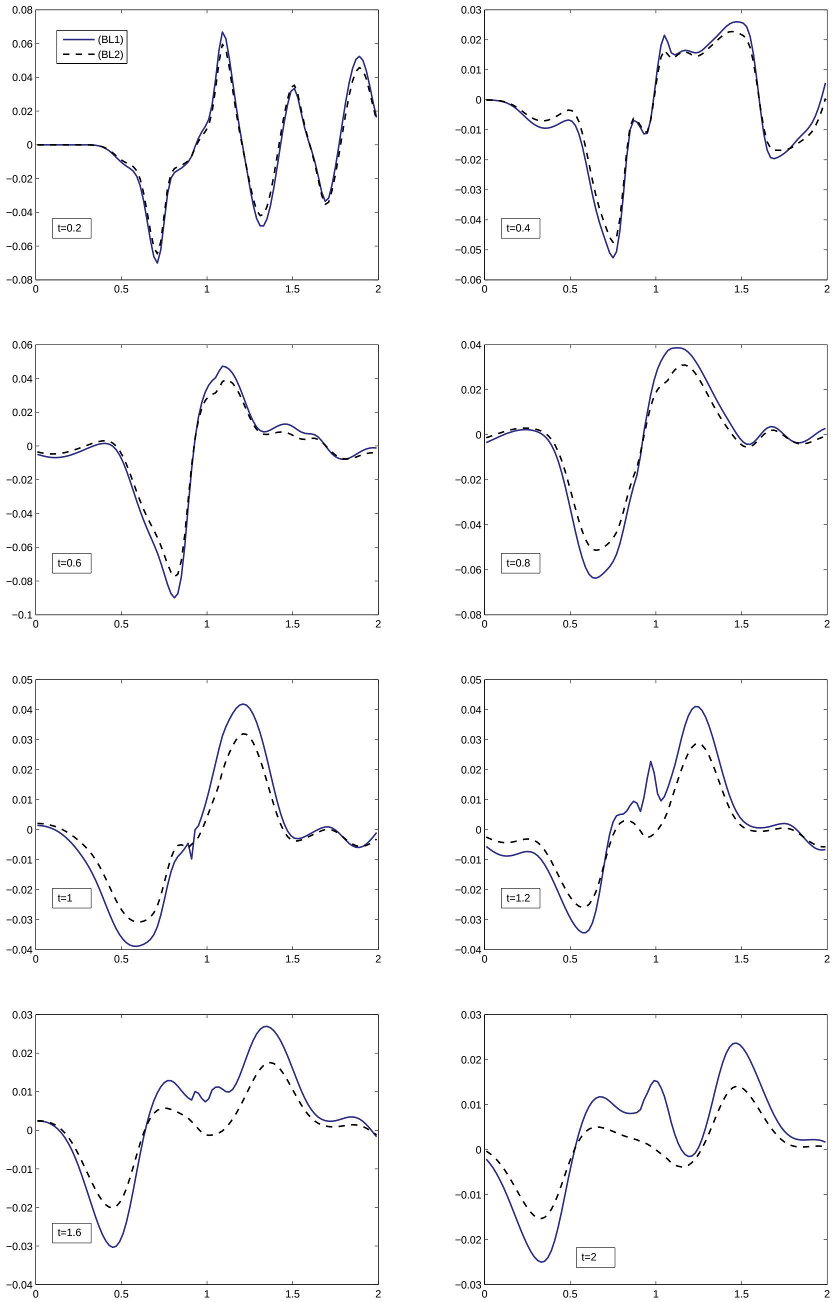

Figure 8: Discharge layer 1.

