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On the Solution of a Coefficient Inverse Problem for the Non-stationary Kinetic Equation

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Abstract: The solvability conditions of an inverse problem for the non-stationary kinetic equation is formulated and a new numerical method is developed to obtain the approximate solution of the problem. A comparison between the approximate solution and the exact solution of the problem is presented.

Keywords: Kinetic Equation, Inverse Problem, Galerkin Method, Symbolic Computation

1 Introduction

Inverse problems appear in many important applications of physics, geophysics, technology and medicine. One of the characteristic features of these problems for differential equations is their being ill-posed in the sense of Hadamard. The general theory of ill-posed problems and their applications is developed by A. N. Tikhonov, V. K. Ivanov, M. M. Lavrent'ev and their students [Ivanov et al (1978), Lavrent'ev (1967), Lavrent'ev et al (1980), Tikhonov and Arsenin (1979), Tikhonov et al (1987)]. Inverse problems for kinetic equations are important both from theoretical and practical points of view. Interesting results in this field are presented by Amirov [Amirov (1985), Amirov (1987), Amirov (2001)], Anikonov, Kovtanyuk and Prokhorov [Anikonov et al (2002)], Anikonov and Amirov [Anikonov and Amirov (1983)], Anikonov [Anikonov (2001)], Hamaker, Smith, Solmon and Wagner [Hamaker et al (1980)]. Some recent works devoted to numerical solution of inverse problems can be found in [Ling and Atluri (2006); Huang and Shih (2007); Ling and Takeuchi (2008); Liu (2008); Marin (2008); Beilina and Klibanov (2008); Amirov et al (2009)].

In this paper, the existence and uniqueness of the solution of a non-linear inverse problem for the non-stationary kinetic equation is proven in the case where the values of the solution are known on the boundary of a domain. A new numerical

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method based on the Galerkin method is developed to obtain the approximate solution of the problem. A comparison between the computed approximate solution and the exact solution of the problem is presented.

The main difficulty in studying the solvability of considered problem is overdeterminacy. In the paper, using some extension of the class of unknown functions, the overdetermined inverse problem is replaced by a related determined one, which is a new and interesting technique of investigating the solvability of overdetermined problems. The proposed approximation method for the non-linear inverse problem for the non-stationary kinetic equation is also new and important which is based on this technique.

For a bounded domain G, $C^m(G)$ is the Banach space of functions that are m times continuously differentiable in G; $C^\infty(G)$ is the set of functions that belong to $C^m(G)$ for all $m \geq 0$; $C_0^\infty(G)$ is the set of finite functions in G that belong to $C^\infty(G)$; $L_2(G)$ is the space of measurable functions that are square integrable in G, $H^k(G)$ is the Sobolev space and $\overset{\circ}{H}{}^k(G)$ is the closure of $C_0^\infty(G)$ with respect to the norm of $H^k(G)$. These standard spaces are described in detail, for example, in Lions and Magenes [Lions and Magenes (1972)] and Mikhailov [Mikhailov (1978)].

2 Statement of the Problem

In this work, the kinetic equation

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n}\left(v_i\frac{\partial u}{\partial x_i} + f_i\frac{\partial u}{\partial v_i}\right) - a(x,v,t)\,u = 0 \tag{1}$$

is considered in the domain

$$\Omega = \left\{(x,v,t) : x \in D \subset \mathbb{R}^n,\ v \in G \subset \mathbb{R}^n,\ n \geq 1,\ t \in (0,T)\right\},$$

where the boundaries ∂D, $\partial G \in C^3$, $a(x,v,t)$ is an unknown function and satisfies the equation

$$\left\langle a, \widehat{L}\eta \right\rangle = 0, \quad \widehat{L} = \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i \partial v_i} \tag{2}$$

for any $\eta \in C_0^\infty(\Omega)$, $\langle .,. \rangle$ is a scalar product in $L_2(\Omega)$.

