A Discontinuous Galerkin Meshfree Modeling of Material Interface

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Abstract: A discontinuous Galerkin meshfree formulation is proposed to solve the potential and elasticity problems of composite material where the material interface has to be appropriately modeled. In the present approach the problem domain is partitioned into patches or sub-domains and each patch holds the same material properties. The discretized meshfree particles within a patch are classified as one particle group. Various patches occupied by different particle groups are then linked using the discontinuous Galerkin formulation where an averaged interface flux or traction is constructed based on the fluxes or tractions computed from the adjacent patches. The gradient jump condition across the material interface is accurately captured by the boundary of the neighboring particle groups. The continuity of the primary field variable and the resulting interface flux or traction across the material interface is enforced weakly in the variational form through the corresponding constraints. There are no additional unknowns like Lagrange multipliers and special interface functions as well in the proposed approach. The effectiveness of the present method is demonstrated by several typical numerical examples.

Keywords: meshfree method, discontinuous Galerkin formulation, composite material, material interface, gradient jump

1 Introduction

Significant progress has been made on the class of meshfree or meshless methods since the early 1990s due to their advantages on solving large deformation problems with severe mesh distortion, high order problems like thin plates and shells, and moving boundary problems like crack propagation. These methods with fast growing research interests and applications mainly comprise: smoothed particle hydrodynamics method (SPH) [Lucy (1977), Gingold and Monaghan (1977), Wong and Shie (2008)], radial basis function method (RBF)[Kansa (1990), Kosec and

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Sarler (2008), Wen et al (2008)], diffusive element method (DEM) [Nayroles et al (1992)], element free Galerkin method (EFG) [Belytschko et al (1994, 1996), Chen et al (2009)], reproducing kernel particle method (RKPM) [Liu et al (1995), Chen et al (1996), Wu et al (2008)], partition of unity method (PUM) [Babuška and Melenk (1996), Sethuraman and Rajesh (2009)], meshless local Petrov-Galerkin method (MLPG) and its mixed generalizations [Atluri and Zhu (1998), Atluri et al (2006a, 2006b), Li and Atluri (2008a, 2008b), Mohammadi (2008), Sladek et al (2008), Pini et al (2008)], local boundary integral equation method (LBIE) [Zhu et al (1998), Chen et al (2007), Sellountos et al (2009)], method of fundamental solutions (MFS)[Golberg and Chen (1998), Liu(2008), Marin(2008)], natural element method (NEM) [Sukumar et al (1998)], reproducing kernel element method (RKEM) [Liu et al (2004)], etc. A comprehensive review of the state of the art for meshfree methods can be found in the monographs by Atluri and Shen (2002), Liu (2003), Atluri (2004), Li and Liu (2004), and the articles by Babuška et al (2003), Monaghan (2005).

Among the various types of meshfree methods the moving least square (MLS) [Lancaster and Salkauskas (1983)] and reproducing kernel (RK) [Liu et al (1995)] meshfree approximations are often employed as the shape function. It turns out that the MLS and RK approximations are equivalent if monomial basis is employed [Atluri and Shen (2002)]. The globally conforming and highly smoothing properties of MLS/RK meshfree approximation enable the meshfree methods work particularly well for problems requiring higher order continuity, such as thin plate and shell problems we just mentioned at the beginning. For general problems a smooth stress field can also be directly obtained without needing any special postprocessing. On the other hand this smoothing characteristic could cause severe solution oscillation when dealing with the problems involving material interface such as composite material problems. The reason lies behind this undesired result is that the smooth MLS/RK meshfree shape functions with overlapping local supports can not properly model the gradient jump condition across the material interface and thus special techniques are required for the Galerkin meshfree formulations to properly treat the material interface.

Several methods have been developed to model the material interface. Codes and Moran (1996) introduced the interface continuity conditions into the variational form of meshfree discretization using Lagrange multipliers. This approach leads to more additional unknowns to be solved. Krongauz and Belytschko (1998) modeled the material interface by adding a special jump function into the conventional MLS/RK approximation of the dependent field variable. Additional unknowns associated with the special jump function are required as well in this method. The special jump function was also built into the meshfree shape function by Wang

et al (2003) based upon the consistency conditions and no additional unknowns are introduced in this approach. However in this method the coupled meshfree shape function is much more involved than the conventional MLS/RK approximation. In the category of MLPG methods, the methods of MLPG2 and MLPG5 were combined to effectively treat the material discontinuity by Li et al (2003). In this approach the consistency conditions can not be preserved since different basis functions were used by MLPG2 and MLPG5. Batra et al (2004) used two MLPG formulations and systematically compared the methods of Lagrange multiplier and jump function for material interface treatment. Masuda and Noguchi (2006) employed a discontinuous derivative basis function in the MLS/RK approximation to simulate the interface discontinuity. In this formulation it is noticed the discontinuous basis function actually also serves as a special gradient jump function and the related meshfree shape function has to be modified.

