 CMES, vol.46, no.1, pp.1-19, 2009

Expression for the Gradient of the First Normal Derivative of the Velocity Potential

Zai You Yan[1]

Abstract: It is well-known that the velocity potential and its first normal derivative on the structure surface can be easily found in the boundary element method for problems of potential flow. Based on an investigation in progress, the gradient of the normal derivative of the velocity potential will be very helpful in the treatment of the so-called hypersingular integral. Through a coordinate transformation, such gradient can be expressed by the combination of the first and the second normal derivatives of the velocity potential. Then one interesting problem is how to find the second normal derivative of the velocity potential through the combination of the velocity potential and its first normal derivative on the structure surface. Here a detailed derivation of the second normal derivative of the velocity potential and the gradient of the first normal derivative of the velocity potential are presented for isoparametric curvilinear boundary elements. To validate these expressions, some numerical results for the second normal derivative of the velocity potential are compared to the corresponding analytical solutions.

Keywords: Boundary element method, second normal derivative, velocity potential, gradient

1 Introduction

Boundary element method [Banerjee (1994)] is a very popular numerical approach. It can be applied in many scientific and engineering fields, such as stress analysis [Tan, Shiah & Lin (2009)], cracks [Karlis, Tsinopoulos, Polyzos & Beskos (2008)], electromagnetics [Soares & Vinagre (2008)], acoustics [Yang (2004), Chandrasekhar & Rao (2007)], fluid mechanics [Mantia & Dabnichki (2008)], carbon nano-tube reinforced composite [Wang & Yao (2008)], and so on. Comparing with other numerical methods, such as finite element method, it has the merit that mesh need only to be generated on the boundary. While two main drawbacks greatly affect its

[1] Department of Aerodynamics, Nanjing University of Aeronautics and Astronautics, Yu Dao Street 29, City of Nan Jing, Jiang Su Province, P. R.China, 210016. Email: jutsjtu@yahoo.com.cn

broad application in engineering. One is the fact that the final influence matrices are dense matrices. In the past few years, the fast multipole expansion method and its extensions (FMM) [Liu & Nishimura (2006), He, Lim & Lim (2008)] and pre-corrected fast Fourier transform method (PFFT) [Phillips & White (1997), Ding & Ye (2004)] had been developed to overcome this problem. The other is the singular, strongly singular and hypersingular integrals [De Klerk (2005)] appearing in the boundary integral equations. To date, a lot of research has been done on the treatment of such kind of integrals, especially hypersingular integrals [Tanaka, Sladek & Sladek (1994), Chen & Hong (1999)]. Yan (2000), Yan, Hung & Zheng (2003) and Yan, Cui & Hung (2005) investigated the calculation techniques of the hypersingular integral occurring in acoustic problems. Yan, Cui & Hung(2005) pointed out that taking into account the derivative of the solid angle the hypersingular integral is reduced to an at most strongly singular integral. Qian, Han, Atluri (2004) derived the symmetric Galerkin boundary element formulations of the regularized forms of non-hypersingular boundary integral equations. Meantime, Qian, Han, Ufimtsev, Atluri (2004) presented the non-hypersingular boundary integral equations by the collocation based boundary element method. Recently, Han & Atluri (2007) presented a systematic derivation of the weakly singular boundary integral equations. In their study, hypersingularities are avoided by applying some properties of the fundamental solution. Sanz, Solis & Dominguez (2007) analytically transformed the strongly singular and hypersingular integrals appearing in the mixed boundary element formulation for three-dimensional piezoelectric fracture mechanics problems into weakly singular and regular integrals. Chandrasekhar & Rao (2008), and Chandrasekhar (2008) used some simple vector calculus techniques to circumvent the hypersingularity occurred in the double layer and the so-called combined layer formulation for exterior acoustic scattering. Gao, Yang & Wang (2008) presented an algorithm for the evaluation of some weakly, strongly and hypersingular integrals in 2D problems by a semi-analytical method. Li, Wu & Yu (2009) presented an generalized extrapolation algorithm for the computation of one kind of Hardamard finite part integrals.

In the conventional boundary integral equation(CBIE), generally there exist a pair of variables directly related to the unknown variable. For problems of potential flow, they are the velocity potential and its normal derivative or the normal velocity. While sometimes, the normal derivative of the conventional boundary integral equation (NDBIE) on the surface is required, such as that in the Burton and Miller's (1971) formulation for exterior acoustic problems. For simplicity, only potential flow problem will be investigated in this paper. It is well-known that hypersingular boundary integral occurs in NDBIE. However, a new theory of hypersingular boundary integral in process by the author shows that the same as in the

CBIE, at most weakly singular integrals exist in the NDBIE after an exact derivation. It is very exciting that none strongly singular nor hypersingular boundary integrals ocuur in the new NDBIE formulation. However, another problem is created. That is the appearance of the gradient of the normal velocity. To solve the new NDBIE formulation, this variable must be expressed by the combination of the velocity potential and the normal velocity. Using a transformation between the global coordinate system and the local coordinate system, the gradient of the normal velocity can be expressed by the velocity potential, the normal velocity and the second derivative of the velocity potential. Schulz, Schwabb & Wendland (1998) presented a method to find the potentials near the boundary using Taylor expansion in the normal deriction of the boundary. Second normal derivative of the velocity potential appears in this method. General formulation for the computation of the second normal derivative of the velocity potential was presented. The second normal derivative of coordinates is presented by Lee & Soni (2004) in the enhancement of elliptic grid generation. The governing equation is Poisson equation. Their work can help to understand the application of second normal derivative of variables. Meade, Slade, Peterson & Webb (1995) derived the second normal derivative of the variable in the two-dimensional Helmholtz equation when they investigated on the radiation boundary conditions.

