

Quasilinear Hybrid Boundary Node Method for Solving Nonlinear Problems

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Abstract: A novel boundary type meshless method called Quasilinear Hybrid Boundary Node Method (QHBNM), which combines quasilinearization method, dual reciprocity method (DRM) and hybrid boundary node method (HBNM), is developed to solving a class of nonlinear problems. The nonlinear term of the governing equation is linearized by the generated quasilinearization method, in which the solution of the linearized equation can exactly converge to the solution of original equation at a very wide range initial value, and the convergence rate is quadratic. Then dual hybrid boundary node method is applied to solving the linearized equation, in which DRM is introduced into HBNM to deal with the integral for the inhomogeneous terms of the governing equations. The solution in present method is divided into two parts, i. e., the complementary solution and the particular solution. The complementary solution is solved by HBNM, and the particular one is obtained by DRM. In order to get a generated use, the basis form of particular solution is presented in this paper. So a boundary type truly meshless method QHBNM is proposed, which retains all the advantages of BEM of linear problems. It does not require the ‘boundary element mesh’, either for interpolation of the variables, or for the integration of the ‘energy’. The convergence of present iteration scheme is quadratic, and the initial values can be widely chosen. The computation is small in this method, in which only several matrixes are needed to update on each iteration.

The numerical examples are presented for several nonlinear problems, for which accurate results, quadratic convergence and high stability can be available. It is shown that present method is effective and can be widely applied in practical engineering.

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1 Introduction

A class of nonlinear problems can be written as, $\nabla^2 u = f(x, u)$, in which u is the dependent variable, $f(x, u)$ is a forcing term which is dependent on the variable u . A physical situation where this equation is encountered are the diffusion with simultaneous reaction in a porous catalyst particle [Aris (1975)] and heat conduction with temperature dependent heat generation within the medium. And a Frank-Kamenitski explosion model also belongs to this category. So if $f(x, u)$ is a nonlinear function of u , explicit analytic solutions are rarely possible to get, one needs to exploit numerical method.

Numerical methods have been developed rapidly in the past decades. Those methods include finite element method(FEM)[Zienkiewicz (1977)], boundary element method(BEM) [Brebbia and Dominguez (1992); Colli et al. (2009)], meshless method, etc. And FEM and BEM as two popular tools have been well developed in the past decades. Compared with FEM and BEM, the meshless method does not require elements and thus attracts more and more attention in recent years.

The meshless method has great varieties and can be divided into two types, i.e., the domain type and the boundary type. The domain type has the element free Galerkin (EFG) method [Belytschko et al. (1994)], the meshless local Petrov-Galerkin (MLPG) method [Atluri (2002, 2004); Atluri et al. (2003); Atluri and Zhu (1998, 2000); Atluri and Shen (2002); Han and Atluri (2003a, b); Ma (2007, 2008); Pini et al. (2008); Li and Atluri (2008); Sellountos et al. (2009)] and so on. On the other hand, the boundary type includes the local boundary integral equation (LBIE) method [Atluri and Zhu (2000); Sladek et al. (2001); Chen et al. (2007); Li and Atluri (2008); Sellountos et al. (2009)], the method of fundamental solutions (MFS)[Liu (2008); Shiah et al. (2008); Hu et al. (2008); Young and Ruan (2005); Marin (2008)], the local hypersingular boundary integral equation method (LHBIE) [Vavourakis (2008)], hybrid boundary node method (HBNM) [Zhang and Yao (2002, 2004)] and dual reciprocity hybrid boundary node method (DHBNM) [Yan and Wang (2008a, b, 2009)].

Though all meshless methods do not need the element mesh for the field variable interpolation, some of them require a background mesh for integration. For example, Mukherjee et al. (1994) applied MLS to the boundary integration equations and proposed Boundary node method(BNM). It only requires to discrete the boundary. Although this method does not require an element mesh for the interpolation of the boundary variables, a background element is inevitable for integration.

Based on BNM, Zhang and Yao(2002) proposed another boundary-type meshless method: Hybrid Boundary Node Method (HBNM). It achieves a truly meshless method. However, it has a drawback of serious ‘boundary layer effect’. To avoid this shortcoming, Zhang and Yao(2004) further proposed the Regular Hybrid Boundary Node Method(RHBNM), in which the source points of the fundamental solution are located outside of the domain. Although this method can avoid the singular integration and boundary layer effect, it creates some new problems. For example, how to arrange the positions of the source points?

To overcome these problems, Wang and Yao et al.(2004) presented a meshless singular hybrid node method for 2-D elasticity. And Miao and Wang (2005) proposed the rigid body motion approach to deal with the singular integration and applied an adaptive integration scheme to solve the boundary layer effect.

Those methods, however, can only be used for solving homogeneous problems. For the nonlinear problem, the domain integration is inevitable. DRM was first proposed by Nardini and Brebbia(1983) for elasto-dynamic problems in 1982 and extended by Wrobel and Brebbia(1987) to time dependent diffusion in 1986. Based on HBNM, DRM is first introduced into HBNM by Yan and Wang(2008a, b, 2009), and a new truly meshless method Dual Hybrid Boundary Node Method(DHBNM) is proposed, which can be applied to dynamic problem and nonlinear problem and so on.

For a nonlinear problem, iteration is inevitable for numerical procedure. As a effective iteration scheme, the quasilinearization method is first developed by Bellman and Kalaba(1965), and this method combines linear approximation techniques with capabilities of the digital computer in various adroit fashions. Combined method of upper and lower solutions and monotone iterative technique together with quasilinearization method, the approximations are constructed to yield rapid convergence and monotonicity as well.

In this paper, firstly, the quasilinearization method, DRM and HBNM are combined, and a new truly meshless method for nonlinear problems is proposed, which is named as QHBNM. The generated quasilinearization method is employed to linearize the nonlinear term of the governing equation, in which the solution of the linearized equation can exactly monotonically converge to the solution of original equation at a wide range initial value, and convergence rate is quadratic. Secondly, DHBNM is employed to solve this linearized equation. The solution in this method composes two parts: complementary solution and particular solution. For the first part, HBNM is applied. For the second part, DRM has been used and the radial basis functions are applied to interpolating the nonlinear part of the equations. Because of the nonlinear term of the governing equations, the boundary integral equations obtained by DHBNM are not enough to solve all variables. Some additional

equations are proposed to obtain the relation of the variables in the domain and on the boundary.

This method keeps the ‘boundary-only’ and truly meshless method character of HBNM, and the convergence is quadratic, which is much higher than some other methods. Another important advantage of present method is its insensitivity with initial guesses and high stability. The numerical examples are presented for several nonlinear problems, for which very accurate results, quadratic convergence and high stability can be available. Besides, this method has the advantage of small amount of calculation, in which only several matrixes are needed to update on each iteration, but when QBEM is applied, integral for coefficient matrixes are needed for each iteration. It is shown that the present method is effective and can be widely applied in practical engineering.

2 The generated quasilinearization method

In this section, the quasilinearization method is introduced, and then this method is extended and forms a general method, which named as the generated quasilinearization method.

2.1 The Quasilinearization Method

Let $\Omega \subset R^2$ be a bounded domain with boundary $\partial\Omega$. We consider the following nonlinear elliptic boundary value problem

$$\begin{cases} Lu = -\nabla^2 u + c(x)u = F(x, u) \\ Bu = \phi \end{cases} \quad \text{on } \partial\Omega \quad (1)$$

Assume that α_0, β_0 with $\alpha_0(x) \leq \beta_0(x)$ in Ω satisfy

$$\begin{cases} L\alpha_0 \leq F(x, \alpha_0) & B\alpha_0 \leq \phi & \text{on } \partial\Omega \\ L\beta_0 \geq F(x, \beta_0) & B\beta_0 \geq \phi & \text{on } \partial\Omega \end{cases} \quad (2)$$

And $F(x, u)$ is nondecreasing in u . So α_0, β_0 are the lower and upper solutions of Eq. (1), and there exist monotone sequence $\{\alpha_n(x)\}, \{\beta_n(x)\}$, such that $\alpha_n(x) \leq u \leq \beta_n(x)$ and converge to the unique solution of Eq. (1) [Deo and Pandit (1996)].

