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The Particular Solutions of Chebyshev Polynomials for Reissner Plates under Arbitrary Loadings

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Abstract: Analytical particular solutions of Chebyshev polynomials are obtained for problems of Reissner plates under arbitrary loadings, which are governed by three coupled second-ordered partial differential equation (PDEs). Our solutions can be written explicitly in terms of monomials. By using these formulas, we can obtain the approximate particular solution when the arbitrary loadings have been represented by a truncated series of Chebyshev polynomials. In the derivations of particular solutions, the three coupled second-ordered PDE are first transformed into a single six-ordered PDE through the Hörmander operator decomposition technique. Then the particular solutions of this six-ordered PDE can be found in the author's previous study. These formulas are further implemented to solve problems of Reissner plates under arbitrary loadings in which the homogeneous solutions are complementarily solved by the method of fundamental solutions (MFS). Numerical experiments are carried out to validate these particular solutions. Due to the exponential convergence of both Chebyshev interpolation and the MFS, our numerical results are extremely accurate.

Keywords: Particular solution, Chebyshev polynomials, Reissner plate, Hörmander operator decomposition technique, method of fundamental solutions, method of particular solutions

1 Introduction

There are two main theories to model plate structures: the classical thin plate theory and the shear deformable thick plate theory. Theoretically, the thin plate theory ignores the effect of shear deformation through the thickness and therefore the elastic behaviors are not able to be captured accurately. The thick plate, as well as its generalization to shell, is frequently used for plates and shells of small to moderate thickness. In the last few decades, numerical methods have been well developed

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to solve plate problems, such as the finite element method (FEM), the boundary element method (BEM) and the method of fundamental solutions (MFS).

In this paper we concentrate on obtaining the particular solutions of thick plates under arbitrary loadings when boundary-type numerical methods are utilized. Boundary-type numerical methods have become a popular research field because they can reduce the computational dimensionalities by one and thus can save the computational costs. In these methods, the numerical discretization is performed either on the solution boundary, or on a boundary-like geometry. Examples include the boundary element method (BEM) [Criado, Ortiz, Mantic, Gray, and Paris (2007); Davies, Crann, Kane, and Lai (2007); Sanz, Solis, and Dominguez (2007); He, Lim, and Lim (2008); Lee and Chen (2008); Marin, Power, Bowtell, Sanchez, Becker, Glover, and Jones (2008); Owatsiriwong, Phansri, and Park (2008)], the MFS [Kupradze and Aleksidze (1964); Bogomolny (1985); Liu (2008B); Marin (2008); Young, Chen, Liu, Shen, and Wu (2009)], the Trefftz Methods (TM) [Cozzano, Rodríguez (2007); Liu (2007A); Liu (2007B); Liu (2008A)], the meshless local boundary integral equation method [Zhu, Zhang, and Atluri (1998); Sellountos, Sequeira, and Polyzos (2009)], and others. A comprehensive review can be found in the article of Cheng and Cheng (2005).

Boundary-type numerical methods had also been applied to solve thick plate problems. EL-Zafrany, Debbih and Fadhil (1995) studied the thick Reissner plates in bending by the BEM. On the other hand, Fadhil and EL-Zafrany (1994) and AL-Hosani, Fadhil and EL-Zafrany (1999) developed a BEM model for Reissner plates resting on foundations. Recently, Wen (2008) and Wen, Adetoro and Xu (2008) applied the MFS to solve the problems of thick plates in the Laplace domain. However, these studies were limited to concentrated and uniform loadings. Therefore, the situation of arbitrary loadings should be studied.

When boundary-type numerical methods are applied to solve PDEs, special algorithms are required for the inhomogeneous parts. A usual way to eliminate the inhomogeneous term is by the method of particular solutions (MPS) proposed by Golberg and Chen (1999). The original idea of this formulation stemmed form the dual reciprocity method (DRM) invented by Nardini and Brebbia (1982) which is actually equivalent to the MPS. The applicability of the MPS depends on the availability of the particular solution associated with right hand side function as well as the partial differential operator of the problem. Since analytical particular solutions are rare, approximate methods for the particular solution have to be sought. This can be accomplished by numerically approximating the right hand side function by a summation of basis functions. Once the basis functions are selected, the problem of finding particular solution associated to the right hand side function is reduced to the problem of obtaining the particular solutions of the basis functions. In other word, the success of the MPS depends on the availability of the exact expression of the particular solutions associated to the basis functions and the operator of the problem. In this paper, we will derive analytical particular solutions of Chebyshev polynomials associated to problems of Reissner plates.

In the last few decades, significant progress has been done in obtaining analytical particular solutions for various basis functions. Among these are the radial basis functions [Golberg (1995); Golberg, Chen, and Karur (1996); Golberg and Chen (1999); Muleshkov, Golberg, and Chen (1999); Cheng (2000); Cheng, Chen, Golberg, Rashed (2001); Muleshkov and Golberg (2007); Tsai (2009); Tsai, Cheng, Chen (2009)], the trigonometric functions [Atkinson (1985); Li and, Chen (2004)], the monomials [Janssen and, Lambert (1992); Cheng, Lafe, and Grilli (1994); Golberg, Muleshkov, Chen, and Cheng (2003); Tsai (2008); Tsai, Cheng, Chen (2009)], Chebyshev polynomials [Golberg, Muleshkov, Chen, and Cheng (2003); Reutskiy and Chen (2006); Karageorghis, and Kyza (2007); Ding and Chen (2007); Tsai (2008)] and others. In this study, we consider the analytical particular solution corresponding to Chebyshev polynomials for problems of Reissner plates in bending.

In the original study of particular solutions of Chebyshev polynomials, Golberg, Muleshkov, Chen, and Cheng (2003) utilized the symbol software *Mathematica* to connect monomials with Chebyshev polynomials and used their derived particular solution for floating number computing. However some book keepings are required in their study. Reutskiy and Chen (2006) circumvented the tedious book keeping by using two-stage approximations of trigonometric functions and Chebyshev polynomials. On the other hand, Karageorghis, and Kyza (2007) studied the same issue by directly deriving the particular solutions in terms of Chebyshev polynomials. However matrix inverses are conducted to their final formulas. Alternatively, Ding and Chen (2007) developed a recursive formulation free from book keepings and matrix inverses. Recently, Tsai (2008) analytically derived the particular solutions of Chebyshev polynomials with respect to polyharmonic and poly-Helmholtz operators and implemented the derived formulation free from book keepings and matrix inverses. Compared to the recursive formulation, this analytical formulation is easier and more suitable for higher-ordered partial differential equations (PDEs) and three dimensions.