In this paper, the gradient of the normal velocity for potential problems is investigated. At first, boundary integral equation for three dimensional potential flow is introduced. And then normal derivative of the conventional boundary integral equation is derived. A new variable, the gradient of the normal velocity, is generated in the final boundary integral equation. Hence, calculation of this variable is crucial to the solution of the new NDHIE formulation. Using a transformation matrix, this variable is transformed to the combination of the velocity potential, the normal velocity and the second normal derivative of the velocity potential. Applying the governing equation, the second normal derivative of the velocity potential is expressed by the combination of the velocity potential and its normal derivative. Finally, two examples are computed to validate the expression of the gradient of the normal velocity and the second normal derivative of the velocity potential.

2 Boundary integral equation for potential flow

Consider a uniform stream along z direction with velocity V_∞ passes a three-dimensional rigid object. The normal vector $\vec{n}$ at arbitrary point on the boundary of the object is taken to be the inward normal as shown in Fig. 1. The symbols D, E and S are respectively represents the interior domain, exterior domain and boundary of the object. For this problem, the governing differential equation on the velocity

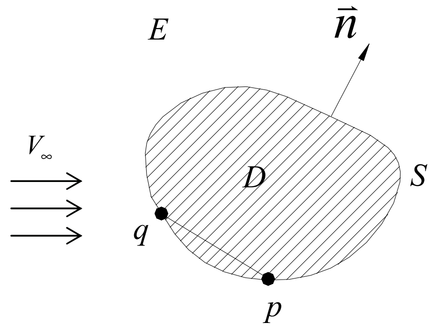

Figure 1: A uniform stream past a rigid object

potential φ in the exterior domain is the well-known Laplace equation,

$$\nabla^2\varphi = 0 \tag{1}$$

Neumann boundary condition or non-penetrating boundary condition on the boundary surface S is given by,

$$\frac{\partial\varphi}{\partial n} = 0 \tag{2}$$

Using the Green's second identity, the conventional boundary integral equation is found as,

$$c\,(p)\,\varphi(p) = \iint \left(\varphi(q)\frac{\partial G(p,q)}{\partial n_q} - G(p,q)\frac{\partial \varphi(q)}{\partial n_q}\right) dS_q + V_\infty z \tag{3}$$

The free-space Green's function $G\,(p,q)$ for the three-dimensional potential flow is just a point source,

$$G(p,q) = 1/4\pi r, \quad r = |p-q| \tag{4}$$

Where p and q are respectively the source point and field point on the surface. r represents the Euclidean distance between the points p and q.
Solid angle $c\,(p)$ is given by

$$c\,(p) = 1 - \iint \frac{\partial G\,(p,q)}{\partial n_q} dS_q \tag{5}$$

Obviously, equation (3) contains at most $1/r$ type weakly singular integrals. This equation can be solved by boundary element method with a treatment technique for the $1/r$ type weakly singular integral.

While sometimes it is also interested in the normal derivative of equation (3) [Burton and Miller (1971), Yan, Hung & Zheng (2003)].

$$\frac{\partial}{\partial n_p}\left[c\left(p\right)\varphi(p)\right] = \frac{\partial}{\partial n_p}\iint\left(\varphi(q)\frac{\partial G(p,q)}{\partial n_q} - G(p,q)\frac{\partial \varphi(q)}{\partial n_q}\right)dS_q + V_\infty n_z \tag{6}$$

In numerical simulation, if the collocation point p does not located on the integrating element, then the normal derivative with respect to p in equation (6) will be easy to be obtained. Otherwise, if p located on the integrating element, it will be very difficult to derive such a normal derivative. Usually, such elements are termed as singular element with respect to the collocation point p. Therefore, in the following derivation, we will focus on the derivation of equation (6) on the singular elements.

For a differentiable function f, the following identity exists,

$$\frac{\partial f}{\partial n} = \nabla f \cdot \vec{n}$$

According to this relation, if the gradient of the integral equation (3) at the point p is obtained, then it will be easy to find the normal derivative of this equation. Based on such an idea, now let us derive the gradient of the integral equation (3) at the point p. On a singular element, taking the partial derivative of equation (3) with respect to x_p, we have,

$$\frac{\partial}{\partial x_p}\left[c\left(p\right)\varphi(p)\right] = \frac{\partial}{\partial x_p}\iint\limits_{\text{singular}}\left(\varphi(q)\frac{\partial G(p,q)}{\partial n_q} - G(p,q)\frac{\partial \varphi(q)}{\partial n_q}\right)dS_q \tag{7}$$

The boundary element applied for three-dimensional problem is an eight-nodded isoparametric curvilinear quadrilateral element [Yan (2000)]. That is the number of nodes in each element is $NE = 8$. The corresponding shape functions $N_i\left(\xi,\eta\right)$ are given by formulation (A.1). Then the right hand side of equation (7) can be expanded on such kind of singular element as,

$$\begin{aligned}\int_{-1}^{1}\int_{-1}^{1}\left[\frac{\partial}{\partial x_p}\left(\varphi(q)\frac{\partial G}{\partial n_q}\right) - \frac{\partial}{\partial x_p}\left(G\frac{\partial \varphi(q)}{\partial n_q}\right)\right]J_{2D}d\xi d\eta \\ + \int_{-1}^{1}\int_{-1}^{1}\left(\varphi(q)\frac{\partial G}{\partial n_q} - G\frac{\partial \varphi(q)}{\partial n_q}\right)\frac{\partial J_{2D}}{\partial x_p}d\xi d\eta\end{aligned} \tag{8}$$

