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An Efficient Response Surface Based Optimisation Method for Non-Deterministic Harmonic and Transient Dynamic Analysis

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Abstract: Deterministic simulation tools enable a very precise simulation of physical phenomena using numerical models. In many real life situations however, a deterministic analysis is not sufficient to assess the quality of a design. In a design stage, some physical properties of the model may not be determined yet. But even in a design ready for production, design tolerances and production inaccuracies introduce variability and uncertainty. In these cases, a non-deterministic analysis procedure is required, either using a probabilistic or a non-probabilistic approach.
The authors developed an intelligent Kriging response surface based optimisation procedure that can be used in combination with any deterministic numerical analysis code to perform efficient interval and fuzzy numerical analyses. The procedure is illustrated on the transient analysis of a cable stayed bridge model and on the harmonic analysis of a benchmark model.

Keywords: interval, fuzzy set, black box optimisation, multiple outputs, response surface

1 Introduction

Non-deterministic approaches are gaining momentum in the field of numerical modelling techniques. The ability to include non-deterministic properties is of great value for a design engineer. It enables realistic reliability assessment that incorporates the uncertain aspects of the design. Furthermore, the design can be optimised for robust behaviour under varying external influences. Recently, criticism has arisen regarding the general application of the probabilistic concept in this

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context. Especially when objective information on the uncertainties is limited, the subjective probabilistic analysis result proves to be of little value, and does not justify its high computational cost [Elishakoff (2000); Moens and Vandepitte (2005)]. Consequently, alternative non-probabilistic concepts have been introduced for non-deterministic numerical modelling.

In this context, interval and fuzzy approaches are becoming increasingly popular for the analysis of numerical models that incorporate uncertainty in their description. In the interval approach, uncertainties are considered to be contained within a predefined range. For each uncertainty, the analyst has to provide the lower and upper bound. The fuzzy approach extends this methodology by introducing a level of membership that represents to what extent a certain value is member of the range of possible input values. This concept provides the analyst with a tool to express a degree of possibility for a certain value. The interval and fuzzy set concepts are described in more detail in sections 2.3.1 and 2.3.3.

In recent literature, the application of both the interval and the fuzzy concept for the representation of parametric uncertainty during a classical numerical analysis has been studied extensively. While the problem at the core of the analysis, that is the solution of a set of interval equations, is easily formulated, the actual solution of this problem was proven to be extremely problematic [Moens and Vandepitte (2005)]. Nevertheless, some solution schemes of fundamentally different nature have been developed. This paper gives a short overview of available solution strategies in section 3 and proposes a novel algorithm for accurate and efficient interval and fuzzy numerical analysis in section 4.

Finally, section 5 applies this novel algorithm on a transient dynamic finite element analysis of a cable stayed bridge model and on a harmonic dynamic finite element analysis of a benchmark model.

2 Non-deterministic numerical analysis

This section starts with an unambiguous definition of terminology relating to non-determinism and compares these definitions to other definitions found in literature. The next two subsections give an overview of probabilistic numerical analysis and non-probabilistic numerical analysis. Each subsection introduces the concept and discusses its applicability. In a final subsection, both concepts are compared to each other.

2.1 Terminology

Oberkampf, DeLand, Rutherford, Diegert, and Alvin (1999) classify non-determinism in three categories: variability, uncertainty and error.

Variability is defined as *the variation which is inherent to the modelled physical system or the environment under consideration*. Variabilities vary both between different nominally identical realisations of a design and from time to time and from place to place within the same realisation of the design. Scionti and Lardeur (2006) classify the former as *inter-variability* and the latter as *intra-variability*. Production tolerances, material characteristics and changing environmental conditions are typical causes of variability.

Uncertainty is defined as *a potential deficiency in any phase or activity of the modelling process that is due to a lack of knowledge*. This means that a deterministic value exists for an uncertain model parameter, but that this deterministic value is not known or not sufficiently accurately known. Uncertainties often arise from decisions that are not made yet, from subjective knowledge of experts or from model properties that are difficult to measure (for example damping characteristics of materials or clamping stiffnesses).

An *error* is defined as *a recognisable deficiency in any phase of modelling or simulation that is not due to a lack of knowledge*. The modelling errors introduced by the mathematical description of the physical reality and the errors introduced by the numerical analysis technique, for example discretisation errors, errors due to incomplete convergence and rounding errors are examples of errors that occur when using the finite element method. Human mistakes also introduce errors that are categorised in this group.

Moens (2002) proposes a refinement on the terms *variability* and *uncertainty*, based on the fact that these terms are not mutually exclusive. A variability is characterised by a range of possible values and the likelihood of each variable within this range. However, a variability can be an uncertainty too when no or limited information is available on this range of possible values or on the likelihood of each value. Moens calls this an *uncertain variability*. A variability of which both the range and the likelihood of each value within this range is known is called a *certain variability*.

The same distinction can be made for uncertainties. An uncertainty that by nature has a deterministic value but cannot be reliably modelled as such due to a lack of knowledge is called an *invariable uncertainty*. On the other hand, an uncertain parameter that exhibits variability is called a *variable uncertainty*, although in general the effect of the variability is negligible compared to the effect of the uncertainty.

2.2 Probabilistic non-deterministic numerical analysis

The probabilistic approach is the basis for most commercially available non-deterministic analysis packages. This section describes the basic properties of the probabilistic concept and its applicability for the modelling of variability and un-

certainty. An elaborate discussion of the probabilistic concept is outside the scope of this work, but can be found in the cited references.

2.2.1 Basic properties of the probabilistic concept

In the probabilistic concept, an uncertain quantity is described by a domain of possible values and the frequency of occurrence or the likelihood of each value in this domain. For a probabilistic quantity X, this domain and likelihood are typically described by a probability density function (PDF) $f_X(x)$. The probability that the quantity lies within a given interval $[a,b]$, indicated by $\mathrm{P}[a \leq X \leq b]$ is directly derived from this PDF:

$$\mathrm{P}[a \leq X \leq b] = \int_a^b f_X(x)dx \tag{1}$$

The expected value, mean value or average of the distribution $f_X(x)$, denoted as m_X is given by

$$m_X = \mathrm{E}[X] = \int_{-\infty}^{\infty} x f_X(x)dx. \tag{2}$$

In general, the expected value of a function $g(X)$ with respect to $f_X(x)$ is defined as

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)dx. \tag{3}$$

Most other often used properties of the probabilistic quantity are described by or derived from the PDF's central moments. The n^{th} central moment, denoted as m_n, is defined as

$$m_n = \int_{-\infty}^{\infty} (x - \mathrm{E}[X])^n f_X(x)dx \tag{4}$$

The second central moment or variance of the distribution, denoted as $\mathrm{Var}[X]$, is the most commonly used. The standard deviation, denoted as σ_X, defined as

$$\sigma_X = \sqrt{\mathrm{Var}[X]}, \tag{5}$$

is a commonly used measure for the scatter of the distribution around the expected value.

For multiple probabilistic quantities, the PDF concept is extended to more dimensions, which gives the joint probability density function (JPDF) $f_{X_1,\ldots,X_n}(x_1,\ldots,x_n)$. The expectation $\mathrm{E}[X_i]$ and the variance $\mathrm{Var}[X_i]$ are defined analogous to the univariate case. The first order joint central moment or covariance, denoted as $\mathrm{Cov}[X_i,X_j]$ gives a measure for the interdependence between the quantities and is defined as

$$\mathrm{Cov}[X_i,X_j] = \mathrm{E}\left[(X_i - m_{X_i})(X_j - m_{X_j})\right]. \tag{6}$$

The variances and covariances are often represented by the covariance matrix

$$\Gamma = \begin{bmatrix} \mathrm{Var}[X_1] & \mathrm{Cov}[X_1,X_2] & \dots & \mathrm{Cov}[X_1,X_n] \\ \mathrm{Cov}[X_2,X_1] & \mathrm{Var}[X_2] & \dots & \mathrm{Cov}[X_2,X_n] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}[X_n,X_1] & \mathrm{Cov}[X_n,X_2] & \dots & \mathrm{Var}[X_n] \end{bmatrix}. \tag{7}$$

Haldar and Mahadevan (2000) and Miller and Freund (1985) give an elaborate overview of the probabilistic framework and discuss a variety of probability distribution functions and their applicability for the description of random quantities.

Most non-deterministic numerical analysis software is based on the probabilistic concept and uses the Monte Carlo method. Rubinstein (1981) and Schuëller, Pradlwarter, and Koutsourelakis (2004) give in-depth discussions of this method. A lot of research was and still is focused on the increase of its efficiency. Schuëller (2001) gives an overview of these recent advances. The efficiency of the Monte Carlo method can sometimes be increased by other techniques, for example adaptive mesh refinements in finite element applications [Manjuprasad and Manohar (2007)].

2.2.2 Applicability of the probabilistic concept

Most available procedures and packages for non-deterministic numerical analysis are based on the probabilistic concept. In these procedures, every non-deterministic property is modelled as a probabilistic quantity. For the study of the applicability of this probabilistic model, distinction between certain variabilities, uncertain variabilities and invariable uncertainties is necessary.