Assume that condition of Eq. (2) holds, and $F_u(x, u), F_{uu}(x, u)$ exist and are continuous and $F_{uu}(x, u) \geq 0$ on $\Omega \times R$, and

$$0 < N \leq c(x) - F_u(x, \beta_0) \quad (3)$$

Then there exist monotone sequences $\{\alpha_n(x)\}, \{\beta_n(x)\}$ such that $\alpha_n \rightarrow r, \beta_n \rightarrow s, r = s = u$ is the unique solution of Eq. (1) satisfying $\alpha_0(x) \leq u \leq \beta_0(x)$ and the

convergence is quadratic[Deo and Pandit (1996)]. And the sequences of iterates are constructed as follows:

$$\begin{cases} L\alpha_{n+1} = F(x, \alpha_n) + F_u(x, \alpha_n)(\alpha_{n+1} - \alpha_n) & B\alpha_{n+1} = \phi & \text{on } \partial\Omega \\ L\beta_{n+1} = F(x, \beta_n) + F_u(x, \alpha_n)(\beta_{n+1} - \beta_n) & B\beta_{n+1} = \phi & \text{on } \partial\Omega \end{cases} \quad (4)$$

But it can be see that the above sequences of iterates is only true when $F(x, u)$ is convex. Fortunately, a dual result when $F(x, u)$ is concave is also true, the results are not stated here.

2.2 The Generated Quasilinearization Method

Now for a function of F , we can divided it into a sum of a concave function g and a convex function f , that is

$$F(x, u) = f(x, u) + g(x, u) \quad (5)$$

Assume that α_0, β_0 with $\alpha_0(x) \leq \beta_0(x)$ are the lower and upper solutions in Ω and satisfy [Lakshmikantham and Vatsala (2000)]

$$\begin{cases} L\alpha_0 \leq f(x, \alpha_0) + g(x, \alpha_0) & B\alpha_0 \leq \phi & \text{on } \partial\Omega \\ L\beta_0 \geq f(x, \beta_0) + g(x, \beta_0) & B\beta_0 \geq \phi & \text{on } \partial\Omega \end{cases} \quad (6)$$

In addition, f_u, g_u, f_{uu}, g_{uu} exist, are continuous, and $f_{uu} \geq 0, g_{uu} \leq 0$. And satisfy

$$0 < N \leq c(x) - [f_u(x, \beta_0) + g_u(x, \alpha_0)] \quad (7)$$

Then there exist monotone sequences $\{\alpha_n(x)\}, \{\beta_n(x)\}$ such that $\alpha_n \rightarrow r, \beta_n \rightarrow s, r = s = u$ is the unique solution of Eq. (1) satisfying $\alpha_0(x) \leq u \leq \beta_0(x)$ and the convergence is quadratic[Lakshmikantham and Vatsala (2000)]. And the sequences of iterates are constructed as follows:

$$\begin{cases} L\alpha_{k+1} = F(x, \alpha_{k+1}, \alpha_k, \beta_k) & B\alpha_{k+1} = \phi & \text{on } \partial\Omega \\ L\beta_{k+1} = G(x, \beta_{k+1}, \alpha_k, \beta_k) & B\beta_{k+1} = \phi & \text{on } \partial\Omega \end{cases} \quad (8)$$

Where

$$F(x, u, \alpha_k, \beta_k) = f(x, \alpha_k) + g(x, \alpha_k) + f_u(x, \alpha_k)(u - \alpha_k) + g_u(x, \beta_k)(u - \alpha_k) \quad (9)$$

$$G(x, u, \alpha_k, \beta_k) = f(x, \beta_k) + g(x, \beta_k) + f_u(x, \alpha_k)(u - \beta_k) + g_u(x, \beta_k)(u - \beta_k) \quad (10)$$

This is the generated quasilinearization method. In order to simplify the description, either convex or concave function is used in the following.

3 Dual reciprocity hybrid boundary node method

Although the present method is fully general in solving general nonlinear problems in this type, only the following nonlinear partial differential equation is used to demonstrate the formulation:

$$\nabla^2 u = Mu^N + p(x, y) \quad (11)$$

Where $p(x, y)$ is a given source function, M, N are constant, and the domain Ω is enclosed by $\Gamma = \Gamma_u + \Gamma_q$, with the boundary conditions are

$$\begin{cases} u = \bar{u} & \text{on } \Gamma_u \\ q = \frac{\partial u}{\partial n} = \bar{q} & \text{on } \Gamma_q \end{cases} \quad (12)$$

In which \bar{u} and \bar{q} are the prescribed potential and normal flux, respectively, on the essential boundary Γ_u and on the flux boundary Γ_q , and n is the outward normal direction to the boundary Γ .

It is show in Eq. (11) that $F(x, u) = Mu^N + p(x, y)$ is either convex or concave function, so applying the generated quasilinearization method, one can get

$$\begin{aligned} \nabla^2 u_{m+1} &= F(x, u_m) + F_u(x, u_m)(u_{m+1} - u_m) \\ &= MN(u_m)^{N-1}u_{m+1} + p(x, y) + (M - MN)(u_m)^N \end{aligned} \quad (13)$$

In which u_m is the potential value of m th iteration.

3.1 Dual reciprocity method

Applying DRM, the $(m+1)$ th iteration variables u_{m+1} can be divided into two parts, i.e., the complementary solution u_{m+1}^c and particular solution u_{m+1}^p , that is

$$u_{m+1} = u_{m+1}^c + u_{m+1}^p \quad (14)$$

The particular solution u_{m+1}^p has to satisfy the inhomogeneous equation on whole space as

$$\nabla^2 u_{m+1}^p = MN(u_m)^{N-1}u_{m+1}^p + p(x, y) + (M - MN)(u_m)^N \quad (15)$$

On the other hand, the complementary solution u_{m+1}^c must satisfy the homogeneous equation in the calculation domain Ω and the modified boundary conditions. It can be written in the form

$$\nabla^2 u_{m+1}^c = 0 \quad (16)$$

$$\begin{aligned} \bar{u}_{m+1}^c &= \bar{u}_{m+1}^c = \bar{u}_- u_{m+1}^p \\ \bar{q}_{m+1}^c &= \bar{q}_{m+1}^c = \bar{q}_- - q_{m+1}^p \end{aligned} \quad (17)$$

In which \bar{u}_-, \bar{q}_- is the boundary node value of each node on the boundary, $\bar{u}_{m+1}^c, \bar{q}_{m+1}^c$ is the generous solution of boundary nodes on the $(m+1)$ th iteration.

The DRM can be used in nonlinear problems to transform the domain integral arising from the application of inhomogeneous into equivalent boundary integrals. Applying interpolation for inhomogeneous term, the following approximation can be proposed as [Nardini and Brebbia (1983); Wrobel and Brebbia (1987)]

$$MN(u_m)^{N-1} u_{m+1}^p + p(x, y) + (M - MN)(u_m)^N \approx \sum_{j=1}^{N_B+N_I} f^j \alpha^j \quad (18)$$

where the α^j are a set of initially unknown coefficients, the f^j are approximation functions. N_B and N_I are the total number of boundary nodes and interior nodes respectively.