The expression for two-dimensional Jacobean determinant J_{2D} is given by formulation (A.2). In formulation (8), we only concern the following term,

$$\frac{\partial}{\partial x_p}\left[G(p,q)\frac{\partial\varphi(q)}{\partial n_q}\right]$$

It can be expanded as,

$$\frac{\partial}{\partial x_p}\left[G(p,q)\frac{\partial\varphi(q)}{\partial n_q}\right]=\frac{\partial G(p,q)}{\partial x_p}\frac{\partial\varphi(q)}{\partial n_q}+G(p,q)\frac{\partial}{\partial x_q}\left[\frac{\partial\varphi(q)}{\partial n_q}\right]\frac{\partial x_q}{\partial x_p} \tag{9}$$

It is worthy to note that $\partial x_q/\partial x_p$ does not equal to zero. In the second term on the right hand side of formulation (9), we can find one term,

$$\frac{\partial}{\partial x_q}\left[\frac{\partial\varphi(q)}{\partial n_q}\right] \tag{10}$$

This is just what to be focused on in this paper. Obviously, for a general case, it is equivalent to the gradient of the first normal derivative of the velocity potential or the gradient of the normal velocity.

$$\nabla_q\left[\frac{\partial\varphi(q)}{\partial n_q}\right] \tag{11}$$

3 The gradient of the normal velocity

First, let us define a local coordinate system $O\xi\eta\zeta$ on a boundary element, where ξ and η are along the tangential directions and ζ coincide with the normal vector $\vec{n}$ [Liu and Rizzo (1992)]. Then the following transformation relation exists,

$$\begin{Bmatrix}\frac{\partial\varphi}{\partial x}\\ \frac{\partial\varphi}{\partial y}\\ \frac{\partial\varphi}{\partial z}\end{Bmatrix}=J^{-1}\begin{Bmatrix}\sum\limits_{i=1}^{NE}N_{i,\xi}(\xi,\eta)\varphi_i\\ \sum\limits_{i=1}^{NE}N_{i,\eta}(\xi,\eta)\varphi_i\\ \frac{\partial\varphi}{\partial n}\end{Bmatrix}=\begin{bmatrix}\xi_{,x} & \eta_{,x} & \zeta_{,x}\\ \xi_{,y} & \eta_{,y} & \zeta_{,y}\\ \xi_{,z} & \eta_{,z} & \zeta_{,z}\end{bmatrix}\begin{Bmatrix}\sum\limits_{i=1}^{NE}N_{i,\xi}(\xi,\eta)\varphi_i\\ \sum\limits_{i=1}^{NE}N_{i,\eta}(\xi,\eta)\varphi_i\\ \frac{\partial\varphi}{\partial n}\end{Bmatrix} \tag{12}$$

The Jacobean matrix J is given by formulation (A.3).

Similarly, we find the transformation relation for the gradient of the normal velocity is,

$$\begin{Bmatrix}\frac{\partial}{\partial x}\frac{\partial\varphi}{\partial n}\\ \frac{\partial}{\partial y}\frac{\partial\varphi}{\partial n}\\ \frac{\partial}{\partial z}\frac{\partial\varphi}{\partial n}\end{Bmatrix}=J^{-1}\begin{Bmatrix}\sum\limits_{i=1}^{NE}N_{i,\xi}(\xi,\eta)\left(\frac{\partial\varphi}{\partial n}\right)_i\\ \sum\limits_{i=1}^{NE}N_{i,\eta}(\xi,\eta)\left(\frac{\partial\varphi}{\partial n}\right)_i\\ \frac{\partial^2\varphi}{\partial n^2}\end{Bmatrix} \tag{13}$$

In this formulation, there exists a new unknown variable $\partial^2\varphi/\partial n^2$. To make sure the considering problem can be solved, this variable must be expressed by the combination of the velocity potential φ and the normal velocity $\partial\varphi/\partial n$.

How to find the second normal derivative of the velocity potential $\partial^2\varphi/\partial n^2$? This is a second order partial derivative, so we think of the governing equation (1). As the local coordinate ζ is defined to coincide with the normal vector, therefore,

$$\frac{\partial^2\varphi}{\partial n^2} \equiv \frac{\partial^2\varphi}{\partial\zeta^2} \tag{14}$$

Now, let us express the governing equation (1) in the local coordinate system $O\xi\eta\zeta$ [Oostendorp & Oosterom (1996)].

$$\Delta\varphi = \frac{1}{H_1H_2H_3}\left[\frac{\partial}{\partial\xi}\left(\frac{H_2H_3}{H_1}\frac{\partial\varphi}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{H_3H_1}{H_2}\frac{\partial\varphi}{\partial\eta}\right) + \frac{\partial}{\partial\zeta}\left(\frac{H_1H_2}{H_3}\frac{\partial\varphi}{\partial\zeta}\right)\right] \tag{15}$$

The Lame coefficientsH_1, H_2, H_3 are given by formulation (A.4). Equation (15) can be expanded as,

$$\begin{aligned}\Delta\varphi = \frac{1}{H_1H_2}\Bigg[&\frac{\partial}{\partial\xi}\left(\frac{H_2}{H_1}\right)\frac{\partial\varphi}{\partial\xi} + \frac{H_2}{H_1}\frac{\partial^2\varphi}{\partial\xi^2} + \\ &\frac{\partial}{\partial\eta}\left(\frac{H_1}{H_2}\right)\frac{\partial\varphi}{\partial\eta} + \frac{H_1}{H_2}\frac{\partial^2\varphi}{\partial\eta^2} + \frac{\partial H_1H_2}{\partial n}\frac{\partial\varphi}{\partial n}\Bigg] + \frac{\partial^2\varphi}{\partial n^2}\end{aligned} \tag{16}$$

Where

$$\begin{aligned}\frac{\partial H_1}{\partial n} &= \frac{x_\xi\frac{\partial n_x}{\partial\xi} + y_\xi\frac{\partial n_y}{\partial\xi} + x_\xi\frac{\partial n_z}{\partial\xi}}{H_2} \\ &= \frac{x_\xi\sum\limits_{i=1}^{NE}N_{i,\xi}n_{ix} + y_\xi\sum\limits_{i=1}^{NE}N_{i,\xi}n_{iy} + x_\xi\sum\limits_{i=1}^{NE}N_{i,\xi}n_{iz}}{H_2}\end{aligned} \tag{17}$$

Similarly, the expression for $\partial H_2/\partial n$ can be obtained.