We select a subset $\{w_1, w_2, ...\}$ of $\widetilde{C}_0^3 = \left\{\varphi : \varphi \in C^3(\Omega),\ \varphi = 0 \text{ on } \partial\Omega\right\}$ which is orthonormal in $L_2(\Omega)$ and the linear span of this set is everywhere dense in $\overset{\circ}{H}_{1,2}(\Omega)$, where $\overset{\circ}{H}_{1,2}(\Omega)$ is the set of all real-valued functions $u(x,v,t) \in L_2(\Omega)$ that have generalized derivatives u_{x_i}, u_{v_i}, $u_{x_i v_j}$, $u_{v_i v_j}$ $(i, j = 1, 2, ..., n)$, which belong to $L_2(\Omega)$ and whose trace on $\partial\Omega$ is zero. Let P_n be the orthogonal projector of $L_2(\Omega)$ onto M_n, where M_n is the linear span of $\{w_1, w_2, ..., w_n\}$.

Eq. 1 is extensively used in plasma physics and astrophysics. In applications, $u(x,v,t)$ represents the number (or the mass) of particles in the unit volume element of the phase space in the neighbourhood of a point (x,v) at the moment t, $a(x,v,t)$ is the absorption term and $f = (f_1, ..., f_n)$ is the force acting on a particle.

Problem 1 *Determine the functions $u(x,v,t)$ and $a(x,v,t)$ defined in Ω from equation (1), provided that $u(x,v,t) > 0$, the function $a(x,v,t)$ satisfies (2) and the trace of $u(x,v,t)$ is known on the boundary, i.e., $u|_{\partial\Omega} = u_0$.*

Remark 1 *It is easy to see that Problem 1 is non-linear because Eq. 1 contains a production of unknown functions $u(x,v,t)$ and $a(x,v,t)$.*

Remark 2 *In practise, the function $a(x,v,t)$ depends only on the argument x and t, i.e. the problem is overdetermined. In [Amirov (2001)], a genereal scheme is presented to overcome this difficulty: It's assumed that the unknown coefficient in the problem depends not only on the variables x and t but also on the direction v in a specific way, that is, $\widehat{L}a = 0$.*

Remark 3 *By introducing a new unknown function $\ln u = y$, Problem 1 can be reduced to the following problem:*

Problem 2 *Find a pair of functions (y,a) defined in Ω satisfying the equation*

$$Ly \equiv \frac{\partial y}{\partial t} + \sum_{i=1}^{n} \left(v_i \frac{\partial y}{\partial x_i} + f_i \frac{\partial y}{\partial v_i} \right) = a(x,v,t), \tag{3}$$

provided that $a(x,v,t)$ satisfies (2) and y is known on $\partial\Omega$: $y|_{\partial\Omega} = \ln u_0 = y_0$.

To formulate the solvability theorem for Problem 2, we need the following notation: $\Gamma(A)$ denotes the set of functions y with the following properties

i) For $y \in \Gamma(A)$, $Ay \in L_2(\Omega)$ in the generalized sense, where $Ay = \widehat{L}Ly$;

ii) There exists a sequence $\{y_k\} \subset \widetilde{C}_0^3$ such that $y_k \to y$ in $L_2(\Omega)$ and $\langle Ay_k, y_k \rangle \to \langle Ay, y \rangle$ as $k \to \infty$.

The condition $Ay \in L_2(\Omega)$ in the generalized sense means that there exists a function $\mathscr{F} \in L_2(\Omega)$ such that for all $\varphi \in C_0^\infty(\Omega)$, $\langle y, A^*\varphi \rangle = \langle \mathscr{F}, \varphi \rangle$ and $Ay = \mathscr{F}$ where A^* is the differential operator conjugate to A in the sense of Lagrange.

3 Solvability of the Problem

Theorem 1 *Let $f \in C^1(\Omega)$ and assume that the following inequality holds for all $\xi \in \mathbb{R}^n$:*

$$\sum_{i,j=1}^{n} \frac{\partial f_i}{\partial x_j} \xi^i \xi^j \geq \alpha_1 |\xi|^2, \tag{4}$$

where α_1 is a positive number. Then Problem 2 has at most one solution (y,a) such that $y \in \Gamma(A)$ and $a \in L_2(\Omega)$.