If the basis form of the particular solution \bar{u}^j is defined and satisfies the following equations, one can obtain [Yan and Wang (2008a, b, 2009)]

$$\nabla^2 \bar{u}^j = f^j \quad (19)$$

Substituting Eqs. (18) and (19) into Eq. (15), the particular solution u^p of origin equation can be interpolated by the basis form of the particular solution, which is shown as

$$u^p = \sum_{j=1}^{N_B+N_I} \bar{u}^j \alpha^j \quad (20)$$

For simplicity, the conical function can be chosen as the interpolation function, $f^j = 1 + r + r^2 + \dots$. And the basis form of particular solution \bar{u} satisfying Eq.(19) can be obtained as[Nardini and Brebbia (1983); Wrobel and Brebbia (1987)]

$$\bar{u} = \frac{r^2}{4} + \frac{r^3}{9} + \frac{r^4}{16} + \dots \quad (21)$$

The corresponding expression for the normal flux \bar{q} is

$$\bar{q} = (r_{,x} \frac{\partial x}{\partial n} + r_{,y} \frac{\partial y}{\partial n}) (\frac{1}{2} + \frac{r}{3} + \frac{r^2}{4} + \dots) \quad (22)$$

Solving Eq. (18), one can get

$$\alpha = MN\mathbf{F}^{-1}(\mathbf{u}_m)^{N-1} \mathbf{u}_{m+1} + (M - MN)\mathbf{F}^{-1}(\mathbf{u}_m)^N + \mathbf{F}^{-1} \mathbf{p} \quad (23)$$

In which each column of \mathbf{F} consists of a vector f^j containing the values of the function f^j at the DRM collocation nodes.

$$(\mathbf{u}_m)^{N-1} = \begin{bmatrix} (u_m^1)^{N-1} & 0 & 0 & 0 \\ 0 & (u_m^2)^{N-1} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & (u_m^{N_B+N_I})^{N-1} \end{bmatrix}$$

$$(\mathbf{u}_m)^N = [(u_m^1)^N \quad (u_m^2)^N \quad \cdots \quad (u_m^{N_B+N_I})^N]^T$$

$$\mathbf{p} = [p(x_1, y_1) \quad p(x_2, y_2) \quad \cdots \quad p(x_{N_B+N_I}, y_{N_B+N_I})]^T$$

$$\mathbf{u}_{m+1} = [u_{m+1}^1 \quad u_{m+1}^2 \quad \cdots \quad u_{m+1}^{N_B+N_I}]^T$$

In which the subscript is the number of iteration and the superscript is the nodes number in u_m^j . If assuming that

$$\mathbf{S}_1 = M\mathbf{N}\mathbf{F}^{-1}(\mathbf{u}_m)^{N-1} \tag{24}$$

$$\mathbf{S}_2 = (M - MN)\mathbf{F}^{-1}(\mathbf{u}_m)^N + \mathbf{F}^{-1}\mathbf{p}$$

Substituting Eqs. (23) and (24) into Eq. (20), the particular solution can be written as

$$\mathbf{u}_{m+1}^p = \bar{\mathbf{U}}\mathbf{S}_1\mathbf{u}_{m+1} + \bar{\mathbf{U}}\mathbf{S}_2 \tag{25}$$

$$\mathbf{q}_{m+1}^p = \bar{\mathbf{Q}}\mathbf{S}_1\mathbf{u}_{m+1} + \bar{\mathbf{Q}}\mathbf{S}_2 \tag{26}$$

In which $\bar{\mathbf{U}}$ and $\bar{\mathbf{Q}}$ are matrixes of the basic form of the particular solution.

3.2 Hybrid boundary node method

The Hybrid BNM is based on a modified variational principle and moving least square(MLS). The functions in the modified principle assumed to be independent are: potential field within the domain, u , boundary potential field \tilde{u} and boundary normal flux \tilde{q} [Zhang and Yao (2002)].

According to MLS, one can approximate \tilde{u} and \tilde{q} at the boundary Γ , as

$$\tilde{u}(s) = \sum_{I=1}^{N_B} \Phi_I(s)\hat{u}_I \tag{27}$$

$$\tilde{q}(s) = \sum_{I=1}^{N_B} \Phi_I(s)\hat{q}_I \tag{28}$$

where N_B stands for the number of nodes located on the surface; \hat{u}_I and \hat{q}_I are nodal values, and $\Phi_I(s)$ is the shape function of the MLS approximation, corresponding to node S_I , which is given by[Lancaster and Salkauskas (1981)]

$$\Phi_I(s) = \sum_{j=1}^m p_j(s) \left[\mathbf{A}^{-1}(\mathbf{s}) \mathbf{B}^{-1}(\mathbf{s}) \right]_{jI} \quad (29)$$

In the above equation, $p_j(s)$ provide a basis function of order m . In this study, we take m to 4, namely, $\mathbf{p}^T(\mathbf{s}) = [1, s, s^2, s^3]$. Matrixes $\mathbf{A}(\mathbf{s})$ and $\mathbf{B}(\mathbf{s})$ are defined as

$$\mathbf{A}(\mathbf{s}) = \sum_{I=1}^{N_B} w_I(s) \mathbf{p}(\mathbf{s}_I) \mathbf{p}^T(\mathbf{s}_I) \quad (30)$$

$$\mathbf{B}(\mathbf{s}) = [w_1(s)p(s_1), w_2(s)p(s_2), \dots, w_{N_B}(s)p(s_{N_B})] \quad (31)$$

In Eqs.(30) and (31), $w_I(s)$ are weight functions. Gaussian weight function corresponding to node S_I can be written as

$$w_I(\mathbf{s}) = \begin{cases} \frac{\exp[-(d_I/c_I)^2] - \exp[-(r_I/c_I)^2]}{1 - \exp[-(r_I/c_I)^2]} & 0 \leq d_I \leq r_I \\ 0 & d_I \geq r_I \end{cases} \quad (32)$$

where $d_I = |s - s_I|$ is the distance between an evaluation point and node S_I , c_I is a constant controlling the shape of the weight function $w_I(s)$, and \hat{d}_I is the size of the support for the weight function $w_I(s)$ and determines the support of node S_I .

According to HBNM formulation, the domain variables \mathbf{u} and \mathbf{q} are interpolated by the fundamental solution, and can be written as

$$\begin{aligned} u &= \sum_{I=1}^{N_B} u_I^S x^I \\ q &= \sum_{I=1}^{N_B} \frac{\partial u_I^S}{\partial n} x^I \end{aligned} \quad (33)$$

In which x^I is the unknown parameter, $u_I^S = \frac{\ln(r)}{2\pi}$ is the fundamental solution, where r is the distance between source point S_I and field point Q .

Applied modified variational principle, the corresponding variational function Π_{AB} is defined as[Zhang and Yao (2002)]

$$\Pi_{AB} = \int_{\Omega} \frac{1}{2} u_{,i} u_{,i} d\Omega - \int_{\Gamma} \tilde{q}(u - \tilde{u}) d\Gamma - \int_{\Gamma_q} \tilde{q} \tilde{u} d\Gamma \quad (34)$$

where, u is the internal node value, and the boundary displacement \tilde{u} satisfies the essential boundary condition, i.e. $\bar{u} = \tilde{u}$, on Γ_u .

Taking the variational of Eq. (34) and with the vanishing of $\delta\Pi_{AB}$ over the domain and its boundary, the following equivalent integral can be obtained

$$\int_{\Gamma} (q - \tilde{q}) \delta u d\Gamma - \int_{\Omega} u_{,ii} \delta u d\Omega = 0 \quad (35)$$

$$\int_{\Gamma} (u - \tilde{u}) \delta \tilde{q} d\Gamma = 0 \quad (36)$$

$$\int_{\Gamma} (\tilde{q} - \bar{q}) \delta \tilde{u} d\Gamma = 0 \quad (37)$$

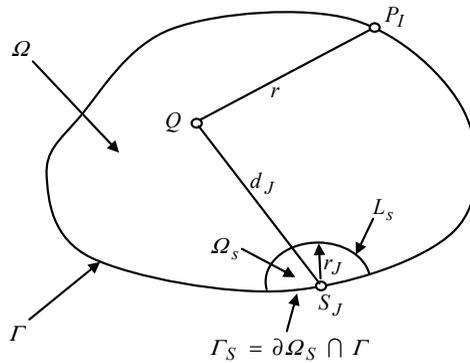


Figure 1: Local domain and source point of fundamental solution corresponding to S_J

Eq. (37) will be satisfied if the flux boundary condition, $q = \bar{q}$, is imposed. So it will be ignored in the following. Eqs. (35) and (36) hold for any portion of the domain Ω , for example, a sub-domain Ω_s , which is defined as an intersection of a domain and a small circle centered at node S_J , and its boundary Γ_s and L_s (see Fig.1). So, we can use the following weak form for the sub-domain and its boundary to replace Eqs. (35) and (36) [Zhang and Yao (2002)]:

$$\int_{\Gamma_s + L_s} (q - \tilde{q}) h d\Gamma - \int_{\Omega_s} u_{,ii} h d\Omega = 0 \quad (38)$$

$$\int_{\Gamma_s + L_s} (u - \tilde{u}) h d\Gamma = 0 \quad (39)$$

where h is a test function.