Because on each element,

$$\frac{\partial\varphi}{\partial\xi} = \sum_{i=1}^{NE}N_{i,\xi}(\xi,\eta)\varphi_i, \quad \frac{\partial\varphi}{\partial\eta} = \sum_{i=1}^{NE}N_{i,\eta}(\xi,\eta)\varphi_i$$

and

$$\frac{\partial^2\varphi}{\partial\xi^2} = \sum_{i=1}^{NE}N_{i,\xi\xi}(\xi,\eta)\varphi_i, \quad \frac{\partial^2\varphi}{\partial\eta^2} = \sum_{i=1}^{NE}N_{i,\eta\eta}(\xi,\eta)\varphi_i$$

the second normal derivative of the velocity potential $\partial^2\varphi/\partial n^2$ now can be expressed by the combination of the velocity potential φ and its first order normal parital derivative $\partial\varphi/\partial n$ through the equation (16). That is,

$$\frac{\partial^2\varphi}{\partial n^2} = \sum_{i=1}^{NE}\left[A_i\varphi_i + B_i\left(\frac{\partial\varphi}{\partial n}\right)_i\right] \tag{18}$$

Where

$$A_i = \frac{1}{H_1H_2}\left[\frac{\partial}{\partial\xi}\left(\frac{H_2}{H_1}\right)N_{i,\xi} + \frac{H_2}{H_1}N_{i,\xi\xi} + \frac{\partial}{\partial\eta}\left(\frac{H_1}{H_2}\right)N_{i,\eta} + \frac{H_1}{H_2}N_{i,\eta\eta}\right]$$

and

$$B_i = \frac{1}{H_1H_2}\frac{\partial H_1H_2}{\partial n}N_i$$

Then the gradient of the normal velocity can be obtained by equation (13) as,

$$\begin{aligned}
\left\{\begin{matrix}\frac{\partial}{\partial x}\frac{\partial\varphi}{\partial n}\\ \frac{\partial}{\partial y}\frac{\partial\varphi}{\partial n}\\ \frac{\partial}{\partial z}\frac{\partial\varphi}{\partial n}\end{matrix}\right\} &= J^{-1}\left\{\begin{matrix}\sum\limits_{i=1}^{NE}N_{i,\xi}(\xi,\eta)\left(\frac{\partial\varphi}{\partial n}\right)_i\\ \sum\limits_{i=1}^{NE}N_{i,\eta}(\xi,\eta)\left(\frac{\partial\varphi}{\partial n}\right)_i\\ \sum\limits_{i=1}^{NE}\left[A_i\varphi_i + B_i\left(\frac{\partial\varphi}{\partial n}\right)_i\right]\end{matrix}\right\}\\
&= J^{-1}\begin{bmatrix}N_{1,\xi} & N_{2,\xi} & \cdots & N_{NE,\xi}\\ N_{1,\eta} & N_{2,\eta} & \cdots & N_{NE,\eta}\\ B_1 & B_2 & \cdots & B_{NE}\end{bmatrix}\left\{\begin{matrix}\left(\frac{\partial\varphi}{\partial n}\right)_1\\ \left(\frac{\partial\varphi}{\partial n}\right)_2\\ \vdots\\ \left(\frac{\partial\varphi}{\partial n}\right)_{NE}\end{matrix}\right\}\\
&\quad + J_3^{-1}\begin{bmatrix}A_1 & A_2 & \cdots & A_{NE}\end{bmatrix}\left\{\begin{matrix}\varphi_1\\ \varphi_2\\ \vdots\\ \varphi_{NE}\end{matrix}\right\}
\end{aligned} \tag{19}$$

Where J_3^{-1} represents the third column vector of the inverse Jacobean matrix J^{-1}.

4 Numerical examples

To show the feasibility that the gradient of the normal velocity on the boundary can be expressed by the combination of the velocity potential and the normal velocity and to validate the above derivation, two examples about a uniform stream with velocity $V_\infty = 100\, m/s$ past a rigid object are presented. One is the uniform stream along x direction past an unlimited cylinder as shown in Fig. 2. Obviously, this is a two-dimensional exterior problem for potential flow. Another one is the uniform stream along z direction past a sphere as shown in Fig.3. Of course, this is a three-dimensional exterior problem for potential flow. Both of these cases have analytical solutions.