Meanwhile the discontinuous Galerkin (DG) methods have also attracted a lot of research effort and have been applied to different problems [Oden at al (1998), Cockburn et al eds (2000), Zienkiewicz et al (2003), Engel et al (2004), Liu et al (2009)]. In the DG formulation the continuity of the field variable and its resulting interface flux or traction across the element boundary is imposed in the weak form. In this paper the Galerkin meshfree methods such as EFG and RKPM are reformulated under the discontinuous Galerkin framework to treat the interface problems of composite material. Both potential and elasticity problems are discussed. Within this approach the problem domain is decomposed into patches or sub-domains and the construction of meshfree approximation over each patch is standard and fully independent. The meshfree particles in each patch are named as one particle group. The domains occupied by different particle groups are then combined together using the discontinuous Galerkin formulation where an averaged interface flux or traction is constructed based on the fluxes and tractions obtained from the neighboring patches. No additional unknowns like Lagrange multipliers as well as special interface functions appear in this formulation. The gradient jump of the dependent field variable is properly captured by the boundary of the adjacent patches and the continuity of the dependent field variable and the associated flux or traction crossing the material interface is realized weakly through an augmented variational form including the corresponding continuity constraints.

This paper is organized as follows. The MLS/RK approximation is briefly summarized and the incompatibility of the patch-based MLS/RK approximation is discussed in Section 2. In Section 3 the discontinuous Galerkin meshfree formulations are presented for potential and elasticity problems to treat the material interface. Section 4 presents several benchmark numerical examples to examine the effectiveness of the proposed approach. Finally conclusions are drawn in Section 5.

2 MLS/RK Approximation and Incompatibility of Patch-based Approximation

2.1 MLS/RK Approximation

In MLS/RK approximation, the problem domain Ω is discretized by a set of particles \mathbf{x}_I , I = 1, 2, ..., NP, and the approximation of the field variable $u(\mathbf{x})$, denoted by $u^h(\mathbf{x})$, can be expressed as:

$$u^{h}(\mathbf{x}) = \sum_{I=1}^{NP} \Psi_{I}(\mathbf{x}) d_{I}$$
(1)

with $\Psi_I(\mathbf{x})$ and d_I being the MLS/RK shape function and nodal coefficient associated with the particle *I*, respectively. The shape function $\Psi_I(\mathbf{x})$ takes the following form:

$$\Psi_I(\mathbf{x}) = \mathbf{h}^T(\mathbf{x}_I - \mathbf{x})\mathbf{b}(\mathbf{x})\phi_a(\mathbf{x}_I - \mathbf{x})$$
(2)

where $\mathbf{b}(\mathbf{x})$ is a position dependent coefficient vector. $\phi_a(\mathbf{x}_I - \mathbf{x})$ is the kernel function that centers at \mathbf{x}_I and has a compact support *a*, and $\mathbf{h}(\mathbf{x}_I - \mathbf{x})$ is a vector of *n*-th order monomial basis defined as:

$$\mathbf{h}^{T}(\mathbf{x}_{I} - \mathbf{x}) = \left\{1, (x_{I} - x), (y_{I} - y), (x_{I} - x)^{2}, ..., (x_{I} - x)^{n}, ..., (y_{I} - y)^{n}\right\}$$
(3)

The coefficient vector can be obtained by imposing the following *n*-th order reproducing conditions:

$$\sum_{I=1}^{NP} \Psi_I(\mathbf{x}) x_I^i y_I^j = x^i y^j, \quad i, j = 0, 1, 2, ..., n$$
(4)

Eq. (4) can be further recast into a vector form as:

$$\sum_{I=1}^{NP} \Psi_I(\mathbf{x}) \mathbf{h}(\mathbf{x}_I - \mathbf{x}) = \mathbf{h}(\mathbf{0})$$
(5)

Introducing Eq. (2) into Eq.(5) yields:

$$\mathscr{M}(\mathbf{x})\mathbf{b}(\mathbf{x}) = \mathbf{h}(0) \tag{6}$$

where $\mathcal{M}(\mathbf{x})$ is the moment matrix given by:

$$\mathscr{M}(\mathbf{x}) \equiv \sum_{I=1}^{NP} \mathbf{h}(\mathbf{x}_I - \mathbf{x}) \mathbf{h}^T (\mathbf{x}_I - \mathbf{x}) \phi_a(\mathbf{x}_I - \mathbf{x})$$
(7)

Solving $\mathbf{b}(\mathbf{x}) = \mathcal{M}^{-1}(\mathbf{x})\mathbf{h}(0)$ from Eq.(6) and thus the MLS/RK shape function finally becomes:

$$\Psi_I(\mathbf{x}) = \mathbf{h}^T(\mathbf{0}) \mathscr{M}^{-1}(\mathbf{x}) \mathbf{h}(\mathbf{x}_I - \mathbf{x}) \phi_a(\mathbf{x}_I - \mathbf{x})$$
(8)