The probabilistic model is perfectly suited to model certain variabilities: the available range of possible values and likelihood of each value within this range translates unambiguously to a probability density function. It is however important that all information is available to acquire reliable results. For example, if more than one non-deterministic parameter is present, the joint probability density function describing the likelihood and interdependence of all non-deterministic model properties is required. If this information is not available, the accuracy and reliability of the results is limited.

Uncertain variabilities or variable uncertainties can be modelled using the probabilistic concept by performing the analysis with different probabilistic descriptions of the uncertain variability, each consistent with the limited available information. By comparing the results of these analyses, the analyst can estimate the influence of the uncertainty. Although in theory all possible PDFs should be taken into consideration, the analyst will select only a few probabilistic models which he considers

most appropriate to obtain as much information as possible.

The probabilistic approach is least suited to model invariable uncertainties. The analyst models an invariable uncertainty using a probability density function that to his knowledge represents best the uncertain nature of the property, but this description is not based on objective information. Teichert (1998) indicates the difference in the use of the probabilistic model for variabilities on one hand and for invariable uncertainties on the other hand: in the former case, the probability density function represents variability defined as a variation from unit to unit or from time to time for the final product while in the latter case, the probability density function does not represent variability in the physical structure and may not be interpreted as such in an analysis. If the probabilistic model is used for both variabilities and uncertainties, it is imperative to distinguish between both interpretations and treat them differently in the numerical procedure as described by Hoffman and Hammonds (1994). Gao, Song, and Tin-Loi (2009) propose such a method for reliability analysis of structures with both probabilistic and interval properties.

2.3 Non-probabilistic non-deterministic numerical analysis

This section describes the two most frequently used non-probabilistic concepts (the interval concept and the fuzzy set concept) and their applicability.

2.3.1 The interval concept

Although the first use of the interval concept goes all the way back to Archimedes who bounded the irrational number π by the interval $3\frac{10}{71} < \pi < 3\frac{1}{7}$, most recent developments in interval arithmetic are based on the work of Moore (1966) who introduces intervals, vectors and matrices and develops an interval calculus.

An *interval* or *interval scalar* is a convex subset of the domain of real numbers $\mathbb{R}$. By definition, the range of the interval x^{I} is bounded by its lower bound $\underline{x}$ and its upper bound $\overline{x}$. An interval is called *closed* if both the lower and the upper bound are a member of the interval, that is

$$x^{\mathrm{I}} = [\underline{x}, \overline{x}] = \{x \in \mathbb{R} \mid \underline{x} \leq x \leq \overline{x}\}. \tag{8}$$

The domain of closed intervals is denoted $\mathbb{IR}$.

An *interval vector* is a vector of which each element is an interval, that is

$$z^{\mathrm{I}} = \begin{Bmatrix} z_1^{\mathrm{I}} \\ z_2^{\mathrm{I}} \\ \vdots \\ z_n^{\mathrm{I}} \end{Bmatrix} = \left\{ z \in \mathbb{R}^n \mid z_i \in z_i^{\mathrm{I}} \right\}. \tag{9}$$

The class of closed interval vectors of size n is denoted $\mathbb{IR}^n$.

Similarly, an *interval matrix* is a matrix of which each element is an interval, that is

$$A^{\mathrm{I}} = \begin{bmatrix} a_{11}^{\mathrm{I}} & a_{12}^{\mathrm{I}} & \dots & a_{1n}^{\mathrm{I}} \\ a_{21}^{\mathrm{I}} & a_{22}^{\mathrm{I}} & \dots & a_{2n}^{\mathrm{I}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}^{\mathrm{I}} & a_{m2}^{\mathrm{I}} & \dots & a_{mn}^{\mathrm{I}} \end{bmatrix} = \left\{ A \in \mathbb{R}^{m \times n} \mid a_{ij} \in a_{ij}^{\mathrm{I}} \right\}. \tag{10}$$

The class of closed interval matrices of size $m \times n$ is denoted $\mathbb{IR}^{m \times n}$.

By definition, the entries of interval vectors and interval matrices are assumed to be independent. Thus, an n-dimensional interval vector describes a hypercube in an n-dimensional space. The vertices of this hypercube are determined by the lower and upper bounds of the vector's elements.

In case the elements of an interval vector or an interval matrix cannot be considered mutually independent, the *convex modelling* approach as described by Ben-Haim and Elishakoff (1990) can be considered. Using this approach, the range of the interval vector or interval matrix is not limited to a hypercube, but any convex region in the multidimensional space is allowed. In practice however, the description of the parameter region is often limited to an elliptical region.

Although the concept of convex modelling is intuitive and sounds attractive, the dependency between the elements of non-deterministic interval vectors and matrices is difficult to handle in a numerical analysis and the data describing this interdependence is generally not available. As a result, only a few applications of convex modelling are described in literature.

2.3.2 Applicability of the interval concept

For the study of the applicability of the interval model, distinction between certain variabilities, uncertain variabilities and invariable uncertainties is necessary.

Certain variabilities are generally described by a probability density function, that is, a range of possible values and a likelihood of each value within this range. The interval concept requires only a range of possible values. Consequently, a conversion from a probabilistic description to an interval description is always possible: the range of the PDF forms the interval. The likelihood of each value within this range is lost in the interval concept. In some cases, variabilities are modelled using an unbounded PDF, for example a Gaussian distribution. In these cases, applying the mentioned modelling rule would require the interval to range over the entire domain of real numbers because the PDF defines these values to be possible. However, the probability of the values located in the tails of the commonly applied

unbounded PDFs is immeasurably low. Therefore, the analyst should use bounds which he thinks are realistic with respect to the PDF. Often, the 3σ-bounds are assumed to be realistic interval bounds. It is clear that this conversion introduces subjectivity into the uncertainty description since the analyst deliberately chooses which information contained in the PDF is used and which is discarded.

If for an uncertain variability the range is known, but no information on the likelihood is available, every PDF over this range becomes equally plausible and should be considered. However, the interval concept does not require information on the likelihood, which makes it perfectly suited to model this kind of non-determinism. Consequently, an interval can be interpreted as a collective description of all possible probability density functions over the considered interval.

For uncertain variabilities without objective information on the actual range and for invariable uncertainties, a subjective interval has to be chosen in order to apply the interval concept. When applying the interval concept to describe a lack of knowledge, the interval quantities represent the values the analyst considers possible at the time the analysis is performed and with the information and knowledge available at that time. When the uncertainty is used to model an open design decision, it will be changed based on optimisation of personal preference of the designer in a later stage of the design. When the uncertainty is used to describe a complex property of a model, the exact value inside the interval can remain unknown even after finishing the design.

2.3.3 The fuzzy set concept

The fuzzy set concept was first introduced by Zadeh (1965) as a scientific tool to express often vague linguistic information. Mainly Dubois and Prade (1980, 1988) contributed to the application of the fuzzy set concept in numerical analysis.

A *fuzzy set* can be interpreted as an extension of a conventional crisp set. Where a crisp set clearly distinguishes between members and non-members of the set, a fuzzy set introduces a gradual, soft transition from members to non-members by introducing a membership level. The *membership function* $\mu_{\tilde{x}}(x)$ describes the membership level of each element x in the domain X to the fuzzy set $\tilde{x}$:

$$\tilde{x} = \{(x, \mu_{\tilde{x}}(x)) \mid x \in X \wedge \mu_{\tilde{x}}(x) \in [0,1]\}. \tag{11}$$

If $\mu_{\tilde{x}}(x) = 1$, x is definitely a member of the fuzzy set $\tilde{x}$. If $\mu_{\tilde{x}}(x) = 0$, x is definitely not a member of the fuzzy set $\tilde{x}$. If $0 < \mu_{\tilde{x}}(x) < 1$, the membership is uncertain.

A *normal fuzzy number* is a fuzzy set in $\mathbb{IR}$ with at least one point where the membership function is equal to one and for which the membership function is strictly increasing and decreasing to the left respectively right of that point. The most frequently applied membership functions are the triangular and the Gaussian shape

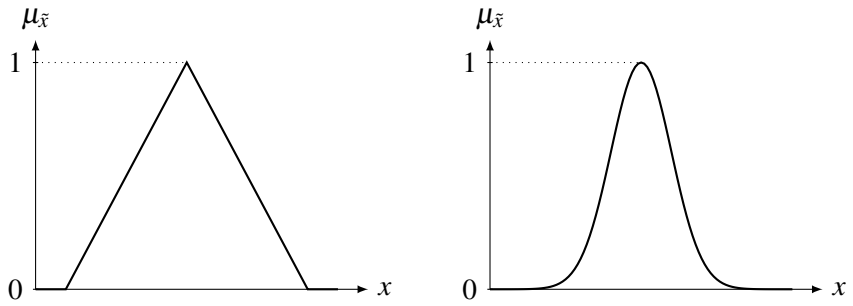


Figure 1: Triangular (left) and Gaussian (right) fuzzy membership functions.

functions, illustrated in Fig. 1. A normal fuzzy number with a triangular shape function, support (the interval for which $\mu_{\tilde{x}}(x) > 0$) (a,b) and top (the value for which $\mu_{\tilde{x}}(x) = 1$) c is often denoted $(a/c/b)$.