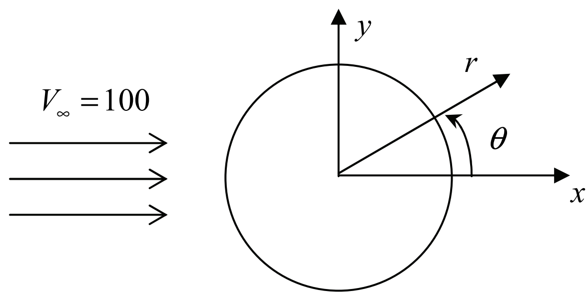

Figure 2: A uniform stream past an unlimited cylinder

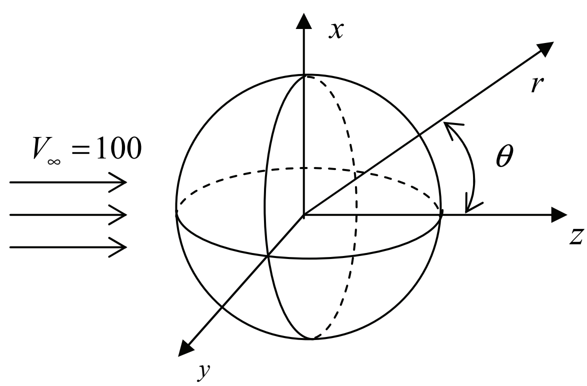

Figure 3: A uniform stream past a sphere

4.1 A uniform stream past an unlimited cylinder

This is a classical two-dimensional problem for potential flow. In section 2, the derivation is about three-dimensional problems of potential flow. For two-dimensional problem of potential flow, the derivation of the gradient of the normal velocity on the boundary can be achieved similarly except that the free-space Green's function is replaced by

$$G(p,q) = -\frac{1}{2\pi}\ln r \tag{20}$$

and the boundary element reduced to a three-nodded isoparametric curvilinear element which is just composed by the first three elemental nodes in the three-dimensional eight nodded boundary element by setting $\eta \equiv -1$. That is for two-dimensional problem the number of element nodes is $NE = 3$. Such treatments make sure the shape functions for the eight nodded elements can be directly applied in the two-dimensional three-nodded elements.

In the following simulation, the boundary of the cylinder is descretized by 80 elements with 160 nodes in total. The analytical solution of the velocity potential φ for the uniform stream past an unlimited cylinder of radius a can be found in general fluid mechanics textbooks as,

$$\varphi(r,\theta) = V_\infty\left(r + \frac{a^2}{r}\right)\cos\theta \tag{21}$$

Where θ is the angle between the radial vector r and the x direction.

Then, the second normal derivative of the velocity potential on the boundary can be derived as,

$$\frac{\partial^2\varphi}{\partial n^2} = \left.\frac{\partial^2\varphi}{\partial r^2}\right|_{r=a=1} = 2V_\infty\cos\theta \tag{22}$$

Fig.4 displays the comparison between the numerical solutions and the corresponding analytical solutions of the second normal derivative of the velocity potential on the boundary. In the numerical simulation, the velocity potentials are provided by the analytical solutions. The agreement between these solutions proofs the derivation is correct and the second normal derivative of the velocity potential does can be expressed by the velocity potential and its normal derivative.

Similarly, the gradient of the normal velocity or the first order normal derivative of the velocity potential on the boundary is derived as,

$$\begin{aligned}
\frac{\partial^2\varphi}{\partial x\partial n} &= \left.\frac{\partial^2\varphi}{\partial x\partial r}\right|_{r=1} \\
&= V_\infty a^2\left.\left[\frac{1}{r} - \frac{1}{r^3} + \left(\frac{-1}{r} + \frac{3}{r^3}\right)x\cos\theta\right]\right|_{r=a=1} \\
&= 2V_\infty x\cos\theta \\
\frac{\partial^2\varphi}{\partial y\partial n} &= \left.\frac{\partial^2\varphi}{\partial y\partial r}\right|_{r=a=1} = 2V_\infty y\cos\theta
\end{aligned} \tag{23}$$

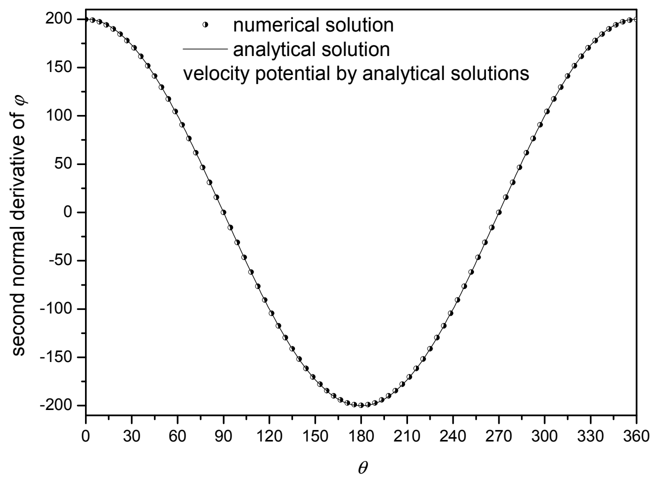

Figure 4: The second normal derivative of the velocity potential as a function of θ

Fig. 5 displays the comparison between the numerical results and the corresponding analytical solutions of the gradient of normal derivative of the velocity potential. The velocity potentials applied in the numerical simulation are given by analytical solution too. Obviously, the numerical results agree quite well with the corresponding analytical solutions.

4.2 A uniform stream past a rigid sphere

The origin of the coordinate system is located at the center of the sphere. Analytical solution of the velocity potential φ on the spherical surface for a uniform stream past a sphere of radius a can also be found in general fluid mechanics textbooks as,

$$\varphi = V_\infty r \cos\theta \left(1 + \frac{a^3}{2r^3}\right) \tag{24}$$

Where θ represents the angle between the radial direction and the z direction.