Figure 1: Meshfree discretization of a two-phase composite

2.2 Incompatibility of Patch-based MLS/RK Approximation

As we know for a given function the MLS/RK approximation defined by Eqns. (1) and (8) produces a approximation which has the same order of smoothness as that of the kernel function $\phi_a(\mathbf{x}_I - \mathbf{x})$. Here the widely used cubic B-spline function is employed as the kernel function and thus the MLS/RK is C^2 continuous over the problem domain. On the other hand, when one treats a heterogeneous material like a two-phase composite as shown in Fig. 1, it is natural to separate the problem domain into two patches or sub-domains $\Omega^{(1)}$ and $\Omega^{(2)}$ with different material properties, i.e., $\Omega = \bigcup_{e=1}^2 \Omega^{(e)}$. After the meshfree discretization the meshfree particles are also arranged into two particle groups, namely $G^{(1)}$ and $G^{(2)}$ for the patches $\Omega^{(1)}$ and $\Omega^{(2)}$, respectively. The number of particles within $G^{(1)}$ and $G^{(2)}$ are denoted by NP_1 and NP_2 , on the martial interface Γ^s there are NS particles that are shared by $G^{(1)}$ and $G^{(2)}$. Clearly the total number of meshfree particles is $NP = NP_1 + NP_2 - NS$. The MLS/RK approximations in each patch are constructed independently as follows:

$$u^{h(1)}(x) = \sum_{I \in G^{(1)}} \Psi_I^{(1)}(\mathbf{x}) d_I^{(1)}, \quad u^{h(2)}(x) = \sum_{J \in G^{(2)}} \Psi_J^{(2)}(\mathbf{x}) d_J^{(2)}$$
(9)

where $\Psi_I^{(1)}$, $d_I^{(1)}$ and $\Psi_J^{(2)}$, $d_J^{(2)}$ are the MLS/RK shape functions and nodal unknowns for the two patches. Note $\Psi_I^{(1)}$'s, $I = 1, 2, ..., NP_1$, are built purely using the particles in $G^{(1)}$ and only the particles in $G^{(2)}$ are employed to construct $\Psi_I^{(2)}$'s, $J = 1, 2, ..., NP_2$. However due to the overlapping effect of the various MLS/RK shape functions, this straightforward patched-based MLS/RK approximation scheme would lead to truncated kernel supports and thus yields an incompatible approximation across the material interface. Next we illustrate this incompatibility through two examples.



Figure 2: Patch-based 1D meshfree approximation

First let's consider a 1D function $f(x) = x^3 - x^2$, $x \in [0, 5]$, the problem domain is partitioned into two patches $\Omega^{(1)} = [0, 2.5]$ and $\Omega^{(2)} = [2.5, 5]$. The meshfree discretization with 11 uniformly spaced particles is shown in Fig. 2, a linear basis function with a normalized support size of 3.0 is employed in the MLS/RK approximation. Here the MLS/RK approximation is carried out in the two patches independently using Eq. (9) and the result as shown in Fig. 2 apparently proves that a discontinuity of the approximate f(x) occurs at the interface of these two patches.

To further look at the incompatibility of patch-based approximation, an approximation of the 2D function: $f(\mathbf{x}) = -[(x-5)^3 - x^2 + y^3 - y^2]$ as shown in Fig. 3(a) is considered. The problem domain is $\Omega = [0, 10] \times [0, 5]$ that is equally partitioned into two patches: $\Omega^{(1)} = [0, 5] \times [0, 5]$, $\Omega^{(2)} = [5, 10] \times [0, 5]$. Similar to the 1D case, the discretization is regular with 10×5 particles. The basis order



Figure 3: Patch-based 2D meshfree approximation: (a) $f(\mathbf{x})$; (b) approximation of $f(\mathbf{x})$

and normalized support size of the MLS/RK approximation in each patch are 1 and 3.0, respectively. The patch-based meshfree approximation of $f(\mathbf{x})$ is plotted in Fig. 3(b). It again demonstrates that at the interface of the two patches there is an incompatibility. To ensure the convergence of the meshfree method, this incompatibility has to be treated properly. In this study the discontinuous Galerkin formulation is employed to remove this incompatibility.

3 Discontinuous Galerkin Meshfree Formulation for Interface Problems

3.1 Potential Problem

Without loss of generality we start with a potential problem with two phase material properties as shown in Fig. 1, where the problem domain Ω consists of two patches $\Omega^{(1)}$ and $\Omega^{(2)}$ that are separated by the material interface Γ^s . The governing equation of a potential problem can be stated as:

$$div\mathbf{q} + s = 0 \text{ in } \Omega$$

$$\mathbf{n} \cdot \mathbf{q} = t \text{ on } \Gamma^{t}$$

$$u = g \text{ on } \Gamma^{g}$$
(10)

where div denotes the divergence operator, *s* is the source term, Γ^t and Γ^g denote the natural and essential boundaries, respectively. **n** is the outward normal of Γ^t . **q** and the dependent scalar field variable *u* are related as: $\mathbf{q} = \mathbf{k}\nabla u$ with ∇ being the gradient operator. **k** is a second order symmetric tensor, for isotropic case one has $k_{ij} = k\delta_{ij}$. On the material interface Γ^s , the field variable *u* and the directional flux **n** · **q** are required to be continuous:

$$\llbracket u(\mathbf{x}) \rrbracket = u^{(2)}(\mathbf{x}) - u^{(1)}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma^s$$
(11)

$$\mathbf{n}^{(1)}(\mathbf{x}) \cdot \mathbf{q}^{(1)}(\mathbf{x}) - \mathbf{n}^{(2)}(\mathbf{x}) \cdot \mathbf{q}^{(2)}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma^s$$
(12)

where $u^{(1)}$ and $u^{(2)}$, $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$, $\mathbf{q}^{(1)}$ and $\mathbf{q}^{(2)}$ represent the values of dependent field variable u, interface outward normal \mathbf{n} , and flux \mathbf{q} taken from the neighboring patches $\Omega^{(1)}$ and $\Omega^{(2)}$ as shown in Fig. 1, respectively. Define an averaged interface flux $\langle \mathbf{q}(\mathbf{x}) \rangle$ as:

$$\langle \mathbf{q}(\mathbf{x}) \rangle = \frac{1}{2} \left[\mathbf{q}^{(1)}(\mathbf{x}) + \mathbf{q}^{(2)}(\mathbf{x}) \right], \quad \mathbf{x} \in \Gamma^s$$
(13)

Thus since $\mathbf{n}^{(1)} = -\mathbf{n}^{(2)}$, the interface flux balance of Eq. (12) is equivalent to say that:

$$\mathbf{n}^{(1)}(\mathbf{x}) \cdot \langle \mathbf{q}(\mathbf{x}) \rangle = 0, \quad \mathbf{x} \in \Gamma^s$$
(14)

Based on the discontinuous Galerkin formulation, the continuity constraints of Eq. (11) and (14) are imposed in a weak sense through the following augmented variational form of Eq. (10):

$$\int_{\Omega} \nabla \delta u \cdot \mathbf{k} \nabla u d\Omega - \int_{\Omega} \delta u s d\Omega - \int_{\Gamma^{h}} \delta u t d\Gamma + \int_{\Gamma^{s}} \mathbf{n}^{(1)} \cdot \langle \delta q \rangle \llbracket u \rrbracket d\Gamma + \int_{\Gamma^{s}} \llbracket \delta u \rrbracket \mathbf{n}^{(1)} \cdot \langle q \rangle d\Gamma = 0 \quad (15)$$

For convenience of development, Eq. (9) is rewritten in a vector form as:

$$u^{h(1)}(\mathbf{x}) = \mathbf{\Psi}^{(1)}(\mathbf{x})\mathbf{d}, \quad u^{h(2)}(\mathbf{x}) = \mathbf{\Psi}^{(2)}(\mathbf{x})\mathbf{d}$$
(16)

where

$$\boldsymbol{\Psi}^{(1)} = \left\{ \Psi_1^{(1)} \quad \Psi_2^{(1)} \quad \dots \quad \Psi_{NP_1}^{(1)} \quad \boldsymbol{0}_{1 \times (NP_2 - NPS)} \right\}_{1 \times NP}$$
(17)

$$\Psi^{(2)} = \left\{ \mathbf{0}_{1 \times (NP_1 - NS)} \quad \Psi_1^{(2)} \quad \Psi_2^{(2)} \quad \dots \quad \Psi_{NP_2}^{(2)} \right\}_{1 \times NP}$$
(18)

$$\mathbf{d} = \left\{ d_1 \quad d_2 \quad \dots \quad d_{NP} \right\}_{1 \times NP}^T \tag{19}$$

Introducing Eq. (16) into Eq. (15) gives the following discretized equilibrium equations:

$$\mathbf{K}\mathbf{d} = \mathbf{f}, \quad \mathbf{K} = \sum_{e=1}^{3} \mathbf{K}^{(e)}, \quad \mathbf{f} = \sum_{e=1}^{2} \mathbf{f}^{(e)}$$
 (20)

where

$$\mathbf{K}^{(e)} = \int_{\mathbf{\Omega}^{(e)}} \mathbf{B}^{(e)^T} \mathbf{k}^{(e)} \mathbf{B}^{(e)} d\mathbf{\Omega}, \quad e = 1, 2$$
(21)

$$\mathbf{f}^{(e)} = \int_{\Omega^{(e)}} \mathbf{\Psi}^{(e)^T} s d\Omega + \int_{\Gamma^h \cap \Gamma^{(e)}} \mathbf{\Psi}^{(e)^T} t d\Gamma, \quad e = 1, 2$$
(22)

$$\mathbf{K}^{(3)} = sym\left[\int_{\Gamma^s} \left(\sum_{e=1}^2 \mathbf{B}^{(e)^T} \mathbf{k}^{(e)}\right) \mathbf{n}^{(1)} (\mathbf{\Psi}^{(2)} - \mathbf{\Psi}^{(1)}) d\Gamma\right]$$
(23)

$$\mathbf{B}^{(e)} = \left\{ \begin{array}{c} \mathbf{\Psi}_{,x}^{(e)} \\ \mathbf{\Psi}_{,y}^{(e)} \end{array} \right\}, \quad e = 1,2$$
(24)

with $sym[\cdot]$ being the symmetric operator given by $sym[\mathbf{A}] = (\mathbf{A} + \mathbf{A}^T)/2$. $\mathbf{K}^{(e)}$ and $\mathbf{f}^{(e)}$, e = 1, 2, are the stiffness and force contributions from the two patches. $\mathbf{K}^{(3)}$ is the material interface stiffness contribution from the last two terms in Eq. (15).