In this paper, we will extend Tsai's study (2008) to problems of Reissner plates in bending which are governed by three coupled second-ordered PDEs. In our derivations, the coupled PDEs are first transformed into a single six-ordered PDE through the Hörmander operator decomposition technique [Hörmander (1963)]. Then the particular solutions of Chebyshev polynomials for problems of Reissner plates can be obtained by utilizing the particular solutions of the single six-ordered PDE addressed in the literature [Tsai (2008)].

After the particular solutions of Reissner plates under arbitrary loading are obtained, the homogeneous solutions are complementarily solved by the method of fundamental solutions (MFS). The MFS was first proposed by Kupradze and Aleksidze (1964) and mathematically established by Bogomolny (1985). Then the MFS have been intensively studied by several researches, such as Tsai, Young, Cheng (2002), Smyrlis, Karageorghis (2003), Chen, Fan, Young, Murugesan, Tsai (2005), Young, Ruan (2005), Young, Chen, Chen, Kao (2007), Hu, Young, Fan (2008) and et al. The combination of Chebyshev interpolation and the MFS forms a boundary-type meshless numerical method which is very accurate since both of them are of exponential convergences.

The contents of this paper are organized as following: the Ressner plate model is stated in Section 2 and the fundamental solutions are reviewed in Section 3. Then Chebyshev interpolation and the corresponding analytical particular solutions are derived in Section 4 and Section 5, respectively. In Section 6, the implementation of the derived particular solutions is explained. Section 7 gives the MFS-MPS formulations for problems of Reissner plates and the numerical results are delineated in Section 8. Finally, the main conclusions of the present study are given in Section 9.

2 Reissner plate model

In the following of this article, the indices i, j, k are presented in the range $\{1,2,3\}$ and the indices α, β, γ are in the range $\{1,2\}$.

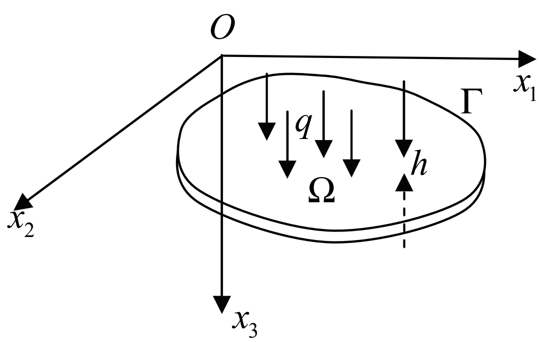

Figure 1: Geometric configuration of the Ressiner plate model.

As described in Fig. 1, we consider a plate of uniform thickness h with its middle plane being a domain Ω with boundary Γ in the $x_1 - x_2$ plane and thickness coordinate x_3. The plate is subjected to normal loading with intensity $q(x_1, x_2)$. Then the equations of equilibrium are given by

$$\begin{cases} \frac{\partial M_{\alpha\beta}}{\partial x_\beta} - Q_\alpha = 0 \\ \frac{\partial Q_\alpha}{\partial x_\alpha} + q = 0 \end{cases} \tag{1}$$

where Q_α are the transverse shears, $M_{\alpha\beta}$ are the bending and twisting moments, all per unit length.

The stress distribution is assumed to follow the Reissner thick plate theory, in which the generalized displacement expressions with a weighted average across the thickness are introduced that u_3 is the lateral deflection of the plate in the middle surface and u_αs represent the average slope angles. According to the Reissner thick plate theory, the constitutive equations written in terms of the displacements and the distributed load $q(x_1, x_2)$ on the plate surface are given by:

$$\begin{cases} M_{\alpha\beta} = \frac{D(1-\nu)}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} + \frac{2\nu}{1-\nu}\frac{\partial u_\gamma}{\partial x_\gamma}\delta_{\alpha\beta}\right) + q\frac{\nu}{\lambda^2(1-\nu)}\delta_{\alpha\beta} \\ Q_\alpha = \frac{D(1-\nu)\lambda^2}{2}\left(u_\alpha + \frac{\partial u_3}{\partial x_\alpha}\right) \end{cases} \tag{2}$$

with

$$\lambda = \sqrt{10}/h \tag{3}$$

Combining Eqs. (1) and (2), we can obtain

$$L_{ij}(\partial)u_j = -E_i(\partial)q \tag{4}$$

with

$$\begin{cases} L_{\alpha\beta} = \frac{D(1-\nu)}{2}(\nabla^2 - \lambda^2)\delta_{\alpha\beta} + \frac{D(1+\nu)}{2}\frac{\partial^2}{\partial x_\alpha \partial x_\beta} \\ L_{\alpha 3} = -L_{3\alpha} = -\frac{D\lambda^2(1-\nu)}{2}\frac{\partial}{\partial x_\alpha} \\ L_{33} = \frac{D\lambda^2(1-\nu)}{2}\nabla^2 \end{cases} \tag{5}$$

and

$$\begin{cases} E_\alpha = \frac{\nu}{(1-\nu)\lambda^2}\frac{\partial}{\partial x_\alpha} \\ E_3 = 1 \end{cases} \tag{6}$$

Eq. (4) are three coupled second-ordered PDEs for the three unknown function u_j and thus three boundary conditions on Γ should be imposed to form a well-posed problem as follows

$$\mathbf{B}_i \cdot \mathbf{u} = b_i \tag{7}$$

where $\mathbf{B}_i$ are the vector-valued boundary differential operators, b_i are given boundary data and $\mathbf{u} = (u_1, u_2, u_3)$. Usually, three of the following six boundary conditions shall be adopted.

$$\begin{cases} \mathbf{B}_{\theta_n} \cdot \mathbf{u} = n_1 u_1 + n_2 u_2 \\ \mathbf{B}_{\theta_t} \cdot \mathbf{u} = -n_2 u_1 + n_1 u_2 \\ \mathbf{B}_w \cdot \mathbf{u} = u_3 \\ \mathbf{B}_{M_n} \cdot \mathbf{u} = n_1^2 M_{11} + 2n_1 n_2 M_{12} + n_2^2 M_{22} \\ \mathbf{B}_{M_t} \cdot \mathbf{u} = -n_1 n_2 M_{11} + (n_1^2 - n_2^2) M_{12} + n_1 n_2 M_{22} \\ \mathbf{B}_{Q_n} \cdot \mathbf{u} = n_1 Q_1 + n_2 Q_2 \end{cases} \tag{8}$$

where n_1 and n_2 are related to the boundary normal and tangential vectors respectively by

$$\mathbf{n} = (n_1, n_2, 0) \quad \mathbf{t} = (-n_2, n_1, 0) \tag{9}$$

For example, the clamped boundary conditions are defined by

$$\begin{cases} \mathbf{B}_1 = \mathbf{B}_{\theta_n} \\ \mathbf{B}_2 = \mathbf{B}_{\theta_t} \\ \mathbf{B}_3 = \mathbf{B}_w \end{cases} \tag{10}$$

and the free boundary conditions are represented by

$$\begin{cases} \mathbf{B}_1 = \mathbf{B}_{M_n} \\ \mathbf{B}_2 = \mathbf{B}_{M_t} \\ \mathbf{B}_3 = \mathbf{B}_{Q_n} \end{cases} \tag{11}$$

In this paper we will develop a meshless numerical method to solve the well-posed PDE system given in Eqs. (4) and (7).