Then, the second normal derivative of the velocity potential on the surface can be derived as,

$$\frac{\partial^2 \varphi}{\partial n^2} = \left.\frac{\partial^2 \varphi}{\partial r^2}\right|_{r=1} = 3V_\infty \cos\theta \tag{25}$$

Fig. 6 displays the comparison between the numerical results and the corresponding analytical solutions of the second normal derivative of the velocity potential on the

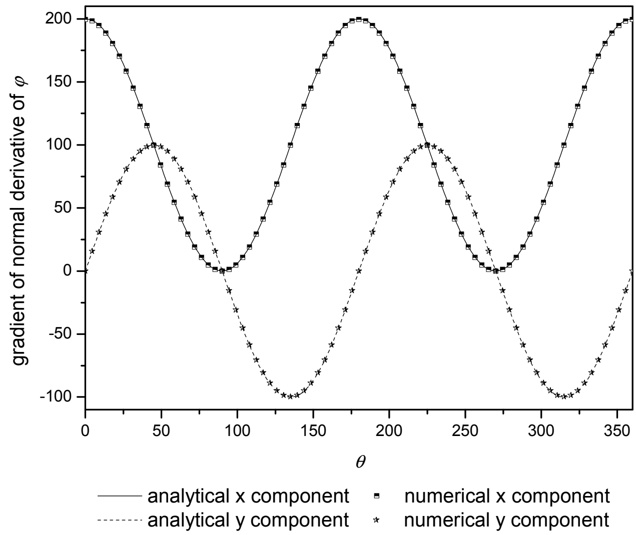

Figure 5: The gradient of the first order normal derivative of the velocity potential as a function of θ

boundary. The velocity potential applied in the numerical simulation are given by analytical formulation. Even though the agreement of these solutions are not as well as that in the previous example, it still can proof the correctness of the derivation and show that the second normal derivative of the velocity potential on the boundary can be expressed by the combination of the velocity potential and its normal derivative.

Similarly, the gradient of the normal derivative of the velocity potential on the surface can be derived as,

$$\frac{\partial^2\varphi}{\partial x\partial n} = \left.\frac{\partial^2\varphi}{\partial x\partial r}\right|_{r=1} = V_\infty\left.\left\{\left[-\frac{1}{r}+\frac{4}{r^4}\right]x\cos\theta\right\}\right|_{r=1} = 3V_\infty x\cos\theta$$

$$\frac{\partial^2\varphi}{\partial y\partial n} = 3V_\infty y\cos\theta$$

$$\frac{\partial^2\varphi}{\partial z\partial n} = V_\infty\left.\left[\frac{1}{r}-\frac{1}{r^4}+\left(\frac{-1}{r}+\frac{4}{r^4}\right)z\cos\theta\right]\right|_{r=1} = 3V_\infty z\cos\theta \tag{26}$$

Fig. 7 displays the comparison between the numerical results and the corresponding analytical solutions of the gradient of the normal derivative of the velocity potential. The velocity potentials applied in the numerical simulation are given by analytical solutions too. Obviously, the numerical results agree with the corresponding analytical solutions.

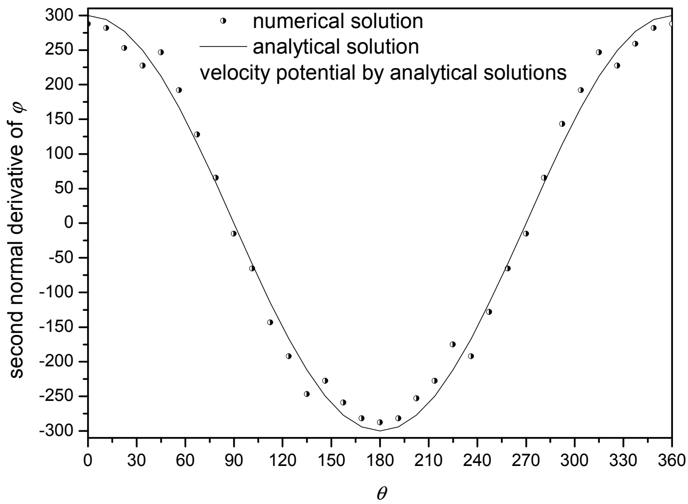

Figure 6: The second normal derivative of velocity potential as a function of θ

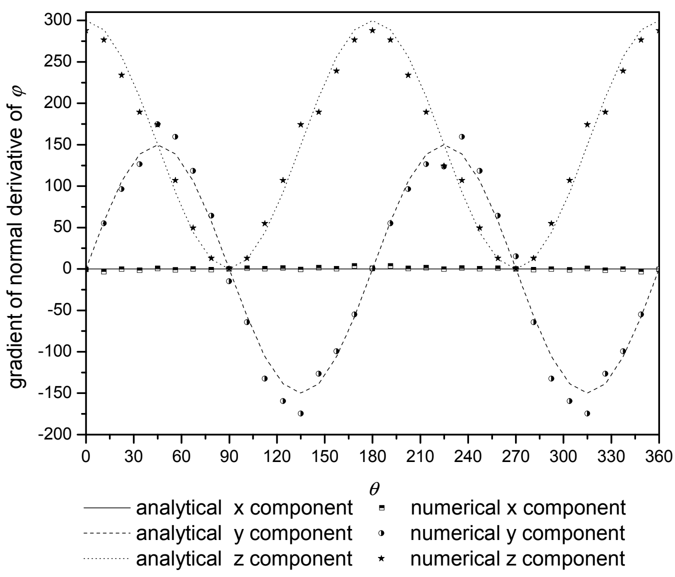

Figure 7: The gradient of the normal velocity on the boundary as a function of θ

5 Conclusion

In the normal derivative of the conventional boundary integral equation for potential flow problems, there exists a new variable $\partial^2\varphi/\partial n^2$ which is the second normal derivative of the velocity potential on the boundary. Accounting for the governing equation, this variable can be expressed by the combination of the velocity potential and the normal velocity. Then the gradient of the normal velocity can be expressed by the combination of the velocity potential and the normal velocity. In fact, the second normal derivative of the velocity potential will be very important in the oncoming treatment technique of hypersingular integrals. Two examples are presented to validate the correctness of the derivation. Under the condition that velocity potentials are provided by analytical formulation, numerical results for the second normal derivative of the velocity potential and the gradient of the normal velocity on the boundary all agree well with the corresponding analytical solutions. The ideal developed here will be combined with the treatment technique of sharp edges & corners [Yan (2006)] to solve complicated acoustic problems.