3.2 Elasticity Problem

In case of elasticity problem where the dependent variable is a vector field of displacement **u**, the governing equation becomes:

$$div\boldsymbol{\sigma} + \mathbf{s} = \mathbf{0} \text{ in } \Omega$$

$$\mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{t} \text{ on } \Gamma^h$$

$$\mathbf{u} = \mathbf{g} \text{ on } \Gamma^g$$
(25)

where $\boldsymbol{\sigma}$ is the tress tensor, **s** denotes the body force, **t** and **g** are the prescribed traction and displacement on the natural and essential boundaries Γ^h and Γ^g , respectively. The stress-strain relationship is given by the Hooke's law:

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} = \frac{1}{2} \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right]$$
(26)

The elastic material properties are $\mathbf{C}^{(1)}$ and $\mathbf{C}^{(2)}$ within $\Omega^{(1)}$ and $\Omega^{(2)}$, respectively. Similar to the preceding formulation of potential problem, by invoking the discontinuous Galerkin formulation for the treatment of the interface incompatibility, the variational statement of the problem of (25) becomes

$$\int_{\Omega} \delta \boldsymbol{\varepsilon} : \mathbf{C} : \boldsymbol{\varepsilon} d\Omega - \int_{\Omega} \delta \mathbf{u} \cdot \mathbf{s} d\Omega - \int_{\Gamma^{h}} \delta \mathbf{u} \cdot \mathbf{t} d\Gamma + \int_{\Gamma^{s}} \mathbf{n}^{(1)} \cdot \langle \delta \boldsymbol{\sigma} \rangle \, [\![\mathbf{u}]\!] d\Gamma + \int_{\Gamma^{s}} [\![\delta \mathbf{u}]\!] \cdot \langle \boldsymbol{\sigma} \rangle \, \mathbf{n}^{(1)} d\Gamma = 0 \quad (27)$$

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where again the last two terms in Eq. (27) imply a weak enforcement of the displacement and traction continuity constraints given by:

$$\llbracket \mathbf{u}(\mathbf{x}) \rrbracket = \mathbf{u}^{(2)}(\mathbf{x}) - \mathbf{u}^{(1)}(\mathbf{x}) = \mathbf{0}, \quad \mathbf{x} \in \Gamma^{\mathbf{s}}$$
(28)

$$\boldsymbol{\sigma}^{(1)} \mathbf{n}^{(1)} - \boldsymbol{\sigma}^{(2)} \mathbf{n}^{(2)} = \mathbf{0}, \quad \mathbf{x} \in \Gamma^s$$
(29)

The 2D MLS/RK approximation for the elasticity problem is given by:

$$\mathbf{u}^{h(1)}(\mathbf{x}) = \mathbf{\Psi}^{(1)}(\mathbf{x})\mathbf{d}, \quad \mathbf{u}^{h(2)}(\mathbf{x}) = \mathbf{\Psi}^{(2)}(\mathbf{x})\mathbf{d}$$
(30)

where

$$\mathbf{\Psi}^{(1)} = \begin{bmatrix} \Psi_1^{(1)} & 0 & \Psi_2^{(1)} & 0 & \dots & \dots & \Psi_{NP_1}^{(1)} & 0 & \mathbf{0}_{1 \times 2(NP_2 - NS)} \\ 0 & \Psi_1^{(1)} & 0 & \Psi_2^{(1)} & \dots & \dots & 0 & \Psi_{NP_1}^{(1)} & \mathbf{0}_{1 \times 2(NP_2 - NS)} \end{bmatrix}_{2 \times 2NP}$$
(31)

$$\boldsymbol{\Psi}^{(2)} = \begin{bmatrix} \boldsymbol{0}_{1 \times 2(NP_1 - NS)} & \boldsymbol{\Psi}_1^{(2)} & 0 & \boldsymbol{\Psi}_2^{(2)} & 0 & \dots & \dots & \boldsymbol{\Psi}_{NP_2}^{(2)} & 0 \\ \boldsymbol{0}_{1 \times 2(NP_1 - NS)} & 0 & \boldsymbol{\Psi}_1^{(2)} & 0 & \boldsymbol{\Psi}_2^{(2)} & \dots & \dots & 0 & \boldsymbol{\Psi}_{NP_2}^{(2)} \end{bmatrix}_{2 \times 2NP}$$
(32)

$$\mathbf{d} = \left\{ d_{11} \quad d_{12} \quad d_{21} \quad \dots \quad d_{NP1} \quad d_{NP2} \right\}_{2 \times NP}^{T}$$
(33)