3 Fundamental solution

In order to apply the MFS for Eq. (4), we need the fundamental solution u_j^* defined by

$$L_{ij}(\partial) u_{jk}^* = -\delta(\mathbf{x} - \mathbf{s}) \delta_{ik} \tag{12}$$

where $\mathbf{x} = (x_1, x_2)$ is the coordinate and $\mathbf{s} = (s_1, s_2)$ is the source point. To derive the fundamental solution, we use the Hörmander operator decomposition technology

[Hörmander (1963)], in which the fundamental solutions should be assumed to be of the following form:

$$u^*_{jk} = L^{adj}_{jk}(\partial)G \tag{13}$$

where G is an unknown function which can be understood later and the adjoint operator $L^{adj}_{jk}(\partial)$ is defined by

$$L^{adj}_{ij}(\partial)L_{jk}(\partial) = L_{ij}(\partial)L^{adj}_{jk} = \det(\mathbf{L})\delta_{ik} \tag{14}$$

with $\det(\mathbf{L})$ being the determinant of the matrix operator $\mathbf{L} = \left\{L_{ij}(\partial)\right\}$. By using the definition of $L^{adj}_{jk}(\partial)$, we have

$$\begin{cases} L^{adj}_{\alpha\beta} = 2\delta_{\alpha\beta}\nabla^4 - \frac{\partial^2}{\partial x_\alpha \partial x_\beta}[(1+\nu)\nabla^2 + (1-\nu)\lambda^2] \\ L^{adj}_{\alpha 3} = (1-\nu)\frac{\partial}{\partial x_\alpha}(\nabla^2 - \lambda^2) \\ L^{adj}_{3\alpha} = -(1-\nu)\frac{\partial}{\partial x_\alpha}(\nabla^2 - \lambda^2) \\ L^{adj}_{33} = (\nabla^2 - \lambda^2)[2\nabla^2 - (1-\nu)\lambda^2]/\lambda^2 \end{cases} \tag{15}$$

and

$$\det(\mathbf{L}) = \frac{D^3\lambda^2(1-\nu)^2}{4}(\nabla^2 - \lambda^2)\nabla^2\nabla^2 \tag{16}$$

Then, we substitute Eq. (13) into Eq. (12) and use Eqs. (14) and (16) to have:

$$\frac{D^3\lambda^2(1-\nu)^2}{4}(\nabla^2 - \lambda^2)\nabla^2\nabla^2 G = -\delta(\mathbf{x}-\mathbf{s}) \tag{17}$$

The solution of Eq. (17) can be found in the literature [Cheng, Antes and Ortner (1994)] as follows:

$$G(\mathbf{x},\mathbf{s}) = \frac{-\lambda^2 r^2 + 4K_0(\lambda r) + (4+\lambda^2 r^2)\log r}{2\pi D^3(1-\nu)^2\lambda^6} \tag{18}$$

where $r = \sqrt{(x_1-s_1)^2 + (x_2-s_2)^2}$. Eqs. (13) and (18) are sufficient for the fundamental solutions as follows:

$$\begin{cases} u^*_{\alpha\beta} = \frac{\delta_{\alpha\beta}[(1-\nu)\lambda^2 r^2 - 8 + 8\lambda^2 r^2 K_0(\lambda r) + 8\lambda r K_1(\lambda r) - 2(1-\nu)\lambda^2 r^2 \log r]}{8\pi D\lambda^2(1-\nu)r^2} \\ \qquad + \frac{x_\alpha x_\beta[8-(1-\nu)\lambda^2 r^2 - 4\lambda^2 r^2 K_2(\lambda r)]}{4\pi D\lambda^2(1-\nu)r^4} \\ u^*_{\alpha 3} = -u^*_{3\alpha} = \frac{x_\alpha(1-2\log r)}{8\pi D} \\ u^*_{33} = \frac{-(1-\nu)\lambda^2 r^2 - [8-(1-\nu)\lambda^2 r^2]\log r}{8\pi D\lambda^2(1-\nu)} \end{cases} \tag{19}$$

Correspondingly, we also have the shears and the moments respectively as follows:

$$\begin{cases} Q^*_{\alpha\beta} = \frac{\delta_{\alpha\beta}[-1+\lambda^2 r^2 K_0(\lambda r)+\lambda r K_1(\lambda r)]}{2\pi r^2} + \frac{x_\alpha x_\beta [2-\lambda^2 r^2 K_2(\lambda r)]}{2\pi r^4} \\ Q^*_{\alpha 3} = -\frac{x_\alpha}{2\pi r^2} \end{cases} \tag{20}$$

$$\begin{cases} \mathrm{M}^*_{\alpha\beta\gamma} = \frac{\delta_{\alpha\beta} x_\gamma [8-(1+\nu)\lambda^2 r^2 - 4\lambda^2 r^2 K_2(\lambda r)]}{4\pi\lambda^2 r^4} + \frac{x_\alpha x_\beta x_\gamma [-16+(1-\nu)\lambda^2 r^2 + 2\lambda^3 r^3 K_3(\lambda r)]}{2\pi\lambda^2 r^6} \\ \qquad + \frac{(\delta_{\alpha\gamma} x_\beta + \delta_{\beta\gamma} x_\alpha)[8-(1-\nu)\lambda^2 r^2 - 4\lambda^2 r^2 K_2(\lambda r) - 2\lambda^3 r^3 K_1(\lambda r)]}{4\pi\lambda^2 r^4} \\ \mathrm{M}^*_{\alpha\beta 3} = \frac{\delta_{\alpha\beta}[(1-\nu)-2(1+\nu)\log r]}{8\pi} - \frac{x_\alpha x_\beta (1-\nu)}{4\pi r^2} \end{cases} \tag{21}$$

which are defined respectively by

$$Q^*_{\alpha i} = \frac{D(1-\nu)\lambda^2}{2}\left(u^*_{\alpha i} + \frac{\partial u^*_{\alpha 3}}{\partial x_\alpha}\right) \tag{22}$$

$$M^*_{\alpha\beta i} = \frac{D(1-\nu)}{2}\left(\frac{\partial u^*_{\alpha i}}{\partial x_\beta} + \frac{\partial u^*_{\beta i}}{\partial x_\alpha} + \frac{2\nu}{1-\nu}\frac{\partial u^*_{\alpha i}}{\partial x_\alpha}\delta_{\alpha\beta}\right) \tag{23}$$

Eqs. (19)–(21) are the fundamental solutions required for the MFS formulation to be introduced later.