Zadeh's extension principle provides a general definition to calculate the fuzzy output $\tilde{y}$ of a crisp function $f(x_1,x_2,\ldots,x_n)$ applied to n fuzzy inputs $\tilde{x_1},\tilde{x_2},\ldots,\tilde{x_n}$. It defines the membership function $\mu_{\tilde{y}}(y)$ as

$$\begin{cases} \mu_{\tilde{y}}(y) = \sup\limits_{\substack{x_1,x_2,\ldots,x_n \\ y=f(x_1,x_2,\ldots,x_n)}} \left(\min\left(\mu_{\tilde{x_1}}(x_1),\mu_{\tilde{x_2}}(x_2),\ldots,\mu_{\tilde{x_n}}(x_n)\right)\right) \\ \mu_{\tilde{y}}(y) = 0 \qquad \text{if } f^{-1}(\{y\}) = \emptyset. \end{cases} \tag{12}$$

This definition implies that the membership value of a fuzzy result $\tilde{y}$ for a specific value y is the largest among the membership values of all realisations of y, that is, all combinations of input parameters $(x_1,x_2,\ldots,x_n)$ that result in y. The possibilistic interpretation of the extension principle is that if a value y has more than one realisation, it will adopt the degree of possibility from the realisation with the highest degree of possibility.

The definition (12) of the extension principle is not readily implementable as for each value y of the output domain, it requires the complete set of realisations to derive the membership value. In general, this is an extremely difficult and computationally expensive procedure.

An alternative approach consists in searching the output domain for sets which have an equal membership level. This is achieved by analysing the input domain at a specific membership level α. At this membership level, the α-*cuts* of the input quantities are defined as

$$x_{i,\alpha}^{\mathrm{I}} = \{x_i \in X_i \mid \mu_{\tilde{x_i}}(x_i) \geq \alpha\} \tag{13}$$

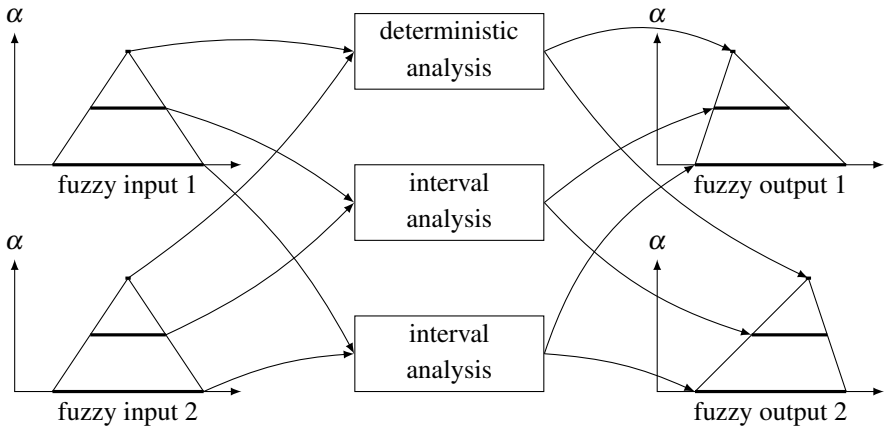


Figure 2: α-cut procedure applied at three membership levels on a function with two inputs and two outputs.

This means that an α-cut is the interval resulting from intersecting the membership function at $\mu_{\tilde{x}_i}(x_i) = \alpha$. After deriving the α-cuts of all input quantities at a specific membership level, an interval analysis is performed on these intervals. The resulting output intervals have two important properties:

- The output intervals result from combining all possible input quantities with a value larger than or equal to α. Following the extension principle, all output quantities have a membership level larger than or equal to α.

- Values outside the output intervals can not be obtained by combinations of input quantities with membership levels larger than or equal to α. Therefore, all values outside the output intervals have membership levels smaller than α.

These two properties state that the obtained output intervals are intersections of the output membership functions at membership level α and consequently an α-cut of the output.

From a practical point of view, this means that a discretised approximation of the output membership functions can be obtained by repeating the α-cut procedure at a number of membership levels. Consequently, a fuzzy numerical analysis can be replaced by a sequence of interval numerical analyses. Fig. 2 illustrates this procedure on a function with 2 fuzzy inputs and 2 fuzzy outputs.

2.3.4 Applicability of the fuzzy set concept

Zadeh (1978) extended the theory of fuzzy sets to a basis for reasoning with possibility. Wang and Klir (1992) developed the theory of fuzzy measures based on this possibilistic interpretation. From this perspective, the membership function is interpreted as a possibility distribution function. The possibility is defined as a subjective measure that expresses the degree to which the analyst considers that an event can occur. As such, it allows to define intermediate possibilities between strictly impossible and strictly possible.

For certain variabilities, Civanalar and Trussell (1986) propose a method to derive a possibility density function corresponding to a given probability density function. The consistency principle states that the degree of possibility of an event is always greater than or equal to its degree of probability. This principle implies that

$$\int_B f_X(x)dx \leq \max_{x\in B}\left(\frac{\mu_{\tilde{x}}(x)}{\max \mu_{\tilde{x}}(x)}\right), \tag{14}$$

where $f_X(x)$ is the probability density function for any set B in the feasible domain. Even for a known probability density function, an infinite number of compatible possibility density functions exists. Therefore, the conversion from a probabilistic density function to a possibilistic density function always relies on some sort of subjective judgement. Consequently, possibilistic analysis results should always be interpreted in a subjective sense.

The choice of the possibility density function for uncertain variabilities and invariable uncertainties is subjective and can only be based on expert knowledge and personal preference. This makes the fuzzy concept a tool for an analyst who wants to study the effect of uncertainties of which he has expert knowledge that is difficult or impossible to express using other modelling techniques.

Alternatively, based on the application of the α-level strategy, a fuzzy analysis can also be interpreted as a large scale sensitivity analysis of the interval results with respect to the input interval widths.

2.4 Comparison between the probabilistic and the non-probabilistic concept

The literature comparing the probabilistic and the non-probabilistic concept and discussing the applicability of both concepts is extensive. Elishakoff (1998), Elishakoff and Zingales (2003), Moens and Vandepitte (2005, 2006) and Zingales and Elishakoff (2000) give an in-depth comparison of both concepts and discuss their applications. When the non-probabilistic methods were developed in the early nineties, both concepts were seen as competing. By now, most authors agree that both methods are quite complementary. Each of the concepts has advantages and disadvantages and is better suited or less suited to model certain types

of non-determinism, but none of the discussed methods offers a solution for all non-deterministic analysis problems.

2.4.1 Based on the type of non-determinism

Sections 2.2.2, 2.3.2 and 2.3.4 discuss the applicability of the considered concepts for the distinguished types of non-determinism.

Certain variabilities are best modelled using the probabilistic concept, as the available range of possible values and likelihood of each value within this range translates unambiguously to a probability density function. All available objective information is used in the non-deterministic model. The interval and fuzzy set concept are less suited to model certain variabilities: the interval concept discards the available information about the likelihood distribution, while the fuzzy set concept replaces the objective probability density function by a subjective possibility density function.

Uncertain variabilities with a known range and an unknown likelihood distribution are best modelled using intervals, because the interval concept does not need the unknown likelihood distribution. Uncertain variabilities with an unknown range but a known likelihood distribution are generally best modelled using the probabilistic concept, since only in this case the available information about the likelihood distribution is included.

Invariable uncertainties are best modelled using the interval or fuzzy set approach. The interval approach requires the least information and consequently leaves the least room for misinterpretation as less subjective information is included. The fuzzy set approach is a valuable tool to study the large scale sensitivity of the results to the input interval widths. In both cases, it is important to be aware that the results obtained using subjective information should be interpreted as such. The probabilistic approach is less suited to model invariable uncertainties since the results can not be interpreted in a frequentist manner due to the use of subjective information. Usually, the limited objective value of the results does not justify the high computational cost that inevitable accompanies a probabilistic analysis.

2.4.2 Based on the design stage

The non-deterministic concept to use can also be guided by the design stage Moens and Vandepitte (2006).

In an initial design stage, when different concepts are compared or when the most appropriate concept is chosen, many design decisions are still open. Some properties have been defined but for some other only expert knowledge (or expert guesses) are available. Most if not all non-determinism is caused by uncertainty rather than

by variability, which makes the non-probabilistic approach better suited than the probabilistic approach. The fuzzy set concept is especially useful, since it handles the linguistic expert knowledge in a mathematically consistent manner and since it allows to study the effect of the interval widths.

As the design evolves, more and more decisions are made and more information is available, the uncertainty decreases. As tolerances are defined, the range of the uncertain parameters is known. Since the details of the actual production process are often still unknown, the likelihood distribution within this range is not yet known. In this stage, mainly interval methods are suited for non-deterministic analysis.

In the final design stages, when the design is finished and decisions about the production processes are made, most numerical analyses are validation and reliability analyses. Variability is the most important source of non-determinism. Consequently, the probabilistic approach is best suited for non-deterministic analyses in this stage.

From this overview, it is clear that the non-deterministic concepts complement each other. Of course, the transition from uncertainty to variability is gradual. The choice of the modelling method depends on the details of the actual design and the personal preference and experience of the analyst too.

3 Implementation strategies for interval and fuzzy numerical analysis

The introduction of the interval and fuzzy set concept in numerical analysis has led to the development of a number of interval and fuzzy numerical analysis procedures. Most of the developed methods are based on the interval concept. Using the α-cut strategy, these methods can be applied to the fuzzy set concept too. Some methods are especially designed for the solution of fuzzy numerical analysis problems.