Proof. The method used here for proving the uniqueness of the solution is similar to that of given in the proof of Theorem 2.2.1 on p. 60 in [Amirov (2001)] which is proved for the stationary kinetic equation. Let (y,a) be a solution to Problem 2 such that $y=0$ on $\partial\Omega$ and $y \in \Gamma(A)$. Eq. 3 and condition (2) imply $Ay=0$. Since $y \in \Gamma(A)$, there exists a sequence $\{y_k\} \subset \widetilde{C}_0^3$ such that $y_k \to y$ in $L_2(\Omega)$ and $\langle Ay_k, y_k \rangle \to 0$ as $k \to \infty$. Observing that $y_k = 0$ on $\partial\Omega$, we get

$$-2\langle Ay_k, y_k \rangle = 2\sum_{i=1}^{n} \left\langle \frac{\partial}{\partial v_i}(Ly_k), y_{k_{x_i}} \right\rangle. \tag{5}$$

We have the following identity for the right-hand side of the last equality:

$$\begin{aligned}
&\sum_{i=1}^{n} 2\frac{\partial y_k}{\partial x_i}\frac{\partial}{\partial v_i}(Ly_k) \\
&= \sum_{i=1}^{n}\left(\frac{\partial y_k}{\partial x_i}\right)^2 + \sum_{i,j=1}^{n}\frac{\partial f_i}{\partial x_j}\frac{\partial y_k}{\partial v_i}\frac{\partial y_k}{\partial v_j} \\
&+ \sum_{i=1}^{n}\frac{\partial}{\partial v_i}\left[\frac{\partial y_k}{\partial t}\frac{\partial y_k}{\partial x_i}\right] + \sum_{i=1}^{n}\frac{\partial}{\partial t}\left[\frac{\partial y_k}{\partial v_i}\frac{\partial y_k}{\partial x_i}\right] - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[\frac{\partial y_k}{\partial t}\frac{\partial y_k}{\partial v_i}\right] \\
&+ \sum_{i,j=1}^{n}\frac{\partial}{\partial v_j}\left(v_i\frac{\partial y_k}{\partial x_i}\frac{\partial y_k}{\partial x_j}\right) + \sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(v_i\frac{\partial y_k}{\partial v_j}\frac{\partial y_k}{\partial x_j}\right) - \sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(v_i\frac{\partial y_k}{\partial x_i}\frac{\partial y_k}{\partial v_j}\right) \\
&+ \sum_{i=1}^{n}\frac{\partial}{\partial v_i}\left[v_i\left(\frac{\partial y_k}{\partial x_i}\right)^2\right] + \sum_{i,j=1}^{n}\frac{\partial}{\partial v_j}\left(f_i\frac{\partial y_k}{\partial v_i}\frac{\partial y_k}{\partial x_j}\right) + \sum_{i,j=1}^{n}\frac{\partial}{\partial v_i}\left(f_i\frac{\partial y_k}{\partial v_j}\frac{\partial y_k}{\partial x_j}\right) \\
&- \sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(f_i\frac{\partial y_k}{\partial v_i}\frac{\partial y_k}{\partial v_j}\right).
\end{aligned} \tag{6}$$

From (6), using the condition $y_k = 0$ on $\partial\Omega$ and the geometry of the domain Ω, we get

$$-\langle Ay_k, y_k \rangle = J(y_k), \tag{7}$$

where

$$J(y_k) \equiv \frac{1}{2}\sum_{i=1}^{n}\int_{\Omega}\left(\left(\frac{\partial y_k}{\partial x_i}\right)^2+\sum_{j=1}^{n}\frac{\partial f_i}{\partial x_j}\frac{\partial y_k}{\partial v_i}\frac{\partial y_k}{\partial v_j}\right)d\Omega. \tag{8}$$

Since Ω is bounded and $y_k = 0$ on $\partial\Omega$, from (4) it follows that

$$J(y_k) > \frac{1}{2}\int_{\Omega}|\nabla_x y_k|^2\, d\Omega \geq c\int_{\Omega}|y_k|^2\, d\Omega,\ c > 0, \tag{9}$$

where $\nabla_x y_k = \left(y_{k_{x_1}},\ldots,y_{k_{x_n}}\right)$. Using definition of $\Gamma(A)$, we have $c\int_{\Omega} y^2 d\Omega \leq 0$. Then (3) implies $a(x,v,t) = 0$. Hence uniqueness of the solution of the problem is proven. ■

Problem 3 *Given the equation*

$$Ly = a + F \tag{10}$$

where the function a satisfies (2) *and F is a known function in* $H^2(\Omega)$, *find the pair of functions* (y,a) *under the condition that* $y|_{\partial\Omega} = 0$.