4 Chebyshev interpolation

In order to apply boundary-type numerical methods to solve Eq. (4), particular solutions should be approximated. Inspired by the MPS based on Chebyshev polynomials [Golberg and Chen (1999); Golberg, Muleshkov, Chen, and Cheng (2003); Tsai (2008)], we need to interpolate the arbitrarily distributed loading $q(x_1,x_2)$ by the bivariate Chebyshev polynomial interpolation as follows:

$$q(x_1,x_2) \cong \tilde{q}(x_1,x_2) = \sum_{m_1=0}^{M_1}\sum_{m_2=0}^{M_2} a_{m_1 m_2} T_{m_1}\left(\frac{2x_1 - x_{1b} - x_{1a}}{x_{1b} - x_{1a}}\right) T_{m_2}\left(\frac{2x_2 - x_{2b} - x_{2a}}{x_{2b} - x_{2a}}\right) \tag{24}$$

where $[x_{1a},\, x_{1b}] \times [x_{2a},\, x_{2b}]$ is a rectangular domain which is big enough to enclose the computational domain. Then, by the theory of Chebyshev interpolation we have

$$a_{m_1 m_2} = \frac{4}{M_1 M_2 c_{M_1,m_1} c_{M_2,m_2}} \sum_{l_1=0}^{M_1}\sum_{l_2=0}^{M_2} \frac{q(x_{1l_1}, x_{2l_2})}{c_{M_1,l_1} c_{M_2,l_2}} \cos\frac{\pi m_1 l_1}{M_1} \cos\frac{\pi m_2 l_2}{M_2} \tag{25}$$

with

$$c_{M_\alpha,0} = c_{M_\alpha,M_\alpha} = 2$$
$$c_{M_\alpha,m_\alpha} = c_{M_\alpha,l_\alpha} = 1 \text{ for } 1 \leq m_\alpha, l_\alpha \leq M_\alpha - 1 \tag{26}$$

and

$$x_{a,l_\alpha} = \frac{(x_{ab} - x_{aa})}{2} \cos \frac{pl_\alpha}{M_\alpha} + \frac{x_{ab} + x_{aa}}{2} \tag{27}$$

Note that $M_\alpha + 1$ is the number of Gauss-Lobatto nodes in the x_α direction. It should be noticed that the use of Gauss-Lobatto nodes ensure the exponential convergence for Chebyshev interpolation. Details of Chebyshev interpolation can be found in an excellent review book by Mason and Handscomb (2003).

Eq. (24) can also be rewritten in terms of monomials of $\frac{2x_\alpha - x_{\alpha b} - x_{\alpha a}}{x_{\alpha b} - x_{\alpha a}}$ as

$$\tilde{q}(x_1, x_2) = \sum_{\check{m}_1=0}^{M_1} \sum_{\check{m}_2=0}^{M_2} \check{a}_{\check{m}_1 \check{m}_2} \left(\frac{2x_1 - x_{1b} - x_{1a}}{x_{1b} - x_{1a}} \right)^{\check{m}_1} \left(\frac{2x_2 - x_{2b} - x_{2a}}{x_{2b} - x_{2a}} \right)^{\check{m}_2} \tag{28}$$

where

$$\check{a}_{\check{m}_1 \check{m}_2} = \sum_{m_1=\check{m}_1}^{M_1} \sum_{m_2=\check{m}_2}^{M_2} a_{m_1 m_2} d_{m_1,\check{m}_1} d_{m_2,\check{m}_2} \tag{29}$$

In Eq. (29), $d_{m_\alpha,\hat{m}_\alpha}$ is defined by

$$T_{m_\alpha} \left(\frac{2x_a - x_{ab} - x_{aa}}{x_{ab} - x_{aa}} \right) = \sum_{\hat{m}_\alpha=0}^{m_\alpha} d_{m_\alpha,\hat{m}_\alpha} \left(\frac{2x_\alpha - x_{\alpha b} - x_{\alpha a}}{x_{\alpha b} - x_{\alpha a}} \right)^{\hat{m}_\alpha} \tag{30}$$

and it can be computed by using the explicit formula in [Mason and Handscomb (2003)]. Furthermore, Eq. (28) can be simplified to

$$\tilde{q}(x_1, x_2) = \sum_{\hat{m}_1=0}^{M_1} \sum_{\hat{m}_2=0}^{M_2} \hat{a}_{\hat{m}_1 \hat{m}_1} x_1^{\hat{m}_1} x_2^{\hat{m}_2} \tag{31}$$

with

$$\hat{a}_{\hat{m}_1 \hat{m}_2} = \sum_{\hat{m}_1=\hat{m}_1}^{M_1} \sum_{\hat{m}_2=\hat{m}_2}^{M_2} \frac{2^{\hat{m}_1} (-x_{1b} - x_{1a})^{\hat{m}_1 - \hat{m}_1} \hat{m}_1!}{(x_{1b} - x_{1a})^{\hat{m}_1} (\hat{m}_1 - \hat{m}_1)! \hat{m}_1!} \frac{2^{\hat{m}_2} (-x_{2b} - x_{2a})^{\hat{m}_2 - \hat{m}_2} \hat{m}_2!}{(x_{2b} - x_{2a})^{\hat{m}_2} (\hat{m}_2 - \hat{m}_2)! \hat{m}_2!} \hat{a}_{\hat{m}_1 \hat{m}_2} \tag{32}$$

where the following binomial expansion is utilized

$$\left(\frac{2x_a - x_{\alpha b} - x_{\alpha a}}{x_{\alpha b} - x_{\alpha a}} \right)^{\hat{m}_\alpha} = \sum_{\hat{m}_\alpha=0}^{\hat{m}_\alpha} \frac{2^{\hat{m}_\alpha} (-x_{\alpha b} - x_{\alpha a})^{\hat{m}_\alpha - \hat{m}_\alpha} \hat{m}_\alpha!}{(x_{\alpha b} - x_{\alpha a})^{\hat{m}_\alpha} (\hat{m}_\alpha - \hat{m}_\alpha)! \hat{m}_\alpha!} x_\alpha^{\hat{m}_\alpha} \tag{33}$$

Eq. (31) is the desired Chebyshev interpolation in terms of monomials.