Acknowledgement: This project was supported by Nanjing University of Aeronautics and Astronautics Talent Development Foundation S0580-12; Y0801-012.

References

Banerjee P.K. (1994): *Boundary element methods in engineering*. McGraw-Hill book Company.

Burton A.J.; Miller G.F. (1971): The application of integral equation methods to the numerical solution of some exterior boundary value problems. *Proc.R.Soc.London Ser.* A323: 201-210.

Chandrasekhar B. (2008): Node based method of moments solution to combined layer formulation of acoustic scattering. *CMES: Computer Modeling in Engineering & Sciences,* 33(3): 243-267.

Chandrasekhar B.; Rao S.M. (2007): Acoustic scattering from fluid bodies of arbitrary shape. *CMES: Computer Modeling in Engineering & Sciences*, 21(1): 67-80.

Chandrasekhar B.; Rao S.M. (2008): A faster method of moments solution to double layer formulation of acoustic scattering. *CMES: Computer Modeling in Engineering & Sciences*, 33(2): 199-214.

Chen J.T.; Hong K.H. (1999): Review of dual boundary element methods with emphasis on hypersingular integrals and divergent series. *Appl. Mech. Rev*.,52(1),17-33.

De Klerk J.H. (2005): Hypersingular integral equations—past, present, future. *Nonlinear Analysis,* 63: 533 –540.

Ding J.; Ye W. (2004): A fast integral approach for drag force calculation due to oscillatory slip stokes flows. *Int. J. Numer. Meth. Engng*, 60:1535–1567.

Gao X.W.; Yang K.; Wang J. (2008): An adaptive element subdivision technique for evaluation of various 2D singular boundary integrals. *Engineering Analysis with Boundary Elements* 32: 692–696.

Han Z.D.; Atluri S.N. (2007):A systematic approach for the development of weakly-singular BIEs, *CMES: Computer Modeling in Engineering & Sciences*, 21(1): 41-52. (9)

He X.F.; Lim K.M.; Lim S.P. (2008): Fast BEM Solvers for 3D Poisson-Type Equations. *CMES: Computer Modeling in Engineering & Sciences*, 35(1): 21-48.

Karlis G.F.; Tsinopoulos S.V.; Polyzos D.; Beskos D.E. (2008): 2D and 3D boundary element analysis of mode-I cracks in gradient elasticity. *CMES: Computer Modeling in Engineering & Sciences*, 26(3): 189-207.

Lee S.H.; Soni B.K. (2004): The enhancement of an elliptic grid using appropriate control functions. *Applied Mathematics and Computation*, 159: 809–821.

Li J.; Wu J.M.; Yu D.H. (2009): Generalized extrapolation for computation of hypersingular integrals in boundary element methods. *CMES: Computer Modeling in Engineering & Sciences*, 42(2): 151-175.

Liu Y.J.; Nishimura N. (2006): The fast multipole boundary element method for potential problems: A tutorial. *Engineering Analysis with Boundary Elements*, 30(5): 371-381.

Liu Y.J.; Rizzo F.J. (1992): A weakly singular form of the hypersingular boundary integral equation applied to 3-D acoustic wave problem. *Comput. Methods Appl. Mech. Eng.*, 96: 271-287.

Mantia M.L.; Dabnichki P. (2008): Unsteady 3D boundary element method for oscillating wing. *CMES: Computer Modeling in Engineering & Sciences*, 33(2): 131-153.

Meade D.B.; Slade G.W.; Peterson A.F.; Webb K.J. (1995): Comparison of local radiation boundary conditions for the scalar Helmholtz equation with general boundary shapes. *IEEE Transactions on Antennas And Propagation*, 43(1): 6-10.

Oostendorp T.F.; Oosterom A.V. (1996): The surface Laplacian of the potential: theory and application. *IEEE Transactions on Biomedical Engineering*, 43(4): 394-405.

Phillips J.R.; White J.K. (1997): A precorrected-FFT method for electrostatic analysis of complicated 3-D structures. *IEEE Transactions on Computer-aided Design of Integrated Circuits and Systems*, 16(10): 1059-1072.

Qian Z.Y.; Han Z.D.; Atluri S.N. (2004): Directly derived non-hypersingular boundary integral equations for acoustic problems, and their solution through Petrov-

Galerkin schemes. *CMES: Computer Modeling in Engineering & Sciences* 5(6): 541-562.

Qian Z.Y.; Han Z.D.; Ufimtsev P.; Atluri S.N. (2004): Non-hypersingular boundary integral equations for acoustic problems, implemented by the collocation-based boundary element method. *CMES: Computer Modeling in Engineering & Sciences* 6(2): 133-144.

Sanz J.A.; Solis M.; Dominguez J. (2007): Hypersingular BEM for piezoelectric solids: formulation and applications for fracture mechanics, *CMES: Computer Modeling in Engineering & Sciences*, 17(3): 215-229.

Schulz H.; Schwabb C.; Wendland W.L. (1998): The computation of potentials near and on the boundary by an extraction technique for boundary element methods. *Comput. Methods Appl. Mech. Engrg*. 157 225-238.

Soares Jr.D.; Vinagre, M. P. (2008): Numerical computation of electromagnetic fields by the time-domain boundary element Method and the complex variable method. *CMES: Computer Modeling in Engineering & Sciences*, 25(1): 1-8.