Subsequently substituting Eq. (30) into Eq. (27) leads to the discrete matrix equations that take the same form of Eq. (20), whereas the stiffness and force contributions now become::

$$\mathbf{K}^{(e)} = \int_{\mathbf{\Omega}^{(e)}} \mathbf{B}^{(e)^T} \mathbf{C}^{(e)} \mathbf{B}^{(e)} d\mathbf{\Omega}, \quad e = 1, 2$$
(34)

$$\mathbf{f}^{(e)} = \int_{\Omega^{(e)}} \mathbf{\Psi}^{(e)^{T}} \mathbf{s} d\Omega + \int_{\Gamma^{h} \cap \Gamma^{(e)}} \mathbf{\Psi}^{(e)^{T}} \mathbf{t} d\Gamma, \quad e = 1, 2$$
(35)

$$\mathbf{K}^{(3)} = sym\left[\int_{\Gamma^s} \left(\sum_{e=1}^2 \mathbf{B}^{(e)^T} \mathbf{C}^{(e)}\right) \mathbf{N}^{(1)^T} (\mathbf{\Psi}^{(2)} - \mathbf{\Psi}^{(1)}) d\Gamma\right]$$
(36)

where

$$\mathbf{B}^{(e)} = \hat{\nabla} \mathbf{\Psi}^{(e)}, \hat{\nabla} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}^T, \quad e = 1, 2$$
(37)

$$\mathbf{N}^{(1)} = \begin{bmatrix} n_x^{(1)} & 0 & n_y^{(1)} \\ 0 & n_y^{(1)} & n_x^{(1)} \end{bmatrix}^T, \quad e = 1, 2$$
(38)

4 Numerical Results

Four typical numerical examples are presented in this section to examine the efficacy of the proposed method. The domain integration is carried out by the 5 by 5 Gauss quadrature for all test examples. To convergence study is performed according to the following standard L2 and H1 error norms of the dependent field variable **u**:

$$L2_{err}(\mathbf{u}) = \int_{\Omega} (u_i u_i) d\Omega, \quad 1 \le i \le n_u$$
(39)

$$H1_{err}(\mathbf{u}) = \int_{\Omega} (u_i u_i + u_{i,j} u_{i,j}) d\Omega, \quad 1 \le i \le n_u, \ 1 \le j \le n_{sd}$$

$$\tag{40}$$

where n_u is the dimension of **u** and n_{sd} denotes the dimension of the problem domain Ω , i.e., $\mathbf{u} : \Omega \to \mathbb{R}^{n_u}$, $\Omega \subset \mathbb{R}^{n_{sd}}$.

4.1 1D Bar Problem

The 1D composite bar problem [Codes and Moran (1996)] is shown in Fig. 4. The geometry and material properties of the two materials are: length L = 10, cross section area A = 1, elastic moduli: $E_1 = 10$, $E_2 = 1$ for $\Omega^{(1)}$ and $\Omega^{(2)}$, respectively. The bar is subjected to a body force: $s(x) = a_0 + a_1x - a_2x^2 + a_3x^3$. In this test, we consider $a_0 = 0, a_1 = 25, a_2 = -7.5, a_3 = 0.5$. The governing equation of this problem is:

$$\frac{d}{dx}\left(E(x)\frac{du}{dx}\right) + s(x) = 0, \quad x \in (0,L) u(0) = 0, \quad u(L) = \bar{u} = 1$$
(41)

$\longrightarrow s(x)$				
Ω ⁽¹⁾	Γ^{s}	$\Omega^{(2)}$	<u> </u>	
L/2		L/2	u = 1	a

Figure 4: 1D composite bar

The corresponding exact solution is given by:

$$u(x) = \begin{cases} \frac{1}{E_1} [E_2 B x + C(x)], & x \in [0, x_s] \\ 1 + B(x - L) + \frac{C(x) - C(L)}{E_2}, & x \in (x_s, L] \end{cases}$$
(42)

with

$$B = \frac{E_1 E_2 - C(x_s)(E_2 - E_1) - C(L)E_1}{E_2[(E_2 - E_1)x_s + 10E_1]}$$

$$C(x) = -\left(a_0 \frac{x^2}{2} + a_1 \frac{x^3}{6} + a_2 \frac{x^4}{12} + a_3 \frac{x^5}{20}\right)$$
(43)

where x_s denotes the material interface, here $x_s = 5$.







Figure 6: Comparison of $u_{,x}(x)$ for 1D composite bar problem

This problem is solved by the conventional meshfree method (MF) and the present discontinuous Galerkin meshfree method (DGMF). A normalized support size of 2.0 is employed in the solution process. The results of both displacement and strain using 21 particles are plotted in Figs. 5 and 6. It can be seen that DGMF can model the material interface with excellent accuracy whereas the solutions with MF show obvious gradient oscillation near the material interface. The convergence study as shown in Figs. 7 and 8 further demonstrates that DGMF produces much smaller solution errors with higher rates of convergence compared to those obtained using MF.