5 Analytical particular solution

Eq. (31) indicates that we need the analytical particular solutions, $u_j^{(\hat{m}_1,\hat{m}_2)}$, corresponding to monomials, $x_1^{\hat{m}_1} x_2^{\hat{m}_2}$, defined by

$$L_{ij}(\partial) u_j^{(\hat{m}_1,\hat{m}_2)} = -E_i(\partial) x_1^{\hat{m}_1} x_2^{\hat{m}_2} \tag{34}$$

The solution of Eq. (34) can also be obtained by using the Hörmander operator decomposition technology [Hörmander (1963)]. First of all, we assume

$$u_j^{(\hat{m}_1,\hat{m}_2)} = \tilde{L}_{jk}^{adj}(\partial) E_k(\partial) F^{(\hat{m}_1,\hat{m}_2)} \tag{35}$$

where $F^{(\hat{m}_1,\hat{m}_2)}$ is a unknown function to be determined. Eq. (35) can also be rewritten in Cartesian coordinates as follows:

$$u_i = \mathbf{\Lambda}_{u_i} F^{(\hat{m}_1,\hat{m}_2)} \tag{36}$$

where $\mathbf{\Lambda}_{u_i}$ are the partial differential operators defined by

$$\mathbf{\Lambda}_{u_\alpha} = -\frac{D^2(1-\nu)}{4} \begin{pmatrix} (1-\nu)\lambda^4 - (1-2\nu)\lambda^2 \frac{\partial^2}{\partial x_1^2} - (1-2\nu)\lambda^2 \frac{\partial^2}{\partial x_2^2} \\ -\nu \frac{\partial^4}{\partial x_1^4} - 2\nu \frac{\partial^4}{\partial x_1^2 \partial x_2^2} - \nu \frac{\partial^4}{\partial x_2^4} \end{pmatrix} \frac{\partial}{\partial x_\alpha} \tag{37}$$

$$\mathbf{\Lambda}_{u_3} = -\frac{D^2(1-\nu)}{4} \begin{pmatrix} -(1-\nu)\lambda^4 + (3-2\nu)\lambda^2 \frac{\partial^2}{\partial x_1^2} + (3-2\nu)\lambda^2 \frac{\partial^2}{\partial x_2^2} \\ -(2-\nu) \frac{\partial^4}{\partial x_1^4} - 2(2-\nu) \frac{\partial^4}{\partial x_1^2 \partial x_2^2} - (2-\nu) \frac{\partial^4}{\partial x_2^4} \end{pmatrix} \tag{38}$$

Then, by using Eq. (2) we also have the corresponding shears and moments as follows:

$$Q_\alpha = \mathbf{\Lambda}_{Q_\alpha} F^{(\hat{m}_1,\hat{m}_2)} \tag{39}$$

$$M_{\alpha\beta} = \mathbf{\Lambda}_{M_{\alpha\beta}} F^{(\hat{m}_1,\hat{m}_2)} + \frac{\delta_{\alpha\beta} \nu x_1^{\hat{m}_1} x_2^{\hat{m}_2}}{\lambda^2(1-\nu)} \tag{40}$$

where $\mathbf{\Lambda}_{Q_\alpha}$ and $\mathbf{\Lambda}_{M_{\alpha\beta}}$ are the partial differential operators defined by

$$\mathbf{\Lambda}_{Q_\alpha} = -\frac{D^3\lambda^2(1-\nu)^2}{4} \left(\lambda^2 \frac{\partial^2}{\partial x_1^2} + \lambda^2 \frac{\partial^2}{\partial x_2^2} - \frac{\partial^4}{\partial x_1^4} - 2\frac{\partial^4}{\partial x_1^2 \partial x_2^2} - \frac{\partial^4}{\partial x_2^4} \right) \frac{\partial}{\partial x_\alpha} \tag{41}$$

$$\mathbf{\Lambda}_{M_{\alpha\beta}} = -\frac{D^3(1-\nu)^2}{4} \left(\delta_{\alpha\beta} \frac{R_1}{(1-\nu)} + R_2 \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \right) \tag{42}$$

with

$$R_1 = \nu(1-\nu)\lambda^4\frac{\partial^2}{\partial x_1^2} + \nu(1-\nu)\lambda^4\frac{\partial^2}{\partial x_2^2} - \nu(1-2\nu)\lambda^2\frac{\partial^4}{\partial x_1^4} - 2\nu(1-2\nu)\lambda^2\frac{\partial^4}{\partial x_1^2\partial x_2^2} - \nu(1-2\nu)\lambda^2\frac{\partial^4}{\partial x_2^4} - \nu^2\frac{\partial^6}{\partial x_1^6} - 3\nu^2\frac{\partial^6}{\partial x_1^4\partial x_2^2} - 3\nu^2\frac{\partial^6}{\partial x_1^2\partial x_2^4} - \nu^2\frac{\partial^6}{\partial x_2^6} \quad (43)$$

and

$$R_2 = (1-\nu)\lambda^4 - (1-2\nu)\lambda^2\frac{\partial^2}{\partial x_1^2} - (1-2\nu)\lambda^2\frac{\partial^2}{\partial x_2^2} - \nu\frac{\partial^4}{\partial x_1^4} - 2\nu\frac{\partial^4}{\partial x_1^2\partial x_2^2} - \nu\frac{\partial^4}{\partial x_2^4} \quad (44)$$

Then we can substitute Eq. (35) into Eq. (34) and use Eqs. (14) and (16) to have the governing equation for Fas follows:

$$-\frac{D^3\lambda^2(1-\nu)^2}{4}(\nabla^2-\lambda^2)\nabla^2\nabla^2 F = x_1^{\hat{m}_1}x_2^{\hat{m}_2} \quad (45)$$