In Eqs.(38) and (39), \tilde{u}_s and \tilde{q}_s at Γ_s can be represented by Eqs.(27) and (28), since Γ_s is a portion of Γ , while \tilde{u}_s and \tilde{q}_s at L_s has not been defined yet. To solve this problem, we select h such that all integrals vanish over L_s . This can be easily accomplished by using the weight function in the MLS approximation for h , with the half-length of the major axis d_J of the support of the weight function being replaced by the radius of the sub-domain Ω_s , i.e.

$$h_J(Q) = \begin{cases} \frac{\exp[-(d_J/c_J)^2] - \exp[-(r_J/c_J)^2]}{1 - \exp[-(r_J/c_J)^2]} & 0 \leq d_J \leq r_J \\ 0 & d_J \geq r_J \end{cases} \quad (40)$$

where d_J is the distance between point Q in the domain and the nodal point S_J . On L_s , $d_J = r_J$, from Eq. (40) it can be seen that $h_J(Q) = 0$, so it vanishes on boundary L_s . Eqs.(38) and (39) can be rewritten as[Zhang and Yao (2002)]

$$\int_{\Gamma_s} (q - \tilde{q})h_J(Q)d\Gamma - \int_{\Omega_s} u_{,ii}h_J(Q)d\Omega = 0 \quad (41)$$

$$\int_{\Gamma_s} (u - \tilde{u})h_J(Q)d\Gamma = 0 \quad (42)$$

By substituting Eqs.(27), (28) and (33) into Eqs. (41) and (42), and omitting the vanishing terms, we have

$$\sum_{I=1}^{N_B} \int_{\Gamma_s} \frac{\partial u_I^s}{\partial n} h_J(Q) x^J d\Gamma = \sum_{I=1}^{N_B} \int_{\Gamma_s} \Phi_I h_J(Q) \hat{q}_I d\Gamma \quad (43)$$

$$\sum_{I=1}^{N_B} \int_{\Gamma_s} u_I^s h_J(Q) x^J d\Gamma = \sum_{I=1}^{N_B} \int_{\Gamma_s} \Phi_I h_J(Q) \hat{u}_I d\Gamma \quad (44)$$

Using the above equations for all nodes, one can get the system equations

$$\mathbf{Ux} = \mathbf{H}\hat{\mathbf{u}} \quad (45)$$

$$\mathbf{Tx} = \mathbf{H}\hat{\mathbf{q}} \quad (46)$$

Where matrix \mathbf{U} , \mathbf{H} , \mathbf{T} can be referred in Zhang and Yao (2002).

3.3 Dual reciprocity hybrid boundary node method

For a well-posed problem, either \hat{u} or \hat{q} is known at each node on the boundary. However, transformation between \hat{u}_I and \tilde{u}_I , \hat{q}_I and \tilde{q}_I is necessary because the MLS approximation lacks the delta function property. For the panels where \tilde{u}_I is prescribed, \tilde{u}_I is related to \hat{u}_I by [Zhang and Yao (2002)]

$$u(\mathbf{s}) = \sum_{J=1}^n \sum_{I=1}^n \Phi_I(\mathbf{s}) R_{IJ} \tilde{u}_J = \sum_{J=1}^n \varphi_J(\mathbf{s}) \hat{u}_J \quad (47)$$

and for the panels where \tilde{q}_I is prescribed, \hat{q}_I is related to \tilde{q}_I by

$$q(\mathbf{s}) = \sum_{J=1}^n \sum_{I=1}^n \Phi_I(\mathbf{s}) R_{IJ} \tilde{q}_J = \sum_{J=1}^n \varphi_J(\mathbf{s}) \hat{q}_J \quad (48)$$

where $R_{IJ} = [\Phi_J(x_I)]^{-1}$, n is the total number on a piece of the edge.

Substituting Eqs.(25) and (26) into Eq. (14), one can get

$$\begin{aligned} \mathbf{u}_{m+1}^c &= \mathbf{u}_{m+1} - \bar{\mathbf{U}}\mathbf{S}_1 \mathbf{u}_{m+1} - \bar{\mathbf{U}}\mathbf{S}_2 \\ \mathbf{q}_{m+1}^c &= \mathbf{q}_{m+1} - \bar{\mathbf{Q}}\mathbf{S}_1 \mathbf{u}_{m+1} - \bar{\mathbf{Q}}\mathbf{S}_2 \end{aligned} \quad (49)$$

Substituting Eq.(49) into Eqs. (45) and (46), we can get

$$\mathbf{U}\mathbf{x} = \mathbf{H}\mathbf{R}(\mathbf{u}_{m+1} - \bar{\mathbf{U}}\mathbf{S}_1 \mathbf{u}_{m+1} - \bar{\mathbf{U}}\mathbf{S}_2) \quad (50)$$

$$\mathbf{T}\mathbf{x} = \mathbf{H}\mathbf{R}(\mathbf{q}_{m+1} - \bar{\mathbf{Q}}\mathbf{S}_1 \mathbf{u}_{m+1} - \bar{\mathbf{Q}}\mathbf{S}_2) \quad (51)$$

where matrix \mathbf{R} is the transition matrix from Eqs. (47) and (48).

From Eq. (50), \mathbf{x} can be expressed as

$$\mathbf{x} = \mathbf{U}^{-1} \mathbf{H}\mathbf{R}(\mathbf{u}_{m+1} - \bar{\mathbf{U}}\mathbf{S}_1 \mathbf{u}_{m+1} - \bar{\mathbf{U}}\mathbf{S}_2) \quad (52)$$

Substituting Eq.(52) into Eq. (51), one can obtain

$$\begin{aligned} \mathbf{T}\mathbf{U}^{-1} \mathbf{H}\mathbf{R}\hat{\mathbf{u}}_{m+1} + (\mathbf{H}\mathbf{R}\bar{\mathbf{Q}}\mathbf{S}_1 - \mathbf{T}\mathbf{U}^{-1} \mathbf{H}\mathbf{R}\bar{\mathbf{U}}\mathbf{S}_1) \mathbf{u}_{m+1} \\ = \mathbf{H}\mathbf{R}\hat{\mathbf{q}}_{m+1} + \mathbf{T}\mathbf{U}^{-1} \mathbf{H}\mathbf{R}\bar{\mathbf{U}}\mathbf{S}_2 - \mathbf{H}\mathbf{R}\bar{\mathbf{Q}}\mathbf{S}_2 \end{aligned} \quad (53)$$

In which

$$\hat{\mathbf{u}}_{m+1} = [u_{m+1}^1 \quad u_{m+1}^2 \quad \cdots \quad u_{m+1}^{N_B} \quad 0 \quad \cdots]^T$$

$$\mathbf{u}_{m+1} = [u_{m+1}^1 \quad u_{m+1}^2 \quad \cdots \quad u_{m+1}^{N_B+N_I}]^T$$

$$\hat{\mathbf{q}}_{m+1} = [q_{m+1}^1 \quad q_{m+1}^2 \quad \cdots \quad q_{m+1}^{N_B} \quad 0 \quad \cdots]^T$$

where the subscript stands for the number of iteration and superscript is the node number.

Assuming that N_B nodes are located on the boundary, we can get N_B equations on the boundary from Eq. (53). However, the equations above include the potential of the N_I internal nodes. So the additional equations are needed.

3.4 Additional equations

The Eq. (53) can not be solved for the variables of the internal nodes, so additional equations for nonlinear problem will be developed in this section.