Problem 2 can be reduced to Problem 3, a similar reduction is presented in [Amirov (2001)] page 65 for an another kinetic equation. For this, consider a new unknown function $\bar{y} = y - \psi$, where ψ is a known function such that $\psi \in C^3(\Omega)$ and $\psi|_{\partial\Omega} = y_0$. Since $y_0 \in C^3(\partial\Omega)$ and $\partial D \in C^3$, $\partial G \in C^3$ the existence of the function ψ follows from Theorem 2, Sec. 4.2., Chapter III in [Mikhailov (1978)]. If we again denote $\bar{y}$ by y, then we obtain Eq. 10 and the condition $y|_{\partial\Omega} = 0$, where $F = -L\psi$. Here, the function $\bar{y}$ depends on F and so, on ψ. From the uniqueness of the solution to Problem 2, a function $y = \bar{y} + \psi$ does not depend on the choice of ψ (also on F) and it depends only on y_0.

Theorem 2 *Under the assumptions of Theorem 1, suppose that* $F \in H^2(\Omega)$. *Then there exists a solution* (y,a) *of Problem* 3 *such that* $y \in \Gamma(A)$, $y \in H^1(\Omega)$, $a \in L_2(\Omega)$.

Proof. We consider the following auxiliary problem

$$Ay = \mathscr{F}, \tag{11}$$

$$y|_{\partial\Omega} = 0, \tag{12}$$

where $\mathscr{F} = \widehat{L}F$. For problem (11)-(12), an approximate solution

$$y_N = \sum_{i=1}^{N} \alpha_{N_i} w_i; \qquad \alpha_N = (\alpha_{N_1}, \alpha_{N_2}, ..., \alpha_{N_N}) \in \mathbb{R}^N \tag{13}$$

is defined as a solution to the following problem:

Find the vector α_N from the system of linear algebraic equations (SLAE)

$$\langle Ay_N - \mathscr{F}, w_i \rangle = 0, \qquad i = 1, 2, ..., N. \tag{14}$$

We will prove that under the hypotheses of Theorem 2, system (14) has a unique solution α_N for any function $F \in H^2(\Omega)$. For this purpose, ith equation of the homogeneous system ($\mathscr{F} = 0$) is multiplied by $-2\alpha_{N_i}$ and sum from 1 to N with respect to i. Hence $-2\langle Ay_N, y_N \rangle = 0$ is obtained. From the equality $-\langle Ay_N, y_N \rangle = J(y_N)$ and the condition (4), we obtain $\nabla y_N = 0$, where $\nabla y_N = (y_{Nx_1}, ..., y_{Nx_n}, y_{Nv_1}, ..., y_{Nv_n})$. So, $y_N = 0$ in Ω as a result of the conditions $y_N = 0$ on $\partial\Omega$ and $y_N \in \widetilde{C}_0^3(\Omega)$. Since the system $\{w_i\}$ is linearly independent, we get $\alpha_{N_i} = 0$, $i = 1, 2, ..., N$. Thus the homogeneous version of the system of linear algebraic equations (14) has only a trivial solution and therefore the original inhomogeneous system (14) has a unique solution $\alpha_N = (\alpha_{N_i})$, $i = 1, ..., N$ for any function $F \in H^2(\Omega)$.

Now we estimate the solution y_N of system (14) in terms of F. We multiply the ith equation of the system by $-2\alpha_{N_i}$ and sum from 1 to N with respect to i. Since $\mathscr{F} = \widehat{L}F$, we obtain

$$-2\langle Ay_N, y_N \rangle = -2\left\langle \widehat{L}F, y_N \right\rangle. \tag{15}$$