The solution of Eq. (45) can be found in Tsai (2008) as follows:

$$F^{(\hat{m}_1,\hat{m}_2)} = \frac{4F_1^{(\hat{m}_1,\hat{m}_2)}}{D^3\lambda^4(1-\nu^2)} + \frac{4F_2^{(\hat{m}_1,\hat{m}_2)}}{D^3\lambda^6(1-\nu^2)} - \frac{4F_3^{(\hat{m}_1,\hat{m}_2)}}{D^3\lambda^6(1-\nu^2)} \quad (46)$$

with

$$\begin{cases} F_1^{(\hat{m}_1,\hat{m}_2)} = \sum\limits_{l=0}^{[\frac{\hat{m}_2}{2}]} \frac{(-1)^l(l+1)\hat{m}_1!\hat{m}_2!x_1^{\hat{m}_1+2l+4}x_2^{\hat{m}_2-2l}}{(\hat{m}_1+2l+4)!(\hat{m}_2-2l)!} \\ F_2^{(\hat{m}_1,\hat{m}_2)} = \sum\limits_{l=0}^{[\frac{\hat{m}_2}{2}]} \frac{(-1)^l\hat{m}_1!\hat{m}_2!x_1^{\hat{m}_1+2l+2}x_2^{\hat{m}_2-2l}}{(\hat{m}_1+2l+2)!(\hat{m}_2-2l)!} \\ F_3^{(\hat{m}_1,\hat{m}_2)} = \sum\limits_{l_1=0}^{[\frac{\hat{m}_1}{2}]}\sum\limits_{l_2=0}^{[\frac{\hat{m}_2}{2}]} \frac{-(l_1+l_2)!\hat{m}_1!\hat{m}_2!x_1^{\hat{m}_1-2l_1}x_2^{\hat{m}_2-2l_2}}{\lambda^{2l_1+2l_2+2}l_1!l_2!(\hat{m}_1-2l_1)!(\hat{m}_2-2l_2)!} \end{cases} \quad (47)$$

which are the solutions for

$$\begin{cases} \nabla^2\nabla^2 F_1^{(\hat{m}_1,\hat{m}_2)} = x_1^{\hat{m}_1}x_2^{\hat{m}_2} \\ \nabla^2 F_2^{(\hat{m}_1,\hat{m}_2)} = x_1^{\hat{m}_1}x_2^{\hat{m}_2} \\ (\nabla^2-\lambda^2)F_3^{(\hat{m}_1,\hat{m}_2)} = x_1^{\hat{m}_1}x_2^{\hat{m}_2} \end{cases} \quad (48)$$

Eq. (36), Eqs. (39)–(40) and Eq. (46) are sufficient for the required analytical particular solutions.

6 Implementation of particular solutions

In the practical implementation of particular solutions, we first approximate the arbitrarily distributed loading $q(x_1,x_2)$ by Chebyshev polynomials in terms of monomials given in Eq. (31). Then in the spirit of the MPS, we can approximate the particular solutions corresponding to $q(x_1,x_2)$ by the following formulas:

$$\tilde{u}_j(x_1,x_2) = \sum_{\hat{m}_1=0}^{M_1}\sum_{\hat{m}_2=0}^{M_2} \hat{a}_{\hat{m}_1\hat{m}_1}\mathbf{\Lambda}_{u_j}F^{(\hat{m}_1,\hat{m}_2)} \tag{49}$$

$$\tilde{Q}_\alpha(x_1,x_2) = \sum_{\hat{m}_1=0}^{M_1}\sum_{\hat{m}_2=0}^{M_2} \hat{a}_{\hat{m}_1\hat{m}_1}\mathbf{\Lambda}_{Q_\alpha}F^{(\hat{m}_1,\hat{m}_2)} \tag{50}$$

$$\tilde{M}_{\alpha\beta}(x_1,x_2) = \sum_{\hat{m}_1=0}^{M_1}\sum_{\hat{m}_2=0}^{M_2} \hat{a}_{\hat{m}_1\hat{m}_1}\left(\mathbf{\Lambda}_{M_{\alpha\beta}}F^{(\hat{m}_1,\hat{m}_2)} + \frac{\delta_{\alpha\beta}\nu x_1^{\hat{m}_1}x_2^{\hat{m}_2}}{\lambda^2(1-\nu)}\right) \tag{51}$$

In Eqs. (49)–(51), we shall use the analytical particular solutions derived in the previous section. Although the explicit forms of $\hat{a}_{\hat{m}_1\hat{m}_2}$ and the analytical particular solutions are very complex, its coding is not very difficult. In Fig. 2, we give the flowchart for the implementation of $\tilde{u}_1(x_1,x_2)$. Furthermore, it should be understood that terms, such like $\mathbf{\Lambda}_{u_1}\left(x_1^{\hat{m}_1+2l+4}x_2^{\hat{m}_2-2l}\right)$, can be computed by coding a subroutine for handling $\frac{\partial^{l_2}\left(x^{l_1}\right)}{\partial x^{l_2}}$ when a real number x and non-negative integers l_1 & l_2 are given. In the solution procedure, there are several summations which can be easily coded by the multiple loops. For example, Eq. (25) is able to be implemented by the pseudo codes depicted in Fig. 3. In the whole computation procedure of the MPS, it can be observed that no book keepings and matrix inverses are required.

7 MFS-MPS formulation

After the particular solutions and the fundamental solutions are introduced, we are in a position to develop the MFS-MFS formulation for problems of Reissner plates governed by Eq. (4) with boundary conditions given in Eq. (7). As usual, the MFS-MPS formulation begins with the principle of superposition