4.2 Rectangular Plate with Inclined Straight Material Interface

Consider a rectangular plate with unit thickness as shown in Fig. 9. The plate is under plane stress condition and the two materials are separated by an inclined straight material interface. The geometry and material properties are: $a = 1, b = 2, E_1 = 2, E_2 = 10$, the Poisson ratio is the same for both materials, i.e., v = 0.3. The plate is subjected to the following displacement boundary conditions: $u_x(0,y) = 0, u_y(0,0) = 0, u_x(2,y) = 0.02$.

The meshfree discretizations for this problem are shown in Fig. 10. A normalized support size of 2.0 is employed for the meshfree approximation. Since no analytical



Figure 7: Comparison of L2 error for 1D composite bar problem



Figure 8: Comparison of H1 error for 1D composite bar problem



Figure 9: A rectangular plate with inclined straight material interface



Figure 10: Meshfree Discretizations for rectangular plate problem

solution is available for this problem, a finite element solution obtained by 5000 bi-linear quadrilateral elements is taken as the reference or "exact" solution. The x-direction displacement and strain solutions at the line of y = 0.5 based different methods with 231 particles are shown in Figs. 11 and 12, where better solutions are observed for DGMF approach. The convergence results as shown Figs. 13 and 14 also favor DGMF with lower errors and superior convergence rates for both H1 and L2 error measures.



Figure 11: Comparison of u_x for rectangular plate problem



Figure 12: Comparison of ε_x for rectangular plate problem



Figure 13: Comparison of L2 error for rectangular plate problem



Figure 14: Comparison of H1 error for rectangular plate problem

4.3 Hollow Cylinder Potential Problem

Another problem we consider here is a scalar potential problem for the two-phase composite as shown in Fig. 15. The geometry and material properties are: $r_i = 1$,



Figure 15: Hollow cylinder problem

 $r_o = 4$, $r_s = 2$, $k_1 = 10$, $k_2 = 1$. The problem are subjected to the essential boundary conditions at the inner and outer boundaries: as: $u(r_i) = 10$, $u(r_o) = 40$. The governing equation of this axisymmetric problem is given by

$$\frac{d}{dr}(kr\frac{du}{dr}) = 0$$

$$u(r_i) = u_i = 10$$

$$u(r_o) = u_o = 40$$
(44)

The exact solution of this problem can be obtained as:

$$u(r) = \begin{cases} \frac{u_o - u_i}{\ln r_s - \ln r_i - \ln(r_s \frac{k_1}{k_2}) + \ln(r_o \frac{k_1}{k_2})} (\ln r - \ln r_i) + u_i & r \in [r_i, r_s] \\ \frac{(u_o - u_i)k_1}{[\ln r_s - \ln r_i - \ln(r_s \frac{k_1}{k_2}) + \ln(r_o \frac{k_1}{k_2})]k_2} (\ln r - \ln r_o) + u_o & r \in (r_s, r_o] \end{cases}$$
(45)

Due to the two-fold symmetry only a quarter model is used for the meshfree solution. Fig.16 lists the meshfree discretizations employed in the computation. The normalized support size for this example is 2.0 for the meshfree approximation. Figs. 17-18 shows that the solutions obtained by DGMF and MF along the symmetric line of the meshfree model with 289 particles. Clearly the solutions by DGMF match very well the analytical solutions and this is not the case for the solutions solved by MF. To see the performance of the present method in a more clear and straightforward way, Figs. 19 and 20 show the distributions of temperature and its radial gradient from both three dimensional and plane views. The convergence rates are compared in Figs. 21-22. They again demonstrate the DGMF can model the interface problem with much better solution accuracy than MF.



Figure 16: Meshfree discretizations for hollow cylinder problem



Figure 17: Comparison of *u* for hollow cylinder potential problem



Figure 18: Comparison of $u_{,r}$ for hollow cylinder potential problem

4.4 Hollow Cylinder Elasticity Problem

The hollow cylinder problem as shown in Fig. 15 is re-considered here for the elasticity case under plane strain condition. The geometric parameters are the same as those of the previous example and the elastic material properties are given by: $E_1 = 2, E_2 = 10$, and Poisson ratio v = 0.3 for both materials. The corresponding Lame's constants are: $\lambda_1 = 1.15, \lambda_2 = 5.77$ and $\mu_1 = 0.77, \mu_2 = 3.85$. The inner and outer boundary conditions are: $u_r(r_i) = 0, u_\theta(r_i) = 0; u_r(r_o) = r_o, u_\theta(r_o) = 0$.