Observing that $y_N = 0$ on $\partial\Omega$, the right-hand side of (15) is estimated as follows:

$$\begin{aligned} -2\left\langle \widehat{L}F, y_N \right\rangle &= 2\int_\Omega \sum_{i=1}^{n} \frac{\partial F}{\partial v_i}\frac{\partial y_N}{\partial x_i} d\Omega \\ &\leq \beta \int_\Omega |\nabla_v F|^2 d\Omega + \beta^{-1} \int_\Omega |\nabla_x y_N|^2 d\Omega, \end{aligned} \tag{16}$$

for a sufficiently large $\beta > 0$ and $\nabla_v F = (F_{v_1}, ..., F_{v_n})$. Since left hand-side of (15) is equal to $2J(y_N)$, from the assumption of the theorem we have

$$\int_\Omega |\nabla_x y_N|^2 d\Omega + \alpha_1 \int_\Omega |\nabla_v y_N|^2 d\Omega \leq \beta \int_\Omega |\nabla_v F|^2 d\Omega + \beta^{-1} \int_\Omega |\nabla_x y_N|^2 d\Omega. \tag{17}$$

Since Ω is bounded and $y_N = 0$ on $\partial\Omega$, from (17)

$$\|y_N\|_{\overset{\circ}{H}{}^1(\Omega)} \leq C \| |\nabla_v F| \|_{L_2(\Omega)}, \tag{18}$$

is obtained, where the constant $C > 0$ does not depend on N.

Thus, the set of functions y_N, $N = 1,2,3,...$ is bounded in $\overset{\circ}{H}{}^1(\Omega)$. Since $\overset{\circ}{H}{}^1(\Omega)$ is a Hilbert space, there exists a subsequence in this set that is denoted again by $\{y_N\}$ converging weakly in $\overset{\circ}{H}{}^1(\Omega)$ to a certain function $y \in \overset{\circ}{H}{}^1(\Omega)$. From inequality (18) and weak convergence of $\{y_N\}$ to y in $\overset{\circ}{H}{}^1(\Omega)$, it follows that

$$\|y\|_{\overset{\circ}{H}{}^1(\Omega)} \leq \underline{\lim}_{N\to\infty} \|y_N\|_{\overset{\circ}{H}{}^1(\Omega)} \leq C\,\||\nabla_\nu F|\|_{L_2(\Omega)}. \tag{19}$$

From estimate (18), it is easy to prove that there exists a subsequence of $\{y_N\}$ and

$$\left\langle Ly_N - F, \widehat{L}w_i \right\rangle = 0. \tag{20}$$

Since the linear span of the functions w_i, $i = 1,2,3,...$ is everywhere dense in $\overset{\circ}{H}_{1,2}(\Omega)$ passing to the limit as $N \to \infty$ in (20), we obtain

$$\left\langle Ly - F, \widehat{L}\eta \right\rangle = 0, \tag{21}$$

for any $\eta \in \overset{\circ}{H}_{1,2}(\Omega)$. If we set $a = Ly - F$, from (21) we see that the function a satisfies the condition (2) and the following estimate is valid:

$$\|a\|_{L_2(\Omega)} \leq C\,\|y\|_{\overset{\circ}{H}{}^1(\Omega)} + \|F\|_{L_2(\Omega)}. \tag{22}$$

Consequently, using the inequality

$$\|y\|_{\overset{\circ}{H}{}^1(\Omega)} \leq C\,\||\nabla_\nu F|\|_{L_2(\Omega)},$$

we obtain

$$\|a\|_{L_2(\Omega)} \leq C\,\|\nabla_\nu F\|_{L_2(\Omega)} + \|F\|_{L_2(\Omega)}, \tag{23}$$

where C stands for different constants that depend only on the given functions and the size of the domain Ω.

Thus we have found a solution (y,a) to Problem 3, where $y \in \overset{\circ}{H}{}^1(\Omega)$ and $a \in L_2(\Omega)$. Now we will show that $y \in \Gamma(A)$. Since $y \in L_2(\Omega)$ and $F \in H^2(\Omega)$, it follows that $\mathscr{F} = Ay \in L_2(\Omega)$ in the generalized sense. For any $\eta \in C_0^\infty(\Omega)$, the following equalities hold.

$$\langle y, A^*\eta \rangle = \left\langle y, L^*\widehat{L}\eta \right\rangle = \left\langle Ly, \widehat{L}\eta \right\rangle = \left\langle F, \widehat{L}\eta \right\rangle = \langle \mathscr{F}, \eta \rangle. \tag{24}$$