$$u_j = u_j^h + u_j^p \tag{52}$$

In which the particular solution u_j^p satisfies

$$L_{ij}(\partial)u_j^p = -E_i(\partial)q \tag{53}$$

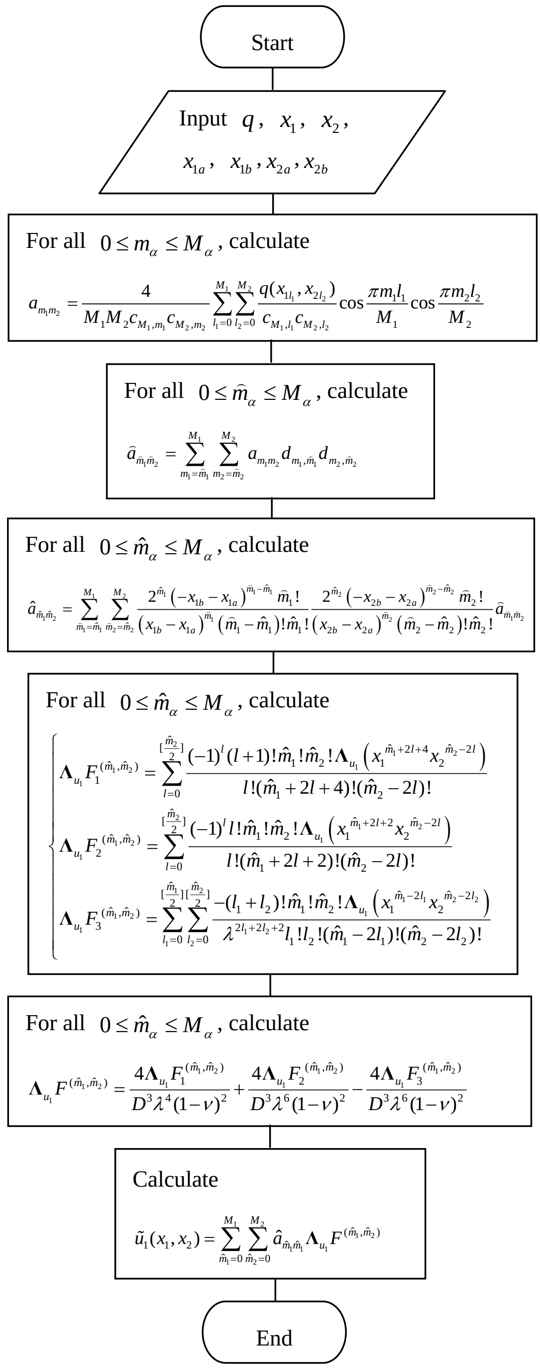

Figure 2: Flowchart for the implementation of $\tilde{u}_1(x_1, x_2)$

```
for m_1 = 0 to M_1
{
 for m_2 = 0 to M_2
 {
  a_{m_1 m_2} = 0
  for l_1 = 0 to M_1
  {
   for l_2 = 0 to M_2
   {
```

$$x_{1l_1} = \frac{(x_{1b}-x_{1a})}{2}\cos\frac{\pi l_1}{M_1} + \frac{x_{1b}+x_{1a}}{2}$$

$$x_{2l_2} = \frac{(x_{2b}-x_{2a})}{2}\cos\frac{\pi l_2}{M_2} + \frac{x_{2b}+x_{2a}}{2}$$

$$a_{m_1m_2} = a_{m_1m_2} + \frac{4}{M_1M_2c_{M_1,m_1}c_{M_2,m_2}}\frac{q(x_{1l_1},x_{2l_2})}{c_{M_1,l_1}c_{M_2,l_2}}\cos\frac{\pi m_1 l_1}{M_1}\cos\frac{\pi m_2 l_2}{M_2}$$

```
   }
  }
 }
}
```

Figure 3: The pseudocode for computing $a_{m_1m_2}$ for all $0 \le m_\alpha \le M_\alpha$.

without specifying any boundary condition. Then the homogeneous solution u_j^h requires

$$L_{ij}(\partial)u_j^h = 0 \tag{54}$$

with the following boundary conditions:

$$\mathbf{B}_i \cdot \mathbf{u}^h = b_i - \mathbf{B}_i \cdot \mathbf{u}^p \tag{55}$$

where $\mathbf{u}^h = (u_1^h, u_2^h, u_3^h)$ and $\mathbf{u}^p = (u_1^p, u_2^p, u_3^p)$.
By applying the MPS based on Chebyshev interpolations, the particular solution governed by Eq. (53) should be approximated by

$$u_j^p \cong \tilde{u}_j(x_1, x_2) \tag{56}$$

where $\tilde{u}_j(x_1,x_2)$ have been introduced in Section 5 and Section 6. Before the introduction of the MFS to solve the homogeneous solution governed by Eqs. (54) and (55), we also need $\mathbf{B}_i \cdot \mathbf{u}^p$ which can be obtained by using Eqs. (49)–(51).

Now, we are ready to solve the homogeneous solution by the MFS. Formally in the applications of the MFS the homogeneous solution can be approximated by

$$u_j^h \cong \sum_{k=1}^{3} \sum_{l=1}^{N} A_l^k u_{jk}^*(\mathbf{x},\mathbf{s}_l) \tag{57}$$

where $u_{jk}^*(\mathbf{x},\mathbf{s}_l)$ is the fundamental solution derived in Section 4, and $\mathbf{s}_l$ are N sources points, which are typically distributed away from the boundary to avoid the singularity when $\mathbf{x}$ and $\mathbf{s}_l$ coincide. Here, A_l^k are the coefficients to be determined. If we collocate Eq. (57) to the boundary conditions in Eq. (55) at N boundary points, we obtain a linear system with $3 \times N$ unknowns, A_l^k, and $3 \times N$ equations, which can be solved if the system is nonsingular. After, A_l^k are solved, we have the homogeneous solution as addressed in Eq. (57). Details of the MFS can be found in the theoretical work of Bogomolny (1985). Also Tsai, Lin, Young, and Atluri (2006) discussed the locations of the source and boundary collocation points. Fig. 4 gives a typical geometrical configuration of the MFS-MPS formulation, where $\mathbf{x}_l$ is the collocation points corresponding to the source point, $\mathbf{s}_l$.

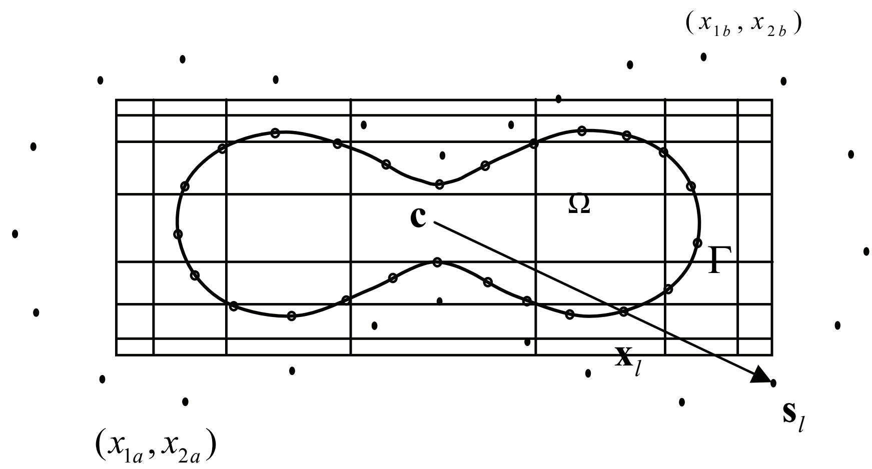

Figure 4: Geometry configuration of the MFS-MPS.

After the homogeneous solution u_j^h and the particular solution u_j^p are solved, we can apply the principle of superposition (Eq. (52)) to get the sought solution.