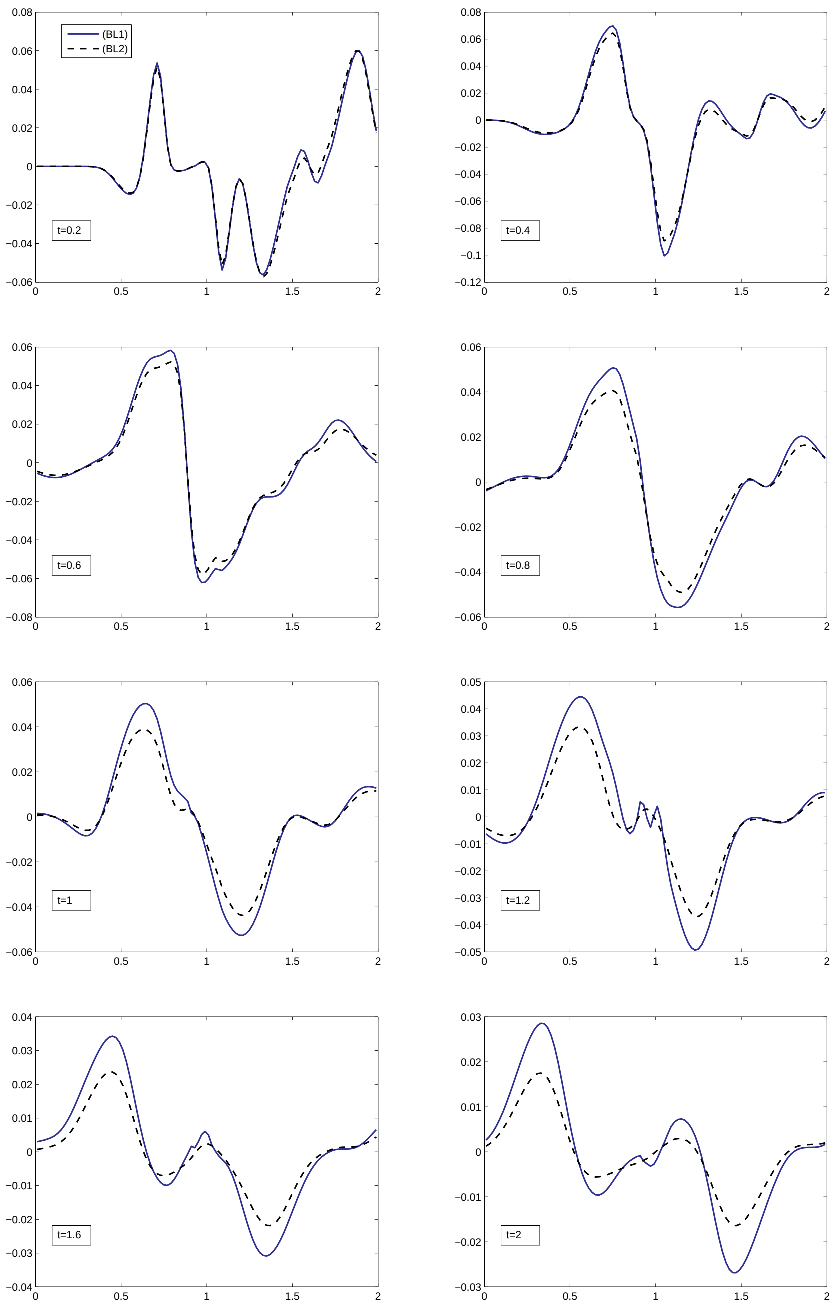

Figure 9: Discharge layer 2.

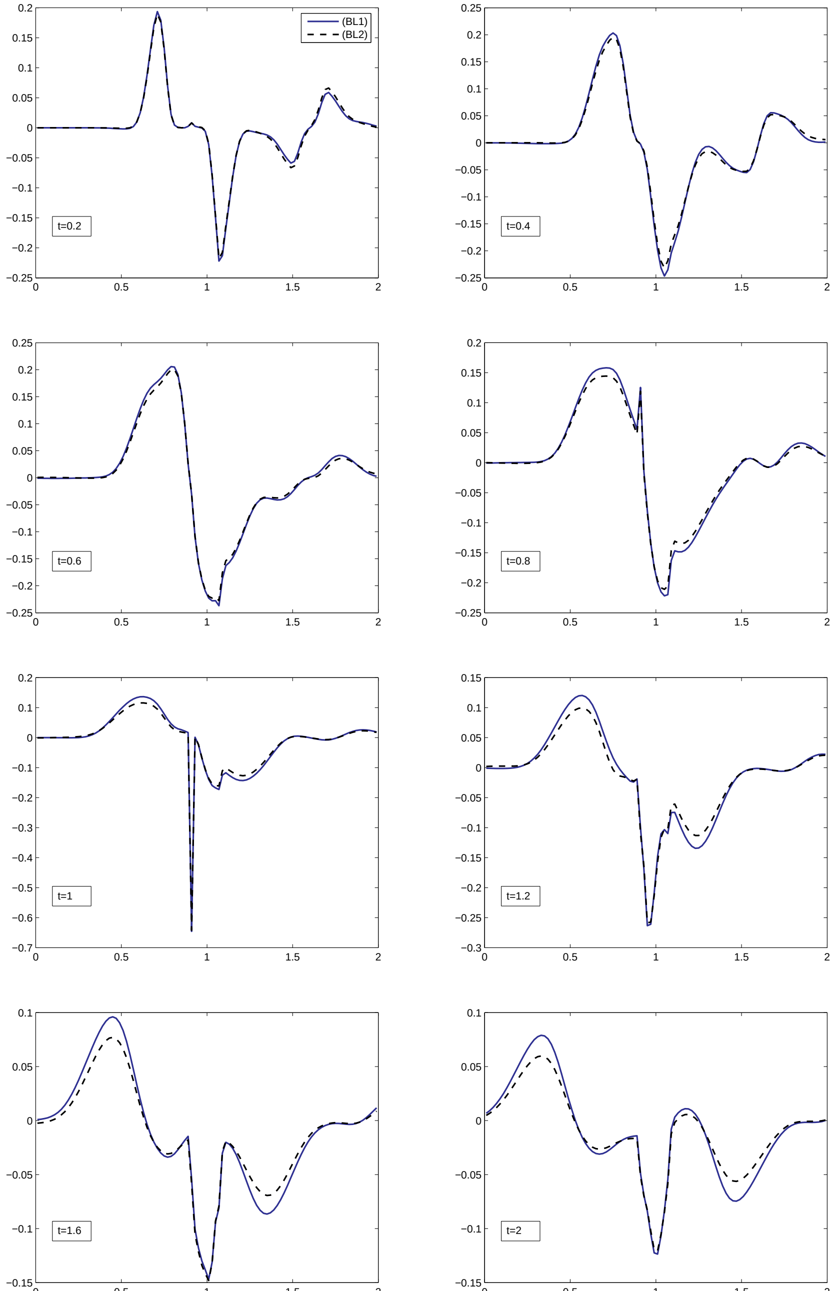

Figure 10: Difference between velocities: $v_{1x} - v_{2x}$.

interface and at the bottom, that we write in function of the average velocities. So we obtain the correction for both friction terms: at the interface and at the bottom. And, in particular we can observe in the model the influence of the friction at the bottom in the upper layer.