Consider a deterministic numerical analysis $y = f(x)$, subject to multiple inputs and resulting in multiple outputs. Using the interval concept, the deterministic input vector x is replaced by an interval input vector x^{I}. In this case, the *exact solution set* is defined as

$$\langle y \rangle = \left\{ y \mid y = f(x) \wedge x \in x^{\mathrm{I}} \right\}, \tag{15}$$

that is, the set $\langle y \rangle$ which contains all vectors y which result from applying the deterministic numerical analysis to all vectors x contained in the interval input vector x^{I}. In general, the elements of the output vector y are interdependent, as they are related through the deterministic numerical model and analysis and the non-deterministic inputs. Therefore, the solution set can adopt any shape in the multidimensional output space. In practice, the calculation of this exact solution set is extremely

difficult. A closed form of this solution set can only be calculated if the deterministic numerical analysis function f is described by an explicit analytical expression, which is in general not the case.

Therefore, most research focused on the calculation of a *conservative hypercubic approximation* of the exact solution set, thus ignoring the interdependency between the output quantities. This results in a separate interval range for each individual output quantity. This conservative hypercubic approximation contains solutions that are not part of the exact solution. Consequently, these points do not correspond to the solution of a physical system. To reduce the number of non-physical solutions, interval numerical analysis methods aim for the smallest conservative hypercubic approximation.

Several implementation strategies for interval numerical analysis have been proposed. Based on their solution strategy, they can be divided in two categories. The *interval arithmetic* based solution strategies approximate the smallest hypercubic approximation of the exact solution set from the outside, using interval arithmetic or related methodologies. The *global optimisation* based solution strategies on the other hand approximate the smallest hypercubic approximation of the exact solution set from the inside, using optimisation techniques. The next sections discuss both categories in more detail. An even more extensive overview of available methods, albeit somewhat focused on the finite element method, can be found in [Moens and Vandepitte (2006)] and [Moens, De Munck, Farkas, De Gersem, Desmet, and Vandepitte (2008)]. For some applications, a combination of the interval arithmetic strategy and the global optimisation strategy can be beneficial. Moens, De Munck, and Vandepitte (2007) propose such a method for envelope frequency response function analysis.

3.1 Interval arithmetic based solution strategies

Interval arithmetic is the calculus that defines the addition, subtraction, multiplication and division of intervals. For two intervals x^{I} and y^{I}, these operations are defined as follows:

$$x^{\mathrm{I}} + y^{\mathrm{I}} = \left[\underline{x} + \underline{y}, \overline{x} + \overline{y}\right] \tag{16}$$

$$x^{\mathrm{I}} - y^{\mathrm{I}} = \left[\underline{x} - \overline{y}, \overline{x} - \underline{y}\right] \tag{17}$$

$$x^{\mathrm{I}} \cdot y^{\mathrm{I}} = \left[\min\left(\underline{xy}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{xy}\right), \max\left(\underline{xy}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{xy}\right)\right] \tag{18}$$

$$x^{\mathrm{I}} / y^{\mathrm{I}} = \begin{cases} x^{\mathrm{I}} \cdot \left[\frac{1}{\overline{y}}, \frac{1}{\underline{y}}\right] & \text{if } 0 \notin y^{\mathrm{I}} \\ \text{undefined} & \text{if } 0 \in y^{\mathrm{I}} \end{cases} \tag{19}$$

The main drawback of interval arithmetic is that it introduces conservatism, that

is an overestimation of the width of the output interval, because the dependencies between the intervals are neglected.

The interval arithmetic approach to the solution of an interval numerical analysis method consists of the direct translation of the deterministic numerical analysis procedure to an equivalent interval arithmetic procedure. Each algebraic operation of the deterministic procedure is replaced by its interval counterpart. Thus, the interval arithmetic approach is founded entirely on the interval calculus described above.

As the interval arithmetic approach is founded on the interval calculus described above, it suffers from the problem illustrated above. Each mathematical operation in which the dependency between uncertain parameters, whether direct or indirect through intermediate results, is neglected, causes conservatism on the result. This overestimation of the exact range accumulates through the interval procedure, causing a huge amount of conservatism on the final result of the interval arithmetic numerical analysis. This can only be avoided by taking into account the dependency between the non-deterministic parameters. For the practical application of interval arithmetic approaches in numerical analysis however, it is nearly impossible to track and manage the various parameter dependencies throughout the entire procedure. Consequently, the interval results are severely overestimated and from a practical point of view useless for anything but simple academic examples.

From a numerical point of view however, the interval arithmetic approach is appealing, as the interval arithmetic approach is computationally very efficient. As a result, a substantial research effort is dedicated to the further development of the interval arithmetic approach for the solution of numerical analysis problems, particularly of finite element problems. This research mainly focuses on the parameter dependency problem. On the one hand, it is attempted to limit the conservatism in the interval matrix assembly phase, for example by using the element-by-element approach proposed by Muhanna and Mullen (2001). On the other hand, the solution of the resulting set of interval equations gets quite some attention, for example of Vroman, Deschrijver, and Kerre (2007), who developed a parametric method for the solution of a set of interval equations that contains an explicit dependency of the interval coefficients of the system. So far, the application of these methods is limited to small systems with a limited number of non-deterministic parameters due to its high computational cost.

Manson (2005) proposes affine arithmetic, a more versatile extension of interval arithmetic that keeps track of the dependencies between operands and sub-formulae as a solution to reduce the conservatism of the classical interval arithmetic approach.

3.2 Global optimisation based solution strategies

Global optimisation based solution strategies actively search in the non-deterministic input interval space for the combination that results in the minimum or maximum value of an output quantity [McWilliam (2001)]. As such, the global optimisation approach can be interpreted as a search for the worst case scenario, hence the alternative name *anti-optimisation* approach. All outputs of the numerical analysis are considered independently. For each output quantity of interest, a separate global minimisation and global maximisation are performed. In mathematical form, the result vector y^{I} is described as

$$y^{\mathrm{I}} = \begin{Bmatrix} y_1^{\mathrm{I}} \\ y_2^{\mathrm{I}} \\ \vdots \\ y_n^{\mathrm{I}} \end{Bmatrix}, \tag{20}$$

where

$$\underline{y}_i = \min_{x \in x^{\mathrm{I}}} f_i(x), \tag{21}$$

$$\overline{y}_i = \max_{x \in x^{\mathrm{I}}} f_i(x), \tag{22}$$

x^{I} is the vector defining the input interval space and $f_i(x)$ is the ith output of the deterministic numerical analysis. Note that each function evaluation required by the optimisation procedure requires the solution of a deterministic numerical analysis.

The global optimisation approach results in the exact interval. Because the interdependence between the non-deterministic parameters is implicitly taken into account by performing deterministic numerical analyses for a given set of deterministic values for the non-deterministic parameters, the global optimisation approach does not suffer from conservatism. On the other hand, the number of global optimisations required to solve a full interval or fuzzy numerical analysis problem and the possibly computationally expensive objective function make the global optimisation based approach computationally expensive. Furthermore, the behaviour of the objective function with respect to the non-deterministic parameters can generally not be predicted. To be applicable to all interval and fuzzy numerical analysis problems, the global optimisation procedure should consider the deterministic numerical analysis as a black box function. This eliminates the use of dedicated optimisation procedures such as, for example, convex optimisation. Fortunately, most outputs resulting from real life engineering problems tend to exhibit a relatively smooth behaviour with respect to typical non-deterministic model parameters [De Gersem, Moens, Desmet, and Vandepitte (2005)]. This smooth behaviour facilitates the

determination of the extrema. Because global optimisation based strategies yield physically correct results, unlike interval arithmetic based strategies, they are more and more acknowledged as the standard approach to solve interval numerical analysis problems.

Despite the smooth behaviour of typical objective functions, the computational cost of the global optimisation based approach remains high. Hence, most research on this method focuses on the reduction of its computational cost. In this domain, two different approaches can be distinguished: direct optimisation and response surface based optimisation.

Direct optimisation approaches. Several general black box optimisation algorithms can be applied to interval and fuzzy numerical analysis. Rao [Rao and Sawyer (1995); Rao and Berke (1997); Rao and Chen (1998)], one of the pioneers in fuzzy finite element modelling, uses a directional search based algorithm to solve the optimisation problem. Several authors [Biondini, Bontempi, and Malerba (2004); Catallo (2004); Köylüoğlu and Elishakoff (1998); Möller, Graf, and Beer (2000)] use various other general purpose optimisation algorithms.

Recently, Degrauwe (2007) introduced the innovative *gradual-α-decreasing* or *GαD algorithm* in the context of fuzzy finite element analysis. It is proposed as an efficient method to solve the interval problems at different levels in the fuzzy solution procedure. The approach exploits the high similarity between the optimisation problems that need to be solved, as well as the smoothness that is generally observed in the behaviour of typical FE output quantities with respect to varying physical parameters. The approach is essentially an iterative search for global optima in a search space, the size of which is gradually increasing when the membership level is decreasing. By keeping track of the optima reached in previous optimisations on subspaces of the current iteration, the method gradually builds search paths along which the local optima of the objective function are situated. The search for the global optimum can then be limited to the observation of the intersection of these search paths with the bounding box at the analysed level.