8 Numerical results

In order to validate the proposed MFS-MPS formulation, three numerical cases are considered. Typically, $D = \lambda = 1000$ and $\nu = 0.3$ are selected in these numerical studies. Furthermore, in the applications of the MFS we uniformly locate sixty boundary field points and place the source points stipulated out on a ten times larger artificial boundary as depicted in Fig. 4 according to the study of Tsai, Lin, Young, and Atluri (2006). These configurations are set up through a preliminary study. This typical setup has been done without thorough explorations since we mainly concentrate on the validation of the derived particular solutions in this study. The root mean square errors (RMSEs) of the numerical solutions are defined by

$$u_{\text{RMSE}} = \sqrt{\frac{\sum\limits_{i=1}^{3}\sum\limits_{l=1}^{L}\left(u_{i,l}^{NUM} - u_{i,l}^{ANA}\right)^2}{3L}} \tag{58}$$

where $u_{i,l}^{NUM}$ is the numerical solution obtained by the MFS-MPS formulation at the l-th point of the L considered positions and $u_{i,l}^{ANA}$ is the corresponding analytical solution of the problem. Here, L shall be chosen to be large enough to ensure the utilization of the RMSEs.

On the other hand, a usual way to validate a numerical method is to set up some analytical solution without the enforcement of the boundary conditions. In our numerical experiments, we solve the following Reissner plate problem:

$$L_{ij}(\partial)u_j = -DE_i(\partial)\sin x_1\left(\sinh x_2 + x_2^2\sinh x_2\right) \tag{59}$$

Analytical solutions of the problem are

$$u_1 = \frac{\cos x_1\begin{pmatrix} -6\nu\left(-8x_2 + \left(1+4x_2+2x_2^2\right)\cosh x_2 - 2\left(1+x_2\right)\sinh x_2\right) \\ -\lambda^2\left(1-\nu\right)\left(48x_2 - 6\left(1+x_2\right)^2\cosh x_2\right. \\ \left. + \left(9+9x_2+6x_2^2+2x_2^3\right)\sinh x_2\right) \end{pmatrix}}{48\lambda^2\left(1-\nu\right)} \tag{60}$$

$$u_2 = \frac{\sin x_1\begin{pmatrix} -6\nu\left(-8+2\left(1+x_2\right)\cosh x_2 + \left(-1+4x_2+2x_2^2\right)\sinh x_2\right) \\ -\lambda^2\left(1-\nu\right)\left(48 + \left(-3-3x_2+6x_2^2+2x_2^3\right)\cosh x_2 + 3\sinh x_2\right) \end{pmatrix}}{48\lambda^2\left(1-\nu\right)} \tag{61}$$

$$u_3 = \frac{\sin x_1 \left(\begin{array}{l} -6(2-\nu)\left(-8x_2 + \left(1+4x_2+2x_2^2\right)\cosh x_2 - 2(1+x_2)\sinh x_2\right) \\ -\lambda^2(1-\nu)\Big(-48x_2 + 6(1+x_2)^2\cosh x_2 \\ \quad -\left(9+9x_2+6x_2^2+2x_2^3\right)\sinh x_2\Big) \end{array} \right)}{48\lambda^2(1-\nu)} \tag{62}$$

Example 1: First of all, we solve the problem defined in Eq. (59), in $[-\frac{1}{4}, \frac{1}{4}] \times [-\frac{1}{4}, \frac{1}{4}]$, subjected to clamped boundary conditions, defined in Eqs. (7) and (10) with b_i being set up according to the analytical solutions in Eqs. (60)∼(62).

Table I: The RMSEs for Example 1

	$M_\alpha = 4$	$M_\alpha = 8$	$M_\alpha = 12$	$M_\alpha = 16$	$M_\alpha = 20$
RMSEs	8.15E-12	3.44E-15	3.44E-15	3.43E-15	1.85E-15

Table I gives the RMSEs for different M_1 and M_2 in which excellent accuracy can be observed.

Example 2: Then, we solve the same problem but we change the boundary conditions to the free boundary conditions (Eqs. (7) and (11)) on one side.

Table II: The RMSEs for Example 2

	$M_\alpha = 4$	$M_\alpha = 8$	$M_\alpha = 12$	$M_\alpha = 16$	$M_\alpha = 20$
RMSEs	3.12E-08	1.02E-13	9.77E-14	3.07E-13	3.40E-10

Table II addresses the RMSEs of the numerical results in this example. The accuracy is also great although it is slightly worse than the previous case. This can be expected since the free boundary conditions are imposed. Furthermore, the ill-conditioning of Chebyshev interpolation can be observed for $M_\alpha \geq 16$, which is a usual phenomenon of Chebyshev interpolations. In our further researches, we will try to remedy the ill-conditioning of Chebyshev interpolation by trying the scheme derived by Liu and Atluri (2009) or Liu, Yeih, and Atluri (2009).

Example 3: Finally, we solve a clamped Reissner plate problem on a peanut defined by

$$r(\theta) = 0.3\sqrt{\cos 2\theta + \sqrt{1.1 - \sin^2 2\theta}} \quad 0 \leq \theta \leq 2\pi \tag{63}$$

where (r,θ) is the usual polar coordinate. In this study, we perform Chebyshev interpolation in a rectangular domain defined by $[-0.45,\ 0.45]\times[-0.18,\ 0.18]$. Table III gives the RMSEs for this problem, which also performs nicely. This result demonstrates the applicability of the proposed method for irregular domains.

Table III: The RMSEs for Example 3

	$M_\alpha=4$	$M_\alpha=8$	$M_\alpha=12$	$M_\alpha=16$	$M_\alpha=20$
RMSEs	5.58E-12	9.32E-14	9.34E-14	3.49E-11	5.54E-07

9 Discussions

The method of particular solutions for problems of Reissner plate in bending under arbitrary loading are developed. In the solution procedure, the arbitrary loading term is approximated by Chebyshev polynomials in terms of monomials. Then, the analytical particular solutions corresponding to the monomials are derived. In the derivations of the analytical particular solutions, the three coupled second-ordered governing equations are transformed into a single sixth-order PDE, whose analytical particular solutions have been found in the author's previous study, by the Hörmander operator decomposition technology. In our implementation of the method of particular solutions, no book keepings and matrix inverses are required.

After the particular solution is solved, the homogeneous solution is formally solved by the method of fundamental solutions, in which the fundamental solutions are also derived by the same framework for finding the analytical particular solutions. Numerical examples are carried out to validate the proposed numerical scheme. By the virtue of the exponential convergences of both the method of fundamental solutions and Chebyshev interpolations, our numerical solutions are highly accurate.

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