Now, we have to show that

$$\langle Ay_N, y_N\rangle \to \langle Ay, y\rangle \text{ as } N \to \infty. \tag{25}$$

We have $\mathscr{P}_N Ay_N = \mathscr{P}_N \mathscr{F}$ from (14) and $\mathscr{P}_N \mathscr{F}$ strongly converges to $\mathscr{F}$ in $L_2(\Omega)$ as $N \to \infty$, since $\mathscr{P}_N$ is an orthogonal projector onto M_n. In other words, $\mathscr{P}_N Ay_N \to \mathscr{F} = Ay$ strongly in $L_2(\Omega)$ as $N \to \infty$. Then we have $\langle \mathscr{P}_N Ay_N, y_N\rangle \to \langle Ay, y\rangle$ as $N \to \infty$ because $\{y_N\}$ weakly converges to y and $\{\mathscr{P}_N Ay_N\}$ strongly converges to Ay in $L_2(\Omega)$ as $N \to \infty$. Since the operator $\mathscr{P}_N$ is self adjoint in L_2,

$$\langle Ay_N, y_N\rangle = \langle Ay_N, \mathscr{P}_N y_N\rangle = \langle \mathscr{P}_N Ay_N, y_N\rangle. \tag{26}$$

Consequently, we obtain the convergence $\langle Ay_N, y_N\rangle \to \langle Ay, y\rangle$ as $N \to \infty$. ■

Theorem 3 *Under the hypotheses of Theorem 1, assume that $u_0 \in H^2(\partial\Omega)$ and $u_0 \geq \alpha_0$, where α_0 is a positive number. Then there exists a solution (u, a) of Problem 1 such that $u \in H^2(\Omega)$, $a \in L_2(\Omega)$.*

4 Algorithm of Solving the Inverse Problem

An approximate solution to Problem 3 will be sought in the following form

$$y_N = \sum_{i=0}^{N-1} \alpha_{N_i} w_i. \tag{27}$$

We give the solution algorithm, for the domains

$$D = \{x \in \mathbb{R}^n : |x| < 1\},\ G = \{v \in \mathbb{R}^n : |v| < 1\}$$

and consider the complete systems

$$\left\{x_1^{i_1}, \ldots, x_n^{i_n}\right\}_{i_1,\ldots,i_n=0}^{\infty}, \quad \left\{v_1^{j_1} \ldots v_n^{j_n}\right\}_{j_1,\ldots,j_n=0}^{\infty}, \quad \left\{1, t, t^2, \ldots\right\}$$

in $L_2(D)$, $L_2(G)$ and $L_2(0,T)$ respectively. The approximate solution can be written in the following form:

$$y_N = \sum_{i_1,\ldots,i_n,j_1,\ldots,j_n,k=0}^{N-1} \alpha_{N_{i_1,\ldots,i_n,j_1,\ldots,j_n,k}} w_{i_1,\ldots,i_n,j_1,\ldots,j_n,k} \eta(x)\mu(v)\zeta(t) \tag{28}$$

where

$$w_{i_1,\ldots,i_n,j_1,\ldots,j_n,k} = \left\{x_1^{i_1} \ldots x_n^{i_n} v_1^{j_1} \ldots v_n^{j_n} t^k\right\}_{i_1,\ldots,i_n,j_1,\ldots,j_n,k=0}^{\infty},$$

$$\eta(x) = \begin{cases} 1-|x|^2, & |x|<1 \\ 0, & |x|\geq 1 \end{cases},$$

$$\mu(v) = \begin{cases} 1-|v|^2, & |v|<1 \\ 0, & |v|\geq 1 \end{cases},$$

$$\zeta(t) = \begin{cases} 1-t^2, & |t|<1 \\ 0, & |t|\geq 1 \end{cases}.$$

In expression (28), unknown coefficients $\alpha_{N_{i_1,\dots,i_n,j_1,\dots,j_n,k}}$, $i_1,\dots,i_n,\ j_1,\dots,j_n,\ k = 0,1,\dots,N-1$ are determined from the following system of linear algebraic equations (SLAE):