The unknown variables of the internal nodes can be expressed as follows

$$u_{m+1} = u_{m+1}^c + u_{m+1}^p \quad (54)$$

The complementary solution, u_{m+1}^c , can be interpolated by the fundamental solution which is expressed in Eq. (33) and the particular solution, u_{m+1}^p , can be expressed in Eq.(25). So the Eq.(54) can be rewritten as

$$\begin{aligned} \check{u}_{m+1} &= \mathbf{u}_1^s \mathbf{x} + \bar{\mathbf{U}} \alpha \\ &= \mathbf{u}_1^s \mathbf{U}^{-1} \mathbf{H} \mathbf{R} \hat{\mathbf{u}}_{m+1} - \mathbf{u}_1^s \mathbf{U}^{-1} \mathbf{H} \mathbf{R} \bar{\mathbf{U}} \mathbf{S}_1 \mathbf{u}_{m+1} + \bar{\mathbf{U}} \mathbf{S}_1 \mathbf{u}_{m+1} + \bar{\mathbf{U}} \mathbf{S}_2 - \mathbf{u}_1^s \mathbf{U}^{-1} \mathbf{H} \mathbf{R} \bar{\mathbf{U}} \mathbf{S}_2 \end{aligned} \quad (55)$$

In which $\check{u}_{m+1} = [[u_{m+1}^{N_B+1} \quad u_{m+1}^{N_B+2} \quad \dots \quad u_{m+1}^{N_B+N_I}]]^T$, \mathbf{u}_1^s is a matrix which composes by the value of internal nodes of fundamental solution.

Eq. (55) can be rewritten as

$$\mathbf{u}_1^s \mathbf{U}^{-1} \mathbf{H} \mathbf{R} \hat{\mathbf{u}}_{m+1} + (\bar{\mathbf{U}} \mathbf{S}_1 - \mathbf{u}_1^s \mathbf{U}^{-1} \mathbf{H} \mathbf{R} \bar{\mathbf{U}} \mathbf{S}_1) \mathbf{u}_{m+1} - \check{u}_{m+1} + \bar{\mathbf{U}} \mathbf{S}_2 - \mathbf{u}_1^s \mathbf{U}^{-1} \mathbf{H} \mathbf{R} \bar{\mathbf{U}} \mathbf{S}_2 = 0 \quad (56)$$

With the initial guesses $u = u_0$, $q = q_0$ on unknown nodes, combined Eq. (53) and Eq. (56), the solutions of origin equation on first iteration can be gotten. And update the initial values via the first iteration values, we can get the new iteration values. Assumed the tolerance is ε , which is small enough. Do above iterations, until the following equation is satisfied

$$e_{m+1} = \sqrt{\frac{\sum_{i=1}^{N_B+N_I} (u_{m+1}^i - u_m^i)^2}{N_B + N_I}} \leq \varepsilon \quad (57)$$

In which the subscript of u_{m+1}^i stands by the number of iteration, and the superscript stands by the nodes number.

According to formulation of generated quasilinearization method, convergence of present method is quadratic, which is higher than some other iteration scheme. And most important advantage is that this method is insensitive to the initial guesses, and has rapid convergence, high stability and less computation.

4 Numerical examples

In all examples, the support size for the weight function \hat{d}_I is set to be $3.5h$, with h being the average distance of adjacent nodes. The parameter c_I is taken to be that $d_I/c_I = 0.5$. In this paper, $r_J = 0.85h$ is chosen as the radius of the sub-domain, and the parameter c_J is taken to be $r_J/c_J = 1.1$. In order to deal with the normal flux discontinuities at the corners, the nodes are not arranged at these places and the support domain for interpolation is truncated.

In order to demonstrate the high accuracy and high convergence rate and insensitivity with the initial guesses of present method, the following three iteration schemes and methods are used for comparison.

Iteration scheme 1:

$$\nabla^2 u_{m+1} \cong M(u_m)^N + p(x, y) \quad (58)$$

Iteration scheme 2:

$$\nabla^2 u_{m+1} \cong M(u_m)^{N-1} u_{m+1} + p(x, y) \quad (59)$$

For above two iteration schemes, DHBNM is applied to solve the problems.

Quasilinear boundary element method(QBEM) scheme:

According to Quasilinearization method, we can get

$$\nabla^2 u_{m+1}^p = MN(u_m)^{N-1} u_{m+1}^p + p(x, y) + (M - MN)(u_m)^N \quad (60)$$

The above equation can be rewritten as

$$\begin{aligned} \nabla^2 u_{m+1}^p - MN(\bar{u}_m)^{N-1} u_{m+1}^p \\ \cong MN u_m^p ((u_m)^{N-1} - (\bar{u}_m)^{N-1}) + p(x, y) + (M - MN)(u_m)^N \end{aligned} \quad (61)$$

Using above equation, we can use the formulation of Helmholtz equation to solve it. In the following calculation, the initial values are 0.0 except that some special description is made.

4.1 patch test

Consider a standard patch test in a domain of dimension 1×1 as shown in Fig. 2. we consider a problem for a known coefficient $M=4.0$, $N=3.0$. With the exact solution

$$u = \frac{1}{x+y} \quad (62)$$

$$q = -\frac{1}{(x+y)^2} \left(\frac{\partial x}{\partial n} + \frac{\partial y}{\partial n} \right) \tag{63}$$

where the essential boundary condition is prescribed on all boundary nodes according to Eq. (62) satisfaction of patch test requires that the value of u at any interior node be given by the same function (62), and that the derivatives of the computed solution along the boundary satisfy the Eq. (63).

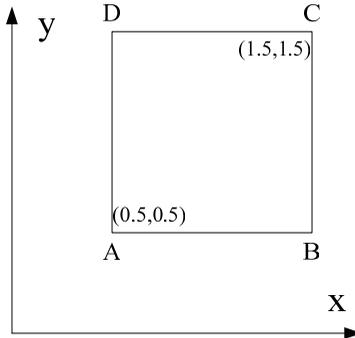


Figure 2: Calculation model

4.1.1 Computational accuracy

For comparison, QBEM is applied, which is mentioned above. In QBEM calculation, we used one subdomain and each side of boundary is divided into 10 constant elements, and 81 internal nodes are arranged in the domain. For present method, 8 boundary nodes are arranged on the boundary and 51 internal nodes are used for interpolation.

Table 1: Potential on internal nodes and flux on boundary nodes

Quantity	Coordinate	QHBNM results	QBEM results	Analytical results
u	(0.625,0.625)	0.799730	0.800887	0.800000
	(0.75,1.00)	0.571434	0.571398	0.571428
	(1.125,1.25)	0.421085	0.421092	0.421052
	(1.375,1.375)	0.363649	0.363668	0.363636
q	(0.5,0.5109)	0.98143	0.98776	0.97861
	(0.5109,1.5)	-0.25083	-0.27789	-0.24730
	(1.5,1.4891)	-0.11356	-0.11786	-0.11192
	(1.4891,0.5)	0.25838	0.29081	0.25273

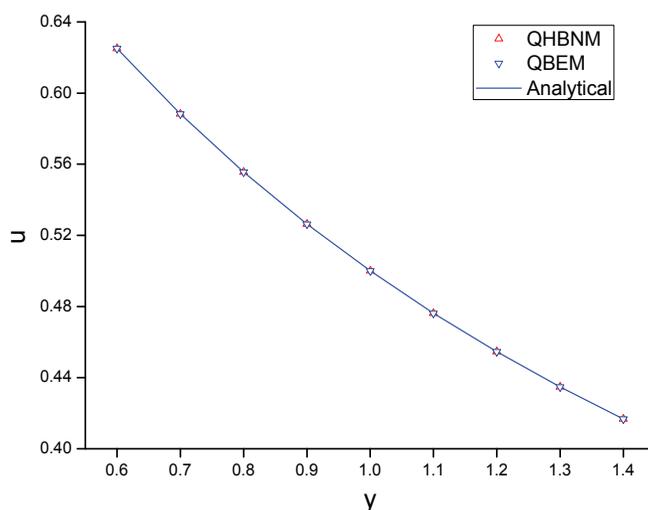


Figure 3: The internal potential results obtained by present method and QBEM

Potential values on internal nodes and flux values on the boundary nodes are shown in Table 1, in addition, the results obtained by QBEM are also shown in this table for comparison. It can be seen that the result obtained by QHBNM is much more accurate than that of QBEM. Additionally, the internal node potential on $x=1.0$ and boundary flux on AB are shown in Fig. 3 and 4 respectively. It is shown that the results obtained by present method is very close to the analytical results, which is accurate than the results obtained by QBEM.