The method starts from the crisp solution at the midpoint of the interval parameter space. This point is also the starting point for all search paths. In an iterative procedure, the interval problem is solved at decreasing levels, using a local optimisation approach. If no local optima are detected, the vertex points at the corresponding level represent the positions of the extreme objective function values inside the domain, and are added to the search paths. If at a certain level, a local optimum is detected on the vertex, i.e. through a sign switch of the local derivative at that point, a new search path is initiated. This path will follow the location of this local optimum throughout the next steps of the procedure where the search domain is

increasing. This is achieved by performing an additional local optimisation step, with starting values chosen at the local optimum of the previous iteration.

Response surface optimisation approaches. Response surface based optimisation approaches consist of two steps. In the first step, they create approximations of the objective functions, based on function evaluations in well-chosen points. In the second step, these approximations or response surfaces are used to find the optimum of the objective function. In engineering applications, especially in the context of modelling the behaviour of a physical system based on measurements, this approach is also known as design of experiments (DOE). Some authors limit the term DOE to linear regression based response surfaces.

The *vertex method* [Dong and Shah (1987)] is in fact the simplest form of response surface, as it performs a linear interpolation between the response points obtained at the vertices of the search domain. This approach needs 2^n deterministic function evaluations to evaluate an interval function with n interval inputs. It has long been very popular for the implementation of interval numerical methods [Abdel-Twab and Noor (1999); Chen and Rao (1997); Kulpa, Pownuk, and Skalna (1998); Wasfy and Noor (2000)] and is the basis for the *transformation method* for fuzzy analysis developed by Hanss (2002). The transformation method performs a vertex analysis at each membership level of interest, and thus requires $2^n \cdot l$ deterministic function evaluations to evaluate a fuzzy function with n fuzzy inputs at l membership levels. It is clear that the accuracy of both the vertex method and the transformation method is limited by the few observation points. However, if the behaviour of the objective function with respect to the uncertain parameters can be guaranteed to be monotonic, these approaches yield the exact solution.

Several variants of the transformation method exist. If the objective function is not monotonic, the *extended transformation method* of Hanss (2003), which essentially adds more observation points to the search domain, can be applied. In its basic form, this method requires $3^n \cdot l$ deterministic function evaluations to evaluate a fuzzy function with n fuzzy inputs at l membership levels. Donders, Vandepitte, Van de Peer, and Desmet (2005) proposed the *short transformation method*, which reduces the computational cost of the standard transformation method by only performing a full vertex analysis at the lowest membership level of interest. This variant of the transformation method requires $2^n + 2 \cdot (l-1)$ deterministic function evaluations to evaluate a fuzzy function with n fuzzy inputs at l membership levels.

Local expansion approaches approximate the objective function of an optimisation procedure using a higher-order expansion scheme, generally around the midpoint of the intervals at the analysed level. This approximated objective function is then used to perform the many objective function evaluations generally required to per-

form an interval analysis.

Massa, Tison, and Lallemand (2006) use higher-order Taylor series constructed at the level midpoints to represent the objective function behaviour with respect to the uncertain parameters. In the proposed TEEM (Taylor expansion with extrema management), this approximation is not used to find the exact optimum at each level. It is included in a procedure that searches for extrema in the interval domains at each level by observation of the function evaluations in the vertices as well as the local derivatives in these points, which are used to detect possible non-monotonic influences. The Taylor expansion is introduced in this procedure to speed up the evaluation of the exact objective function and its derivatives in the vertex points. As the number of required function evaluations increases exponentially with the number of uncertainties, the time for a single function evaluation indeed is crucial for the efficiency of the procedure. Later, this method was extended to use Padé approximates (approximates derived by expanding a function as a ratio of two power series and determining both the numerator and denominator coefficients) in the context of modal analysis [Massa, Tison, and Lallemand (2008)].

In general, a *response surface* is a solution of the problem in which one tries to determine a continuous function f based on a limited number of function evaluations $f(x_i)$. In general, these function evaluations do not uniquely identify the continuous function f; multiple response surfaces are consistent with the available data. When using a response surface based optimisation approach, one should first solve two problems: constructing a response surface $\tilde{f}$ based on the available data $f(x_i)$ (model estimation) and estimating the error ε on the response surface (model appraisal) [Queipo, Haftka, Shyy, Goel, Vaidyanathan, and Tucker (2005)]. For application in the framework of interval numerical analysis, the main challenge is to construct accurate response surface approximations of the output quantities of interest, but still limiting the number of response points, as each response point requires a possibly expensive deterministic numerical analysis. A dedicated response point selection procedure designed for optimisation is required to enable the analysis of industrially sized models. This paper focuses on the response surface based approach and proposes a novel algorithm based on Kriging response surfaces.

The application of response surface methods, including the novel algorithm proposed in this paper, is not limited to interval and fuzzy numerical analysis. Panda and Manohar (2008) for example use a response surface method for stochastic reliability analysis and Jiang and Han (2007) uses response surfaces (approximations) for uncertain optimisation.

4 A Kriging response surface based optimisation method

This section introduces a response surface based optimisation algorithm for interval and fuzzy numerical analysis. The algorithm, based on Kriging, takes advantage of the special properties of the optimisation problem. This section starts with an overview of the procedure. Next, the procedure is developed for the optimisation of one objective function. Finally, the procedure is extended to multiple objective functions.

4.1 Overview of the procedure

This algorithm approximates the objective functions by a Kriging response surface, as described in [Matheron (1963); Sacks, Welch, Mitchell, and Wynn (1989); Jones, Schonlau, and Welch (1998); Lophaven, Nielsen, and Sondergaard (2002a,b)]. Kriging response surfaces are especially suited to model deterministic computer codes because of their flexibility to approximate many different and complex functions [Schonlau (1997); Martin and Simpson (2005)] and because they interpolate the response points.

The general idea of this procedure is that to minimise and maximise a function accurately using a response surface, the response surface should approximate the function well in the regions where the minimum and maximum are expected. Approximation uncertainties and errors outside the regions containing the minima and maxima do not affect the optimisation error.

The procedure starts by generating a small space filling design, evaluating the objective functions in these response points and creating the initial Kriging response surfaces based on these function evaluations. In the second step of the algorithm, these response surfaces are iteratively improved by selecting the most promising response points from a large space filling design, evaluating the objective functions only in these few most promising response points and updating the Kriging response surfaces. Thus, the initial response surfaces are only improved in regions where a minimum or a maximum can be expected. This second step is repeated until no more improvement is possible. Finally, the optima of the objective functions are determined using the response surfaces. Because of the low computational cost of a single evaluation of the response surface and because the response surface can have several local minima and maxima, a global optimisation on the response surface is advisable.

4.2 Procedure for the optimisation of a single objective function

4.2.1 Construction of the initial response surface.

In the first step of the procedure, a small space filling design (for example a Latin hypercube design) is generated and the objective function is evaluated at these response points by the deterministic solver. Using this information, the initial response surface is created. Since this response surface will be improved in the second step, one should not use too many response points. The author achieved good results with five times the number of uncertain parameters. Additional points are best selected by the adaptive procedure in the next step instead of being randomly selected in this step. In the authors experience, the procedure is not very sensitive to the regression and correlation models of the Kriging response surfaces. On average, a constant regression term and a Gaussian or generalised exponential correlation function yield excellent results.

Especially when the objective function has more than two or three parameters, the best results are achieved by using a space filling design that fills a ten to twenty percent larger area than the largest area that will be used in the actual optimisation. This reduces extrapolation, which introduces a much larger approximation error than interpolation. When using such an extended initial space filling design, significantly less response points are required to achieve the same accuracy of the final response surface.

4.2.2 Iterative improvement of the response surface.

In the second step of the procedure, a large space filling design is calculated. These points are not yet response points; only the few most promising points from this set will become real response points for which a deterministic numerical analysis will be performed. For each of these candidate response points, the function value and the expected error on the function value are estimated using the constructed response surface and the *maximum improvement* or *MI* is calculated as

$$\mathrm{MI} = \frac{\min\left(\tilde{f}(x)\right) - \left(\tilde{f}(x_{\mathrm{new}}) - \Delta\tilde{f}(x_{\mathrm{new}})\right)}{\min\left(\tilde{f}(x)\right)} \tag{23}$$

for minimisation and

$$\mathrm{MI} = \frac{\left(\tilde{f}(x_{\mathrm{new}}) + \Delta\tilde{f}(x_{\mathrm{new}})\right) - \max\left(\tilde{f}(x)\right)}{\max\left(\tilde{f}(x)\right)} \tag{24}$$

for maximisation. In these formulae, $\tilde{f}$ is an approximation of an output parameter, $\min(\tilde{f}(x))$ is the current minimum value of the approximation, $\tilde{f}(x_{\mathrm{new}})$ is the