$$\sum_{i_1,\dots,i_n,j_1,\dots,j_n,k=0}^{N-1} \left(A\left(\alpha_{N_{i_1,\dots,i_n,j_1,\dots,j_n,k}} w_{i_1,\dots,i_n,j_1,\dots,j_n,k}\right)\eta\mu\zeta, w_{i'_1,\dots,i'_n,j'_1,\dots,j'_n,k'}\right)_{L_2(\Omega)}$$
$$= \left(\mathscr{F}, w_{i'_1,\dots,i'_n,j'_1,\dots,j'_n,k'}\right)_{L_2(\Omega)}, \quad i'_1,\dots,i'_n,\ j'_1,\dots,j'_n,\ k' = 0,1,\dots,N-1. \quad (29)$$

Left side of each equation in (29) is constructed.

Algorithm 1 $(LeftSLAE)$

INPUT: N, i'_1, ..., i'_n, j'_1,..., j'_n, k', $w_{i'_1,\dots,i'_n,j'_1,\dots,j'_n,k'}$

OUTPUT: Left hand side of each equation in (29): LeftSum

Set LeftSum=0;

For $i_1 = 0,\dots,N-1$ do ... For $i_n = 0,\dots,N-1$ do

For $j_1 = 0,\dots,N-1$ do ... For $j_n = 0,\dots,N-1$ do For $k = 0,\dots,N-1$ do

$LeftSum = LeftSum + \left(A\left(\alpha_{N_{i_1,\dots,i_n,j_1,\dots,j_n,k}} w_{i_1,\dots,i_n,j_1,\dots,j_n,k}\right)\eta\mu\zeta, w_{i'_1,\dots,i'_n,j'_1,\dots,j'_n,k'}\right)_{L_2(\Omega)}$

end k end j_n ...end j_1 end i_n ...end i_1

STOP (The procedure is complete.)

Algorithm 2

INPUT: N, $F(x,v,t)$, $f(x,v,t)$

OUTPUT: Approximate solution u_N and the coefficient a

$SLAE = \{\}$, $y_N = 0$,

For $i'_1 = 0,...,N-1$ do ... For $i'_n = 0,...,N-1$ do

For $j'_1 = 0,...,N-1$ do For $j'_n = 0,...,N-1$ do For $k' = 0,...,N-1$ do

$SLAE = SLAE \cup \left\{ LeftSLAE\left(i'_1,...,i'_n, j'_1,...,j'_n, k', N, \eta, \mu, \zeta, w_{i'_1,...,i'_n,j'_1,...,j'_n,k'}\right)\right\}$

$= \left(\mathscr{F}, w_{i'_1,...,i'_n,j'_1,...,j'_n,k'}\right)_{L_2(\Omega)}$

end k' end j'_n ...end j'_1 end i'_n ...end i'_1

Solve $\left(SLAE, \left\{\alpha_{N_{i_1,...,i_n,j_1,...,j_n,k}}\right\}\right)$

Principle Part

For $i_1 = 0,...,N-1$ do ... For $i_n = 0,...,N-1$ do

For $j_1 = 0,...,N-1$ do ... For $j_n = 0,...,N-1$ do For $k = 0,...,N-1$ do

$y_N = y_N + \left(\alpha_{N_{i_1,...,i_n,j_1,...,j_n,k}} w_{i_1,...,i_n,j_1,...,j_n,k}\right) \eta(x)\, \mu(v)\, \zeta(t)$

end k end j_n ...end j_1 end i_n ...end i_1

$u_N(x,v,t) = e^{y_N}$, $a(x,v,t) = L(y_N) - F(x,v,t)$

End of the Algorithm 2.

This algorithm computes the approximate solution using Algorithm 1.

The algorithms have been implemented in the computer algebra system Maple and tested for several inverse problems. Two examples are presented below where U_N shows the computed solution at N and N is the order of sum in (28).