Figure 19: Comparison of u for hollow cylinder potential problem via three dimensional and plane views

The equilibrium equation of this elasticity problem is:

$$\frac{d}{dr} \begin{bmatrix} \frac{1}{r} \frac{d}{dr}(ru_r) \end{bmatrix} = 0$$

$$u_r(r_i) = 0; \quad u_\theta(r_i) = 0$$

$$u_r(r_o) = r_o; \quad u_\theta(r_o) = 0$$
(46)

The exact solution of this problem is

$$u_{r}(r) = \begin{cases} \alpha r - \alpha r_{i}^{2} \frac{1}{r} \\ \frac{\alpha r_{i}^{2} + r_{o}^{2} - \alpha r_{s}^{2}}{r_{o}^{2} - r_{s}^{2}} r + \frac{(\alpha - 1)r_{s}^{2}r_{o}^{2} - \alpha r_{i}^{2}r_{o}^{2}}{r_{o}^{2} - r_{s}^{2}} \frac{1}{r} \end{cases}$$
(47)

$$\varepsilon_{rr} = \begin{cases} \alpha + \alpha r_i^2 \frac{1}{r^2} \\ \frac{\alpha r_i^2 + r_o^2 - \alpha r_s^2}{r_o^2 - r_s^2} - \frac{(\alpha - 1)r_s^2 r_o^2 - \alpha r_i^2 r_o^2}{r_o^2 - r_s^2} \frac{1}{r^2} \end{cases}$$
(48)

$$\varepsilon_{\theta\theta} = \begin{cases} \alpha - \alpha r_i^2 \frac{1}{r^2} \\ \frac{\alpha r_i^2 + r_o^2 - \alpha r_s^2}{r_o^2 - r_s^2} + \frac{(\alpha - 1)r_s^2 r_o^2 - \alpha r_i^2 r_o^2}{r_o^2 - r_s^2} \frac{1}{r^2} \end{cases}$$
(49)



Figure 20: Comparison of $u_{,r}$ for hollow cylinder potential problem via three dimensional and plane views



Figure 21: Comparison of L2 error for hollow cylinder potential problem



Figure 22: Comparison of H1 error for hollow cylinder potential problem



Figure 23: Comparison of u_r for hollow cylinder elasticity problem



Figure 24: Comparison of ε_{rr} for hollow cylinder elasticity problem



Figure 25: Comparison of $\varepsilon_{\theta\theta}$ for hollow cylinder elasticity problem

where

$$\alpha = \frac{-2\mu_2 r_o^2 - \lambda_2 r_o^2}{\mu_1 (r_o^2 - r_s^2 - r_i^2 + \frac{r_i^2 r_o^2}{r_s^2}) + \lambda_1 (r_o^2 - r_s^2) + \mu_2 (r_s^2 + r_o^2 - r_i^2 - \frac{r_i^2 r_o^2}{r_s^2}) + \lambda_2 (r_s^2 - r_i^2)}$$
(50)

The meshfree discretizations for this example are also shown in Fig. 16. A normalized support size of 2.0 is adopted for the meshfree approximation. The radial displacement u_r , radial strain ε_{rr} , and hoop strain $\varepsilon_{\theta\theta}$ along the symmetric line of the meshfree model with 289 nodes are compared with the corresponding exact solutions in Figs. 23-25. The results evince that the solutions of DGMF are much more accurate than those of MF. It is noticed that the radial strain jump is well captured in Fig. 24 by DGMF and at the same time the continuous hoop strain $\varepsilon_{\theta\theta}$ is simulated accurately in Fig. 25. The excellent solution accuracy of DGMF is also observed in Figs. 26-28 which are the three dimensional and plane views of the distributions of u_r , ε_{rr} and $\varepsilon_{\theta\theta}$. In the comparisons of the L2 and H1 convergence rates as shown in Figs. 29 and 30, all results proves that DGMF performs superiorly compared to MF.



Figure 26: Comparison of u_r for hollow cylinder elasticity problem via three dimensional and plane views

5 Conclusions

A discontinuous Galerkin meshfree formulation (DGMF) was presented for the accurate modeling and analysis of potential and elasticity interface problems of composite material. In DGMF the problem domain is separated into different patches with the same material properties. The meshfree discretizations and approximations within every patch follow the standard MLS/RK meshfree formulations and thus they can be built up independently using the particles in each patch. The incompatibility across the material interface associated with the patch-based



Figure 27: Comparison of ε_{rr} for hollow cylinder elasticity problem via three dimensional and plane views



Figure 28: Comparison of $\varepsilon_{\theta\theta}$ for hollow cylinder elasticity problem via three dimensional and plane views



Figure 29: Comparison of L2 error for hollow cylinder elasticity problem



Figure 30: Comparison of H1 error for hollow cylinder elasticity problem

MLS/RK approximation was illustrated via two numerical examples. Consequently the discontinuous Galerkin formulation was employed to remedy the interface incompatibility. In this method an averaged interface flux or traction is constructed based on its counterparts from the neighboring patches and it is subsequently used to weakly enforce the continuity constraints of the dependent field variable and the related flux or traction across the material interface in the variational form. The formulations of potential and elasticity problems with the present methodology were given in details. No additional variables as well as special interface enrichment functions are required in the proposed approach. The efficacy of DGMF was demonstrated through various benchmark numerical examples of both potential and elasticity problems. Numerical examples showed that the proposed DGMF is stable and can capture the interface gradient jump very accurately, and therefore yields superior solution accuracy and convergence rates for interface problems compared to the conventional Galerkin meshfree approach.

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