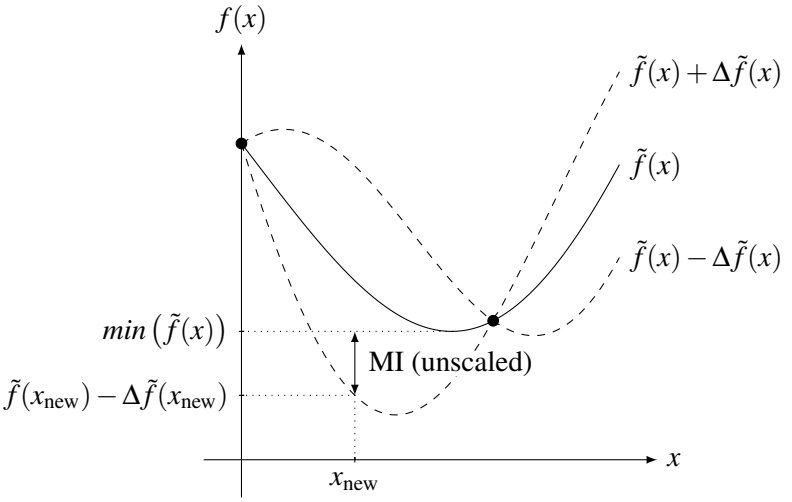


Figure 3: Visualisation of the (unscaled) maximum improvement (MI) in an arbitrary point x_{new} in a minimisation problem. The solid line shows the current approximation of the objective function and the dashed lines show the upper and lower bound defined by the mean squared error of the approximation.

expected value of the approximation in the candidate response point and $\Delta\tilde{f}(x_{\text{new}})$ is the error range on the approximation in this point. The choice of $\Delta\tilde{f}(x_{\text{new}})$ influences the behaviour of the optimiser: a larger value places more emphasis on reducing uncertainty while a smaller value favours the improvement of approximately found minima or maxima. In general, an error range of 3σ, where σ is given by the Kriging response surface, is a good compromise between efficiency and performance. Thus, the maximum improvement gives an indication of the possible improvement in this candidate response point. If an improvement in this candidate response point is unlikely, the maximum improvement will be negative. The maximum improvement is normalised (scaled). Although this scaling is not necessary for the optimisation of a single objective function, it helps monitoring the progress of the response surface improvement procedure.

Fig. 3 illustrates the maximum improvement in an arbitrary point x_{new} in a one-dimensional minimisation problem. Note that the maximum improvement is not scaled, that is, it is defined by the numerator of (23).

The candidate response points which have the highest MI and which satisfy a minimum distance requirement are selected and are added to the response point set. The minimum distance criterion is necessary to prevent the selection of response points

close to each other for which the region of improvement overlaps. The application of the rather crude minimum distance criterion can be avoided by selecting only one new response point in each iteration. In this case, the information given by the new response point is included in the selection procedure for the next response point. In practice, the construction of the Kriging response surfaces is rather expensive, so selecting the new response points in groups and applying the minimum distance criterion gives a better overall performance. The authors achieved good results by adding response points in groups of one to two times the number of uncertain model parameters and a minimum distance of ten to twenty percent of the uncertain parameter range.

Finally, the objective function is evaluated in the new response points and the Kriging response surfaces are updated with the new information. The objective function is only evaluated in these new response points; for all other candidate response points, only cheap response surface evaluations are necessary.

This second step of generating a large set of candidate response points and selecting the most promising points is repeated until a stopping criterion is met. Ideally, one should continue the procedure until no more improvement can be made, that is, until one does not find any more points with a maximum improvement greater than zero. Implicitly, the choice of $\Delta\tilde{f}(x_{\text{new}})$ influences this stopping criterion. Alternatively, this procedure can be used when an optimal approximation of the objective function is needed within a limited computational cost, since it is possible to stop the improvement after a given number of function evaluations and use the response surfaces improved to that point. In addition, it is possible to estimate the error made by prematurely terminating the algorithm by investigating the approximation error estimates. It is not possible to give a general applicable estimate of the total number of response points required since this number is highly dependent on the behaviour of the objective functions. It is clear that more response points are necessary to approximate oscillating functions with several local optima than to approximate smooth monotonous functions.

4.2.3 Optimisation.

Because of the low computational cost of a single evaluation of the response surface and because the response surface can have several local minima and maxima, a global optimisation on the response surface is advisable.

In general, one wants to optimise the function using the expected value given by the Kriging response surface. In some applications, where more conservatism is not a problem, but underestimating the maximum or overestimating the minimum should be avoided, one can choose to optimise the confidence intervals on the approximation instead of the expected value of the approximation. 99.6% confidence

bounds on the result for example are obtained by minimising $\tilde{f} - 3\sigma_{\tilde{f}}$ or maximising $\tilde{f} + 3\sigma_{\tilde{f}}$. Although, due to the nature of response surfaces, it is impossible to absolutely guarantee conservatism, it is possible to increase the confidence on the determined bounds to an arbitrary level.

4.3 Extension to multiple objective functions

As already mentioned before, optimisation based interval and fuzzy numerical analyses generally require a large amount of optimisations. Each of these optimisation procedures is performed using the procedure described above, but two properties of the optimisation problems can be used to significantly reduce the total computational cost compared to the computational cost of individual optimisations:

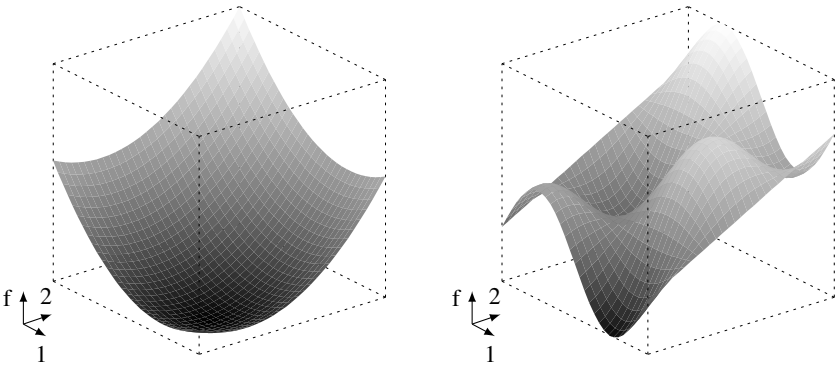
- When a response point is added to a response surface, a deterministic numerical analysis is performed to evaluate that objective function. Since this deterministic numerical analysis evaluates not just this objective function but all objective functions, this response point can be added to the response surface of all objective functions without increasing the computational cost.
- Since the response surface of an objective function at a higher membership level is a subset of the response surface at a lower membership level, only the response surface at the lowest membership level of interest is constructed. All optimisations are done on that response surface; only the bound constraints are different.

With these properties and improvements in mind, the optimisation procedure can be extended efficiently to multiple objective functions.

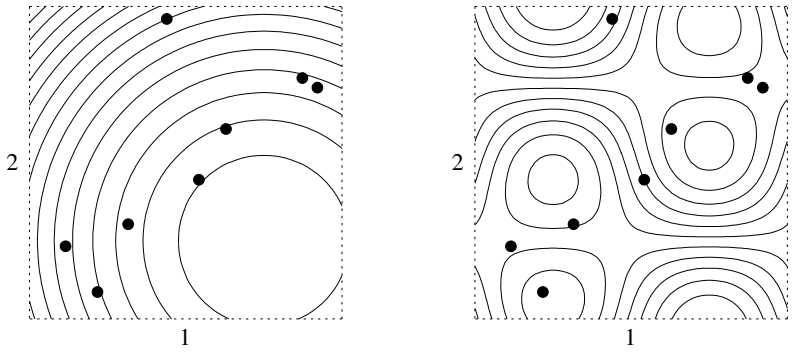
The discussion is guided by the example shown in Fig. 4 in which two objective functions, a convex one and a non-convex one, are simultaneously modelled for minimisation and maximisation. Fig. 4(a) shows surface plots of the two objective functions: the left plot shows the convex objective function while the right plot shows the non-convex objective function.

4.3.1 Construction of the initial response surfaces.

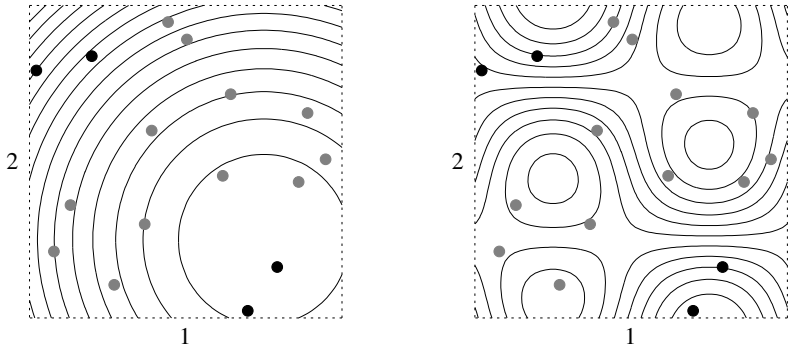
In the single objective function procedure, an initial response surface is constructed based on a small space-filling design. In the multi-objective case, this space filling design is based on the lowest membership level of interest and the response surfaces for all objective functions are based on the same response points. Since the deterministic solver returns all results of interest at once, the computational cost in the multi-objective case is equal to the computational cost in the single-objective case.



(a) Surface plots of the two objective functions.



(b) Initial response points on contour plots of the two objective functions.



(c) Unselected (grey) and selected (black) candidate response points on contour plots of the two objective functions

Figure 4: Illustration of the response point selection process of the response surface based optimisation method described in section 4 on an example with two objective functions and two uncertain parameters.

Fig. 4(b) shows the initial response points on contour plots of the two example objective functions. Note that the same response points are used for both objective functions.