4.1.2 Convergence

The convergence rate of present method, iteration scheme 1, iteration scheme 2 and QBEM are shown in Fig. 5 for comparison. It is clearly seen that present method has the highest convergence rate among them, and via approximation we can get that the convergence is quadratic, where the convergence of other methods mentioned here are near linear, although quasilinearization method is also used in quasilinear boundary element method.

4.1.3 Initial value discussion

Applying Eq. (7), we can see that the iteration can always converge when

$$-g_u(x, \alpha_0) = 12\alpha_0^2 \geq 0 \quad (64)$$

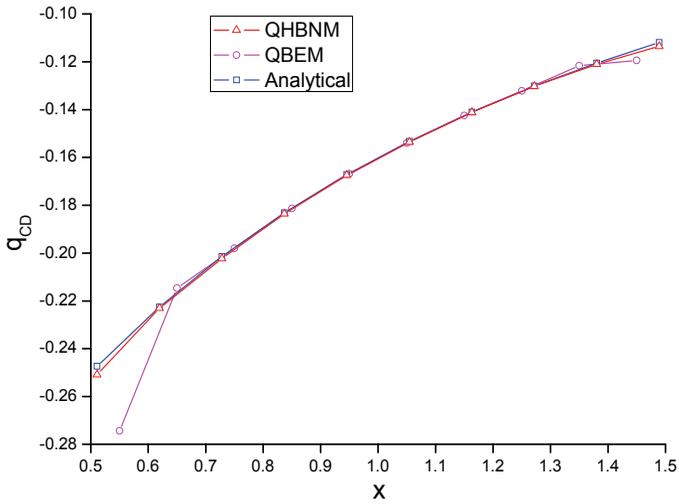


Figure 4: The boundary flux results obtained by present method and QBEM

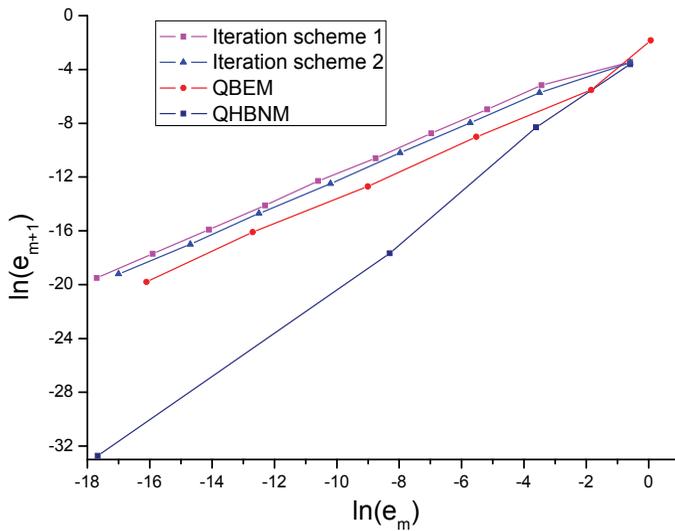


Figure 5: Convergence rate of those methods

So we can get that the initial guesses $\alpha_0 \in (-\infty, +\infty)$, in other word, the present method can converge at any initial values. A wide range of initial guesses are applied to calculation for this example, and results are shown that Eq. (64) is right. The most important advantage is that this method is stable on every initial values,

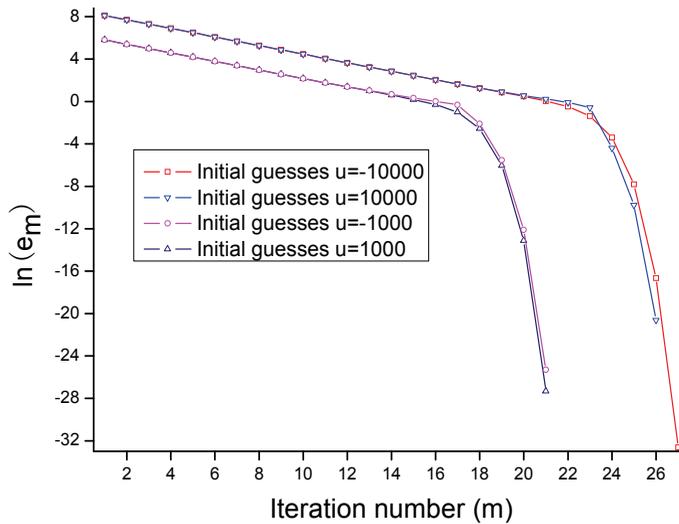


Figure 6: Convergence rate of present method with different initial guesses

which can be seen in Fig. 6, although initial value is -10000.0 , the error is also monotone decreasing. The same calculation is done by QBEM, and the results shown that when $\alpha_0 \in (-\infty, -100)$, the calculation can converge, which may be caused by the some term is omitted in the calculation.

According to iteration scheme 1, the same calculations is done, and results shown that when $\alpha_0 \in (-2.0, 2.3)$ this iteration scheme can converge, it is obvious that the choice of initial value is very small. The same work has been done for iteration scheme 2, it is shown that calculation by this method can converge at any initial value, but the convergence is very slow.

4.1.4 Condition number discussion

Assuming that the finally equation of this problem can be written as $\mathbf{Ax} = \mathbf{y}$. Now the condition number of matrix of \mathbf{A} is discussed in this section. We denote the condition number as $\text{cond}(\mathbf{A}) = \|\mathbf{A}\|_{\infty} \|\mathbf{A}^{-1}\|_{\infty}$.

So we can get the condition number for each iteration steps, for comparison, the results by QBEM is also applied, which are shown in Fig. 7. It is shown that the condition number does not vary greatly with the iteration steps and the value is not very large. One can see that the iteration is stability and the property of matrix is good.

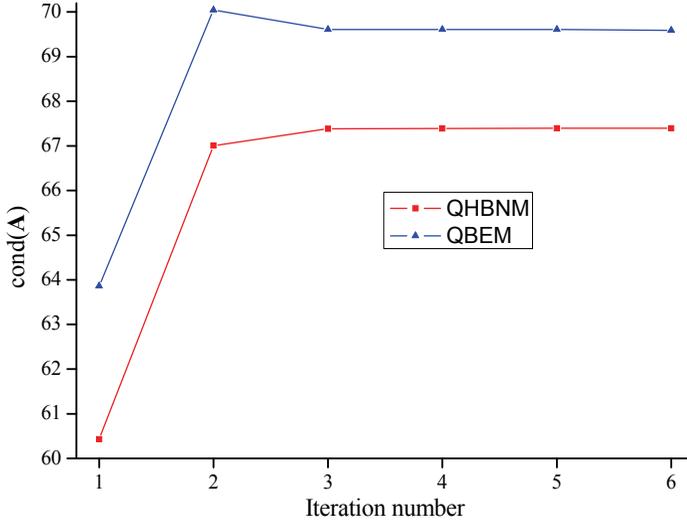


Figure 7: Condition number for each iteration steps

4.2 A cubic solution with different boundary condition

The example here is a nonlinear equation with $M=1.0$, $N=3.0$ in the 2×2 domain shown in Fig. 8, with the exact solution, a cubic polynomial, as [Zhu et al. (1998)]

$$u = -\frac{1}{12}(x^3 + y^3) + \frac{3}{10}(x^2y + xy^2) \tag{65}$$

$$p = -\left[-\frac{1}{12}(x^3 + y^3) + \frac{3}{10}(x^2y + xy^2)\right]^3 + \frac{x+y}{10} \tag{66}$$

A mixed boundary condition is applied on this problem, for which the essential boundary condition is imposed on AB and CD and the flux boundary condition is prescribed on AD and BC, according to Eqs. (65) and (66).

The same as example 4.1, in the computation, 8 boundary nodes are arranged on the boundary and 51 internal nodes are used for interpolation. For comparison, the results obtained by LBIE [Zhu et al. (1998)] are quoted, in which 36 nodes are used.

The values of u , $\partial u/\partial x$, $\partial u/\partial y$ for $x = 1.0$ are depicted in Fig. 9~11. It can be see that the results obtained by present method and LBIE and analytical solution are very close to each other.