Finally we present two numerical tests to check the influence of the viscosity terms and friction corrections in the model. In the first test a one-dimensional internal dam-break problem is presented. We make a comparison between the solution of two models (first and second order) and the solution of Navier-Stokes equations with variable density. In this case we observe that the interface position and velocities computed by the models are comparable with the solution of the Navier-Stokes equations with variable density. Moreover, we can see that the velocity obtained from the second order model is closer to the velocity computed for Navier-Stokes problem.

In the second test, a problem with higher velocities is considered. We set a double circular dam-break problem with a bump in the bottom. The test is designed in order to obtain a great difference between the velocities of the two layers, consequently we find an important influence of friction terms. As motivated in the first test, we show that the effects added in the second order model are significants.

Acknowledgement: The research of E.D. Fernández-Nieto and G. Narbona-Reina to carry on this work was partially supported by the Spanish Government Research project MTM 2006-01275. The authors wish to thank Manuel J. Castro Díaz for interesting discussions about the numerical tests.

References

Audusse, E. (2005): A multilayer Saint-Venant model: derivation and numerical validation. *Discrete Contin. Dyn. Syst. Ser. B*, vol. 5, no. 2, pp. 189–214.

Bresch, D.; Desjardins, B. (2003): Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrophic model. *Comm. Math. Phys.*, vol. 238, no. 1-2, pp. 211–223.

Chueshov, I. D.; Raugel, G.; Rekalo, A. M. (2005): Interface boundary value problem for the Navier-Stokes equations in thin two-layer domains. *J. Differential Equations*, vol. 208, no. 2, pp. 449–493.

Ferrari, S.; Saleri, F. (2004): A new two-dimensional shallow water model including pressure effects and slow varying bottom topography. *M2AN Math. Model. Numer. Anal.*, vol. 38, no. 2, pp. 211–234.

García Rodríguez, J. (2005): *Paralelización de esquemas de volúmenes finitos: aplicación a la resolución de sistemas de tipo aguas someras*. Thesis of the University of Málaga (Spain).

Gerbeau, J.; Perthame, B. (2001): Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation. *Discrete Contin. Dyn. Syst. Ser. B*, vol. 1, no. 1, pp. 89–102.

Ishii, M.; Hibiki, T. (2006): *Thermo-fluid dynamics of two-phase flow*. Springer, New York. With a foreword by Lefteri H. Tsoukalas.

Lucas, C. (2007): Effet cosinus sur un modèle visqueux de type Saint-Venant et ses équations limites de type quasi-géostrophique et lacs. *C. R. Math. Acad. Sci. Paris*, vol. 345, no. 6, pp. 313–318.

Marche, F. (2005): *Theoretical and numerical study of Shallow Water models. Application to Nearshore hydrodinamics*. Thesis of the University of Bordeaux (France).

Ozaki, S.; Hashiguchi, K.; Okayasu, T.; Chen, D. H. (2007): Finite element analysis of particle assembly-water coupled frictional contact problem. *CMES: Computer Modeling in Engineering & Sciences*, vol. 18, no. 2, pp. 101–119.

Parés, C.; Castro, M. (2004): On the well-balance property of Roe's method for nonconservative hyperbolic systems. Applications to shallow-water systems. *M2AN Math. Model. Numer. Anal.*, vol. 38, no. 5, pp. 821–852.

Peybernes, M. (2006): *Analyse de problème mathématiques de la mécanique des fluides de type bi-couche et à frontière libre*. Thesis of the University of Pascal Paoli (France).

Song, Y.; McFarland, D.; Bergman, L.; Vakakis, A. (2005): Stick-Slip-Slap Interface Response Simulation: Formulation and Application of a General Joint/Interface Element. *CMES: Computer Modeling in Engineering & Sciences*, vol. 10, no. 2, pp. 153–170.

Wacher, A.; Givoli, D. (2006): Remeshing and refining with moving finite elements. Application to nonlinear wave problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 15, no. 3, pp. 147–164.

Zabsonré, J. (2008): *Modèles visqueux en sédimentation et stratification. Obtention formelle, Stabilité théorique et schémes volumes finis bien équilibrés*. Thesis of the University of Savoie (France).

Zabsonré, J.; Narbona-Reina, G. (2009): Existence of global weak solution for a 2D viscous bilayer Shallow-Water model. *Nonlin. Anal. Real World Appl.*, vol. 10, no. 5, pp. 2971–2984.