4.3.2 Iterative improvement of the response surfaces.

As in the single-objective case, a large space-filling design of candidate response points is calculated. For each candidate response surface point, the *average maximum improvement* or *AMI* is calculated as

$$\mathrm{AMI} = \sum_k \left(\max \left(\frac{\min\left(\tilde{f}_k(x)\right) - \left(\tilde{f}_k(x_{\mathrm{new}}) - \Delta\tilde{f}_k(x_{\mathrm{new}})\right)}{\min\left(\tilde{f}_k(x)\right)}, 0 \right)^2 \right) \tag{25}$$

for minimisations and

$$\mathrm{AMI} = \sum_k \left(\max \left(\frac{\left(\tilde{f}_k(x_{\mathrm{new}}) + \Delta\tilde{f}_k(x_{\mathrm{new}})\right) - \max\left(\tilde{f}_k(x)\right)}{\max\left(\tilde{f}_k(x)\right)}, 0 \right)^2 \right) \tag{26}$$

for maximisations. This average maximum improvement is a measure for the possible average improvement of the different response surfaces in this candidate response point. The core of the formulae is equal to the maximum improvement of the single-objective case. To acquire the average maximum improvement, the $\max(\cdot, 0)$ operator is added to eliminate negative values and the sum of the squared positive maximum improvements is taken. This formula results in a good balance between large improvements for a single objective function and small improvements for multiple objective functions. As in the single-objective case, it is possible to define an alternative score based on the expected improvement.

As in the single-objective case, the candidate response points which have the highest MI and which satisfy a minimum distance requirement are selected and are added to the response point set. All objective functions are evaluated in these new response points and the Kriging response surfaces are updated with the new information.

Fig. 4(c) shows the unselected candidate response points (grey) and the selected candidate response points which are added to the set of response points (black) on contour plots of the two objective functions. Note again that for both objective functions, the candidate response points and the selected candidate response points are the same. In this simple example, it is clear that the two response points in the upper left corner are selected because they improve both objective functions around their maximum, the upper point in the lower right corner is selected mainly because it improves the approximation of the convex objective function and the lower point

in the lower right corner is selected mainly because it improves the minimum of the non-convex objective function.

In this example, the minima and maxima of both objective functions are located in the same regions. If this is not the case, the procedure is still very efficient, but it is much more difficult to visualise the selection process because the response point selection is highly dependent on the estimated accuracy of the Kriging response surfaces near the local optima. If the response surface of an objective function is relatively accurate around its optima, this objective function will be almost excluded from the selection process.

In fuzzy analyses, all membership levels of interest should be taken into account since a point, which is certainly not a minimum or maximum at the lowest membership level, can be a minimum or a maximum at a higher membership level. This approach yields the fuzzy average maximum improvement

$$\mathrm{AMI} = \sum_{\alpha} \sum_{k} \left(\max \left(\frac{\min\left(\tilde{f}_k(x)\right) - \left(\tilde{f}_k(x_{\mathrm{new}}) - \Delta\tilde{f}_k(x_{\mathrm{new}})\right)}{\min\left(\tilde{f}_k(x)\right)}, 0 \right)^2 \right) \tag{27}$$

for minimisations and

$$\mathrm{AMI} = \sum_{\alpha} \sum_{k} \left(\max \left(\frac{\left(\tilde{f}_k(x_{\mathrm{new}}) + \Delta\tilde{f}_k(x_{\mathrm{new}})\right) - \max\left(\tilde{f}_k(x)\right)}{\max\left(\tilde{f}_k(x)\right)}, 0 \right)^2 \right) \tag{28}$$

for maximisations, where $\sum_{\alpha}$ is the sum over the membership levels of which the point is a member.

Note that in this formula, the AMIs at the different membership levels of interest are simply added; no correction is made to scale the AMI for points that belong to several membership levels of interest. This places a high emphasis on points which could be a minimum or a maximum at several membership levels of interest. In practice, this is beneficial since the accuracy of the approximation in these points is very important for the accuracy of the final result.

4.3.3 Optimisation.

The optimisation step of the Kriging based optimisation method does not introduce new information. The minimisations and maximisations of the different objective functions at the different membership levels of interest can be performed in any order (or even parallel if multiple processors are available).

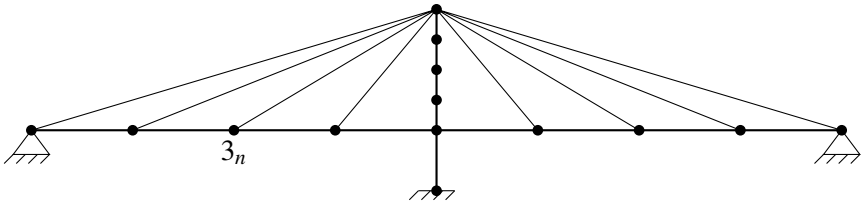


Figure 5: Simplified finite element model of the cable stayed bridge.

5 Applications

5.1 Transient dynamic analysis of a cable stayed bridge model

This section illustrates the applicability of the proposed method on a numerical example.

The considered example is a model of a cable stayed bridge, based on the pedestrian bridge over the Canal du Centre in Obourg (Belgium) [Walther, Houriet, Isler, and Moïa (1988)]. The total span of this bridge is 134 m. The deck slab is made of precast prestressed concrete and has a double T cross-section. The overall width and height of the cross-section are 1.8 m respectively 0.6 m; the web thickness and flange thickness are 0.2 m respectively 0.3 m. The pylon is A-shaped, where each arm has a rectangular cross-section of 0.8 m by 0.6 m. This pylon extends 20 m above the bridge deck and 10 m below it. The deck is supported by stranded steel cables consisting of 37 strands of 12.7 mm each.

The bridge is symmetric along the longitudinal axis. Only one half of the bridge and a single plane of cables are modelled for this analysis. The deck slab and the pylon are modelled using beam elements. The cables are modelled using bar elements. Fig. 5 shows the finite element model of this bridge.

Two uncertain parameters are defined on the model: parameter p_1 defines the deviation of the stiffnesses of concrete and steel from their nominal value and parameter p_2 defines the deviation of the densities of concrete and steel from their nominal value. Both parameters have a maximum deviation of $\pm 5\%$, resulting in fuzzy values $(0.95/1.00/1.05)$. The coupling of the properties of concrete and steel have no physical basis; rather it is convenient to limit the number of uncertain parameters to two to be able to plot the response surfaces to illustrate the behaviour of the different methods.

The bridge is subject to a sudden vertical load of 100 kN for 0.1 s on the bridge deck at node 3, denoted 3_n in the figure. The transient vertical displacement of node 3, resulting from this sudden vertical load, is then computed from 0 s to 2 s in 0.01 s

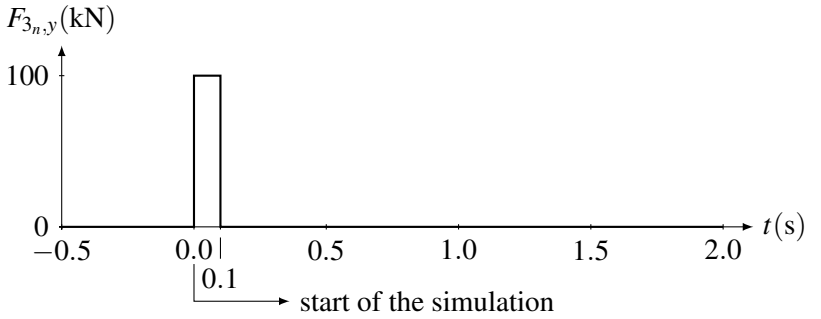


Figure 6: Vertical load on the bridge deck at node 3 as a function of the time.

increments (201 time steps). The deterministic transient vertical displacement is computed using Wilson θ method [Wilson, Farhoomand, and Bathe (1972)], which is an implicit integration method. To compute the displacement at a certain time, it has to be computed at all earlier time steps too. Although it is not necessary to calculate later time steps, the function is always computed at all time steps, to enable the reuse of function evaluations.

The deterministic algorithm is translated to an interval algorithm using the global optimisation based approach. In this approach, the range of the displacement d_3 of node 3 is determined, taking into account that the uncertain parameters p can vary within their intervals p^{I}. This range $\langle d_3 \rangle$ of this displacement is determined by a minimisation and a maximisation over the uncertainty interval p^{I}:

$$\langle d_3 \rangle = \left[\min_{p \in p^{\mathrm{I}}} d_3, \max_{p \in p^{\mathrm{I}}} d_3 \right]. \tag{29}$$

A single interval analysis requires 402 optimisations (a minimisation and a maximisation at each time step). This interval analysis is applied at six membership levels (the support and $0.2, 0.4, \ldots, 1.0$), requiring five interval analyses at the support and at the membership levels $0.2, 0.4, 0.6$ and 0.8 and one deterministic analysis at membership level 1.0. In total, 2010 optimisations and one deterministic analysis are required.

The optimisations are performed using four different optimisation algorithms: MCS [Huyer and Neumaier (1999); Neumaier (2009)] (a MATLAB global optimisation algorithm based on a multilevel coordinate search; note that this algorithm has no relation with Monte Carlo sampling, which is often also abbreviated as MCS), FMINCON [The Mathworks (2008)] (a MATLAB local optimisation algorithm), the Kriging response surface based optimisation method described in section 4 and

a linear regression response surface based optimisation algorithm [De Munck, Moens, Desmet, and Vandepitte (2008)].