The convergence of present iteration scheme, iteration 1 and iteration 2 are shown in Fig. 12. It can be see that the convergence of present method is higher than the

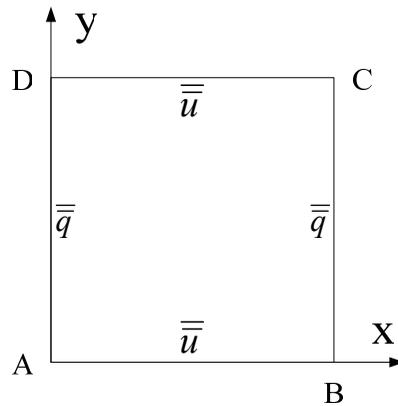
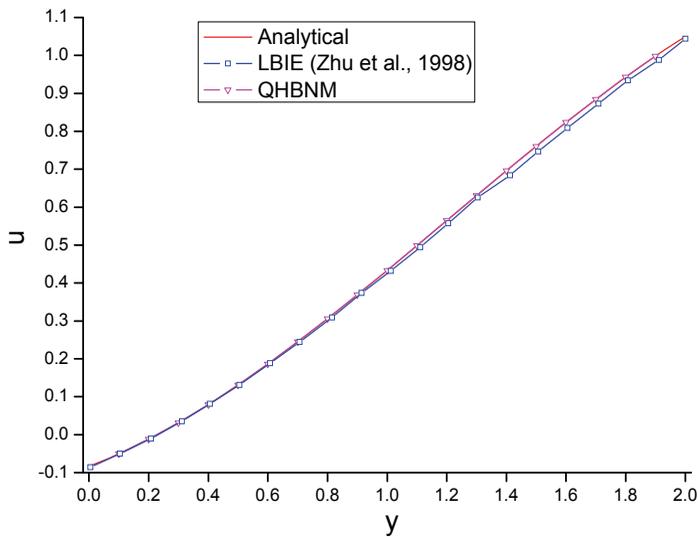


Figure 8: Calculation model

Figure 9: Value of u at $x = 1.0$

other two method, where convergence of present method is close to quadratic and convergence of the other two method is near linear.

The same as example 4.1, the stability and the sensitivity of the iteration scheme are examined in this paper, numerical results shows that the iteration scheme 1 can only converge at $\alpha_0 \in (-4.5, 4.5)$, but present method can monotone converge on any initial values. So the stability and insensitivity with the initial value of present

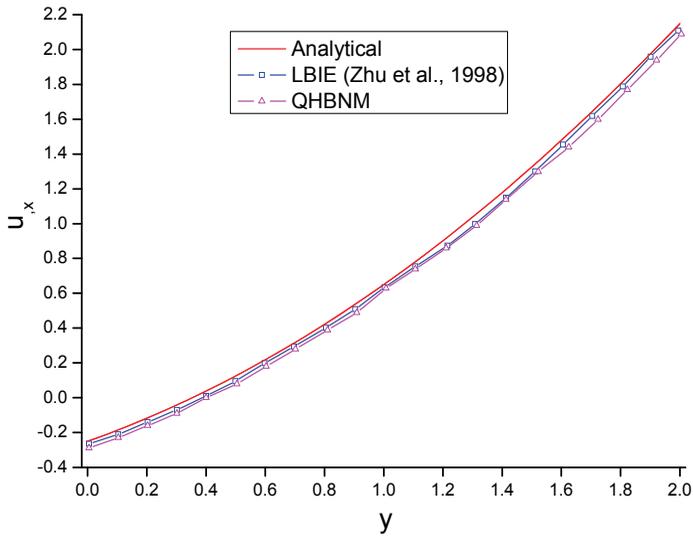


Figure 10: Value of $\partial u/\partial x$ at $x = 1.0$

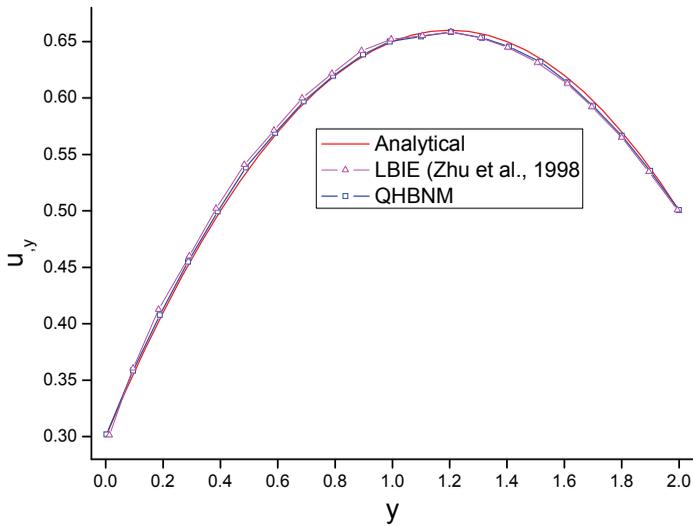


Figure 11: Value of $\partial u/\partial y$ at $x = 1.0$

method are more excellent than the other two methods.

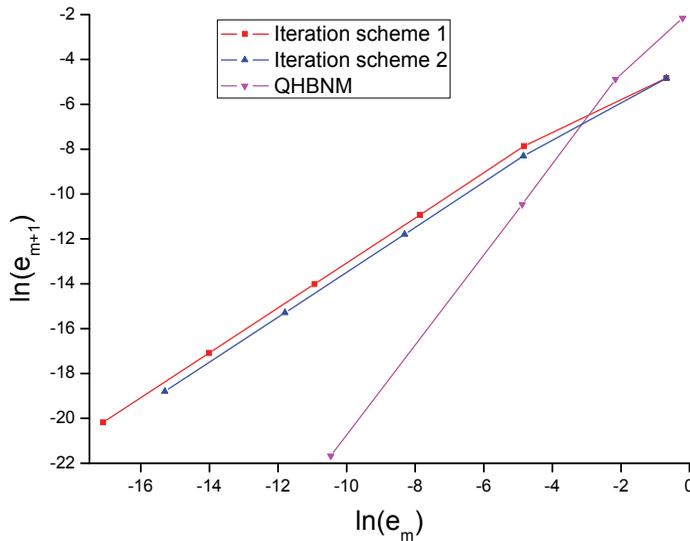


Figure 12: Convergence rate of those methods

4.3 A mixed boundary condition diffusion problem

In this example, a diffusion reaction problem is considered, where u represents the concentration of the diffusing species [Kasab et al. (1995)], and in this case $p(x,y) = 0.0$. It is shown in Fig. 13, in which the Dirichlet condition is prescribed along AB and the Neumann condition (zero flux) on the other boundary.

For comparison, QBEM is applied, in which 12 constant elements are used on each side of the domain. In present method computation, 8 nodes are arranged on each side of object, and 51 internal nodes are used for interpolation. At the same time, the results by QBEM[Kasab et al. 1995] are quoted for comparison.

Table 2 shows the results on different M and N by present method, QBEM, and some other method mentioned by Kasab et al. (1995). It is shown that the results of present method can get the much more accurate results than QBEM and some other numerical method.

Fig 14 shows the convergence of different method and iteration schemes, and it can be see that among those methods, the present method has the most highest convergence, which is quadratic, and convergence of the other methods are near linear. Actually, the iteration scheme 1 does not converge with the initial value of $u = 0$.

The CPU time of different method on the same nodes arrangement is shown in

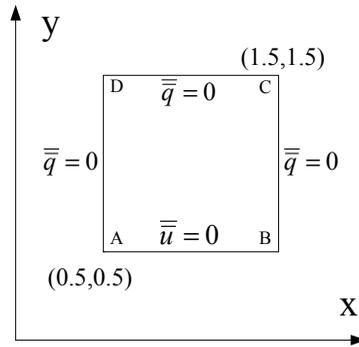


Figure 13: Calculation model

Table 2: Results by different method on different coefficient

Quantity	Method	M=2.0, N=2.0	M=12.0, N=2.0	M=-0.5, N=2.0
$u(CD)$	QHBNM	0.58151	0.25339	1.49911
	QBEM	0.58179	0.253246	1.49770
	QBEM [Kasab et al.(1995)]	0.5896	0.309900	1.486
	DRM [Kasab et al.(1995)]	0.5809	0.250600	1.498
	1D BEM [Kasab et al.(1995)]	0.5813	0.252700	1.500
$q(AB)$	QHBNM	1.03611	2.83739	-0.88824
	QBEM	1.03303	2.802327	-0.88416
	QBEM [Kasab et al.(1995)]	1.0190	2.630900	-0.867
	DRM [Kasab et al.(1995)]	1.0400	2.921000	-0.888
	1D BEM [Kasab et al.(1995)]	1.0350	2.805000	-0.890

Table 3, it can be see that the CPU time of present method is less than that of QBEM. It is obvious that the computation of present method is less than that of QBEM. Especially on case M=-0.5 and N=2.0, the iteration can not converge with initial values $u = 0$.