Tan C.L.; Shiah Y.C.; Lin C.W. (2009): Stress Analysis of 3D Generally Anisotropic Elastic Solids Using the Boundary Element Method. *CMES: Computer Modeling in Engineering & Sciences*, 41(3): 195-214.

Tanaka M.; Sladek V.; Sladek J. (1994): Regularization techniques applied to boundary element methods. *ASME Appl. Mech. Rev.* 47(10): 457-499.

Wang H.T.; Yao Z.H. (2008): A rigid-fiber-based boundary element model for strength simulation of carbon nanotube reinforced composites. *CMES: Computer Modeling in Engineering & Sciences*, 29(1): 1-13.

Yan Z.Y. (2000): *Sound transmission through thin elastic shell with different fluids on both the inside and outside*, Ph. D Thesis, Shanghai Jiao Tong University, P.R. China.

Yan Z.Y.; Hung K.C.; Zheng H. (2003): Solving the hypersingular boundary integral equation in three-dimensional acoustics using a regularization relationship. *J. Acoust. Soc. Am.* 113: 2674-2683.

Yan Z.Y.; Cui F.S.; Hung K.C. (2005): Investigation on the normal derivative equation of Helmholtz integral equation in acoustics. *CMES: Computer Modeling in Engineering & Sciences,* 7(1)**:** 97-106.

Yan Z.Y. (2006): Treatment of sharp edges & corners in the acoustic boundary element method under Neumann boundary condition, *CMES: Computer Modeling in Engineering & Sciences*, 13(2): 81-90.

Yang S.A. (2004): An integral equation approach to three-dimensional acoustic radiation and scattering problems. *J. Acoust. Soc. Am.* 116 (3): 1372-1380.

Appendix: Shape functions and normal vectors

Node distribution for an eight-nodded isoparametric curvilinear quadrilateral element is displayed in Fig. A.1.

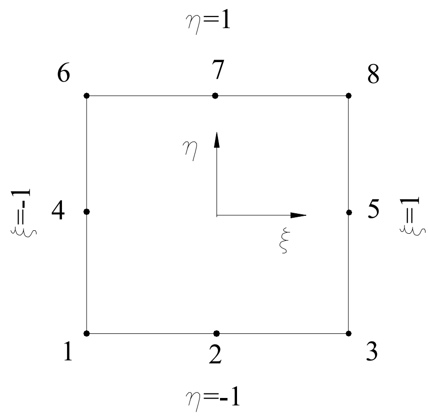

Figure A.1: Parent element for the eight-nodded isoparametric curvilinear quadrilateral element

The eight shape functions are given by,

$$N_1 = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta)$$

$$N_2 = \frac{1}{2}(1-\xi^2)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1)$$

$$N_4 = \frac{1}{2}(1-\xi)(1-\eta^2)$$

$$N_5 = \frac{1}{2}(1+\xi)(1-\eta^2)$$

$$N_6 = \frac{1}{4}(1-\xi)(1+\eta)(-1-\xi+\eta)$$

$$N_7 = \frac{1}{2}(1-\xi^2)(1+\eta)$$

$$N_8 = \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1) \tag{A.1}$$

The global coordinates on the boundary can be found through the combination of the global coordinates of each element node as,

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_{i=1}^{NE} N_i(\xi,\eta) \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}$$

Then the partial derivatives of global coordinates with respect to each elemental local coordinates are,

$$\vec{S} = \begin{Bmatrix} x_{,\xi} \\ y_{,\xi} \\ z_{,\xi} \end{Bmatrix} = \sum_{i=1}^{NE} N_{i,\xi}(\xi,\eta) \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}$$

$$\vec{T} = \begin{Bmatrix} x_{,\eta} \\ y_{,\eta} \\ z_{,\eta} \end{Bmatrix} = \sum_{i=1}^{NE} N_{i,\eta}(\xi,\eta) \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}$$

$$\vec{U} = \begin{Bmatrix} x_{,\zeta} \\ y_{,\zeta} \\ z_{,\zeta} \end{Bmatrix} = \vec{n}, \quad \zeta \equiv n, \quad \left|\vec{U}\right| \equiv 1$$

And the Jacobean determinant occurs in the boundary integration is,

$$J_{2D} = \left|\vec{S} \times \vec{T}\right| \tag{A.2}$$

The normal vector can be expressed using the above formulations as,

$$\vec{n} = \begin{bmatrix} n_x & n_y & n_z \end{bmatrix}$$

$$n_x = \left(y_{,\xi} z_{,\eta} - y_{,\eta} z_{,\xi}\right)/J_{2D}$$

$$n_y = \left(z_{,\xi} x_{,\eta} - z_{,\eta} x_{,\xi}\right)/J_{2D}$$

$$n_z = \left(x_{,\xi} y_{,\eta} - x_{,\eta} y_{,\xi}\right)/J_{2D}$$

The Jacobean matrix J between the global coordinate system and the local coordinate system is,

$$J^T = \begin{bmatrix} \vec{S} & \vec{T} & \vec{U} \end{bmatrix} = \begin{bmatrix} x_{,\xi} & x_{,\eta} & x_{,\zeta} \\ y_{,\xi} & y_{,\eta} & y_{,\zeta} \\ z_{,\xi} & z_{,\eta} & z_{,\zeta} \end{bmatrix} \tag{A.3}$$

The Lame coefficients are given by,

$$H_1 = \left|\vec{S}\right|$$

$$H_2 = \left|\vec{T}\right|$$

$$H_3 = \left|\vec{U}\right| \equiv 1 \tag{A.4}$$