Example 1 *Let us consider Problem 3 on*

$$\Omega = \{(x,v,t) |\ x \in (-1,1),\ v \in (-1,1),\ t \in (-1,1)\},$$

with the given functions

$$F(x,v,t) = -2txv + 2txv^3 + 2tx^3v - 2tx^3v^3 - 3v^2x^2 + 3v^2x^2t^2 + 3x^2v^4 - 3x^2v^4t^2$$

and $f_1(x,v,t) = 0$. *Then, at* $N = 2$ *the method gives the result:*

$$\begin{aligned} U_2 =& e^{(1-x^2)(1-v^2)\left(1-t^2\right)xv}, \\ a_2 =& -2vx(1-x^2)(1-v^2)t + v(1-v^2)(1-t^2)(v(1-x^2)-2vx^2) + 2txv - 2txv^3 \\ &- 2tx^3v + 2tx^3v^3 + 3v^2x^2 - 3v^2x^2t^2 - 3x^2v^4 + 3x^2v^4t^2 \end{aligned}$$

and this is the exact solution.

Example 2 *Consider Problem 3 on*

$$\Omega = \{(x,v,t) |\ x \in (-1,1),\ v \in (1,2),\ t \in (-1,1)\},$$

then we take $\mu(v)$ *as*

$$\mu(v) = \begin{cases} (1-v)(2-v), & v \in (1,2) \\ 0, & v \notin (1,2) \end{cases},$$

so according to the given functions

$$\begin{aligned} F(x,v) =& x^2(-4t + 2t(v-2)^2)/v + (2tx^4)/v \\ &+ x(v-2)^2(-2 + 2t^2 - 3tx + 3v - 3vt^2 + txv - v^2 + v^2t^2) \\ &+ x^3v(6-6t^2) + vx(-6+6t^2) - 2tx^2v + tx^4v - 2x^3v^2 + 2x^3t^2v^2 \\ &+ 2v^2x - 2v^2xt^2 \end{aligned}$$

and $f_1(x,v,t) = 0$, *approximate solution of the problem at* $N = 1$ *is*

$$U_1 = e^{-\frac{1}{2}(1-x^2)(2-3v+v^2)\left(1-t^2\right)},$$

where the exact solution is

$$u(x,v) = e^{\frac{1}{2v}\left(x^2+(2-v)^2-1\right)(1-x^2)(2-3v+v^2)\left(1-t^2\right)}.$$

A comparison between the approximate and the exact solution $u(x,v,t)$ *of the problem is presented on Figure 1.* a_1 *and* a_4 *can be obtained from equation* $Ly = a + F$ *easily.*

In example 1, computed solution at $N = 2$ coincides with the exact solution of the problem and in example 2, as it can be seen from Figure 1b, approximate solution at $N = 4$ is very closed to the exact solution. Consequently, the computational experiments show that the proposed algorithm gives efficient and reliable results.

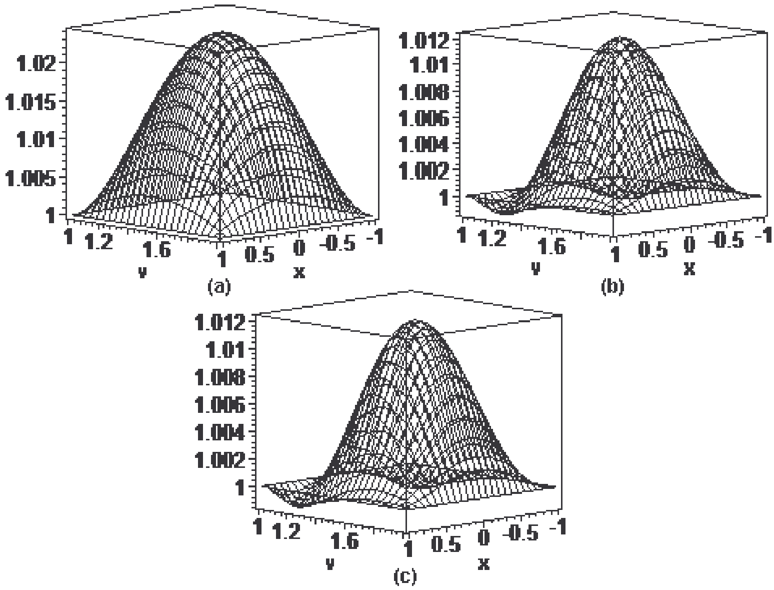

Figure 1: Approximate and exact solution of the problem (a) $N = 1$, (b) $N = 4$, (c) Exact.

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