Table 3: CPU time by different method on different coefficient

Method	M=2.0,N=2.0	M=12.0,N=2.0	M=-0.5,N=2.0
QHBNM	2.75	2.844	2.672
QBEM	39	56	---

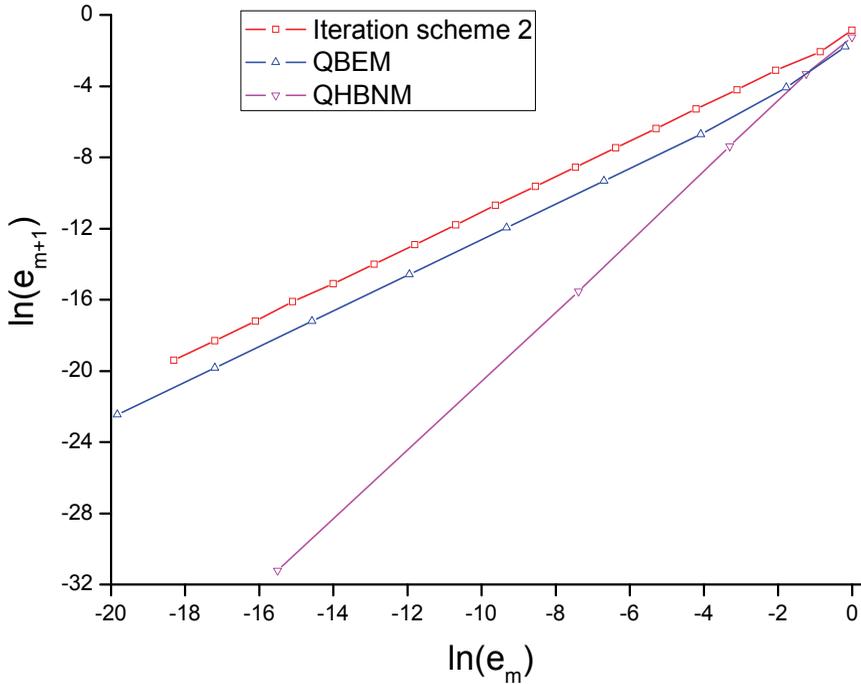


Figure 14: Convergence rate of those methods

Applying Eq. (7), we can see that the iteration can always converge when

$$-g_u(x, \alpha_0) = 4\alpha_0 \geq 0 \quad (67)$$

So we can see that when $\alpha_0 \geq 0$ the present method can converge. The same as before, many initial values are attempted for $M=2.0$ and $N=2.0$, and we can get that the QBEM and QHBNM can converge at $\alpha_0 \in (-100, +\infty)$, and the convergence of QHBNM is much more higher than that of QBEM.

On this case, it is shown that this method is stable on any initial values in $\alpha_0 \in (-100, +\infty)$, which can be seen in Fig. 15, the error is monotone decreasing at any initial values, and this is a very important advantage of present method.

4.4 Complex geometries problem

In this numerical example, a case of diffusion reaction in a trilobe catalyst particle problem is considered, and the catalyst geometry of the schematic is shown in Fig. 16. Dirichlet boundary conditions $u = 1.0$ are imposed along the boundary of the

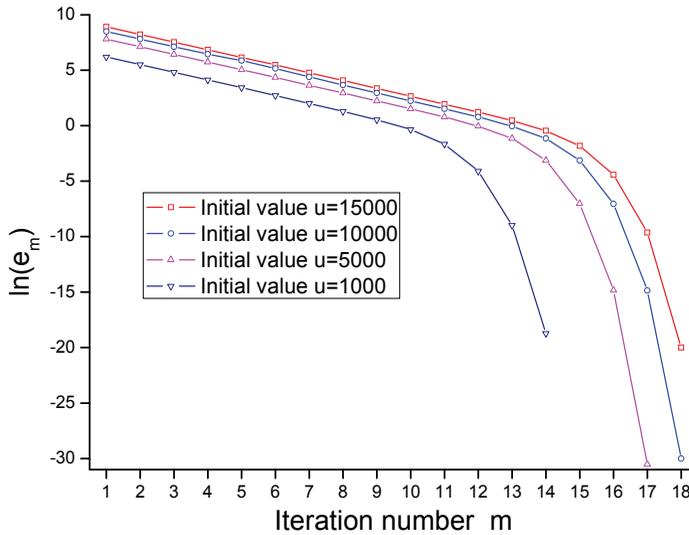


Figure 15: Convergence rate of present method with different initial guesses

catalyst particle. Due to the symmetry, only one sixth of the geometry needs to be discretized.

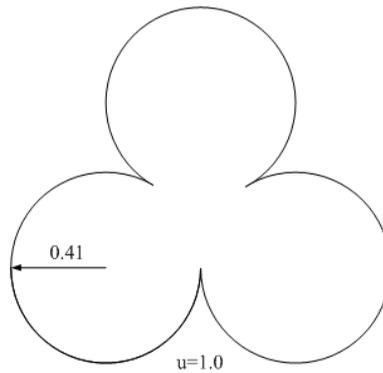


Figure 16: Geometry of a trilobe shaped particle

For this particular problem, M is taken to be 25, and a total of 30 nodes are arranged on the boundary and 20 nodes on the domain. The total normal concentration gradient along the boundary ($\int_{\Gamma} p d\Gamma$) is directly proportional to the effectiveness factor of the catalyst which is a key parameter in the design of heterogeneous reactions, so

Table 4: Total gradient along the boundary for diffusion reaction in a trilobe catalyst particle for different N

Method	N=0.5	N=1.0	N=2.0
Present method	30.579	25.816	20.581
QBEM [Kasab et al.(1995)]	29.870	25.341	20.590
DRM [Kasab et al.(1995)]	31.173	26.069	20.576

this quantity is evaluated. For comparison, the results obtained by QBEM [Kasab et al. (1995)] and DRM are also shown in Table 4. In QBEM, 12 cubic boundary elements are used to discretize the one sixth of the problems. It is shown that a good agreement can be got between those methods.

5 Conclusion

In this paper, firstly, the quasilinearization method, dual reciprocity method and hybrid boundary node method are combined, and a new truly meshless method for nonlinear problems is proposed, which is named as QHBNM. The generated quasilinearization method is employed to linearize the nonlinear term of the governing equation, in which method of upper and lower and monotone iterative technique are applied, and the solution of the linearized equation can exactly converge to the solution of original equation at a very wide range initial value, and convergence rate is quadratic.

Secondly, DHBNM is employed to solve this linearized equation. The solution in this method composes two parts: complementary solution and particular solution. For the first part, HBNM is applied. For the second part, DRM has been used and the radial basis functions are applied to interpolating the nonlinear part of the equations. Because of the nonlinear term of the governing equations, the boundary integral equations obtained by DHBNM are not enough to solve all variables. Some additional equations are proposed to obtain the relation of the variables in the domain and on the boundary.

This method keeps the ‘boundary-only’ and truly meshless method character of HBNM, and the convergence is quadratic, which is much higher than some other methods. The most important advantage of present method is its stability, insensitivity with initial guesses and less computation.

The numerical examples are presented for several nonlinear problems, for which accurate results can be available. The convergence of present method is quadratic, and the stability and its insensitivity with the initial value is very excellent. It is shown that the present method is effective and can be widely applied in practical

engineering. Besides, this method has the advantage of small amount of calculation, in which on each iteration only several matrixes are needed to update, while when QBEM is applied, integral for coefficient matrixes are needed on each iteration. And the present procedure can easily be extended to 3D problems.

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