To reduce the computational cost of the optimisations using the global optimisation algorithm MCS and the local optimisation algorithm FMINCON, all function evaluations of the objective function are stored in a database to enable the optimiser to reuse them for future optimisations without performing the same finite element analysis again. For the local optimisation algorithm FMINCON, this database is also used to start optimisations from the point with the best function value found so far – the lowest function value for minimisations and the highest function value for maximisations. All optimisations are performed from the highest to the lowest membership level.

The linear regression based optimisation algorithm is applied using a quadratic response surface and using a fourth order response surface. The results of the linear regression based optimisation algorithm using the quadratic response surface are denoted *linreg2* in the result graphs while the results of the linear regression based optimisation algorithm using the fourth order response surface are denoted *linreg4*.

Fig. 7 to 9 show the upper and lower bound on the transient dynamics response of node 3 at the support and at membership levels 0.4 and 0.8, calculated using the different optimisation algorithms. From these graphs, it is clear that the global optimisation method, the local optimisation method and the Kriging based optimisation method yield the same results, except for a small difference in the lower bound around $t = 1.9$ s at the support. The linear regression based method however, is somewhat less accurate until around $t = 1.2$ s and has significant errors from there on. Although the error is smaller for the fourth order response surface than for the quadratic response surface – which is clearly visible from $t = 1.5$ s on – the error is still too large for practical applications.

To examine the cause of the increasing error of the solution computed using the linear regression based optimisation method, Fig. 10(a) shows d_3, the displacement at node 3, as a function of p_1 and p_2, the two uncertain parameters, at $t = 0.25$ s (left) and at $t = 1.85$ s (right). At $t = 0.25$ s, the behaviour of d_3 is very smooth. At $t = 1.85$ s, the behaviour is less smooth, with two distinct peaks. Because the uncertain parameters influence the eigenfrequencies of the bridge model, the distance between the peaks can increase or decrease, depending on the value of the uncertain parameters. In the beginning of the simulation, this causes the response to be a bit more up or down the slope, as illustrated in the left figures at $t = 0.25\ s$. As the time increases, the uncertainty on the eigenfrequencies causes more and more uncertainty on the exact response: the response can be on one peak or on the next peak or in the valley in between. This is clearly illustrated on the right figures at $t = 1.85$ s. If the simulation is extended beyond $t = 2$ s, even more peaks will

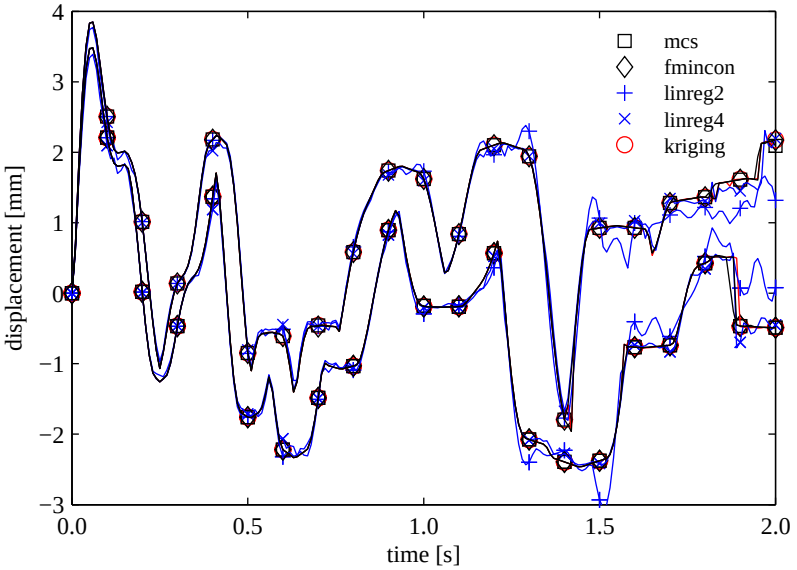


Figure 7: Bounds on the transient response at the support, computed using the discussed algorithms.

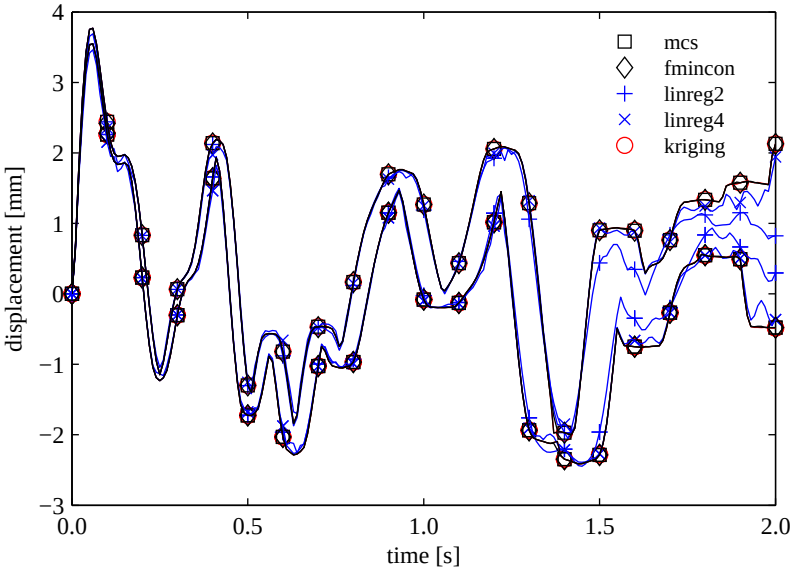


Figure 8: Bounds on the transient response at membership level 0.4, computed using the discussed algorithms.

appear in a single response.

Fig. 10(b) shows the quadratic approximation of the displacement of node 3 at $t = 0.25$ s (left) and at $t = 1.85$ s (right) created and used by the linear regression based optimisation method. The approximated response at $t = 0.25$ s is a good approximation of the real response. However, it is obvious that a quadratic response surface cannot approximate the shape of the real response at $t = 1.85$ s.

To approximate the shape of the response at $t = 1.85$ s, at least a fourth order polynomial is required. Fig. 10(c) shows the fourth order approximations created and used by the linear regression based optimisation method. Because the fourth order polynomial has more parameters than the quadratic polynomial. At $t = 0.25$ s, the quadratic approximation was accurate, and the fourth order approximation does not change much. At $t = 1.85$ s however, the fourth order response surface is a much better approximation, although the difference with the real response is still too large to be neglected.

Fig. 11 shows the response surface created and used by the Kriging based optimisation method. These response surfaces are almost equal to the exact surfaces shown in Fig. 10(a). The contour plots in Fig. 11 show the response points used to create these response surfaces. Note that the same response points are used for all objective functions, not only for the two objective functions illustrated here.

Fig. 14 shows the fuzzy transient response of node 3, assembled from the interval responses at the support and at membership levels $0.2, 0.4, \ldots, 1.0$ computed using the Kriging response surface based optimisation method. The fuzzy responses computed using the global and local optimisation methods are not distinguishable from the fuzzy response computed using the Kriging response surface based optimisation method and are not reproduced here. Fig. 12 and 13 show the fuzzy transient response of node 3, assembled from the interval responses computed using the linear regression based optimisation method using a quadratic respectively fourth order response surface. The same remarks made for the interval responses apply to the fuzzy responses: the accuracy of the linear regression based method is good up to around $t = 1.2$ s, but is problematic from there on.

Until now, only the accuracy of the results is discussed. For practical applications, especially on industrially sized models, the computational cost is also very important. Tab. 1 shows the number of objective function evaluations (finite element analyses) and, for the response surface based methods, the number of approximation function evaluations.

The computational cost of the direct optimisation methods is prohibitively high. Although the database with function values proves to be efficient, reducing the number of objective function evaluations by 75%, the global and local optimisation

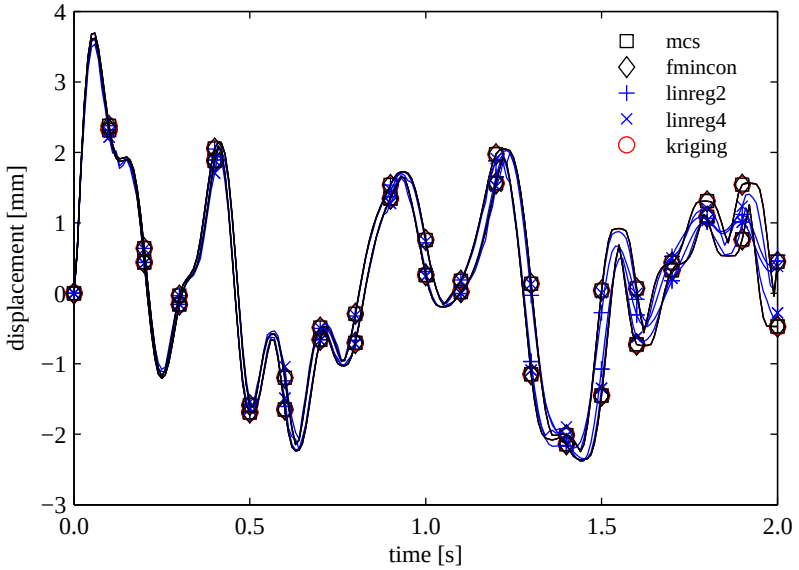
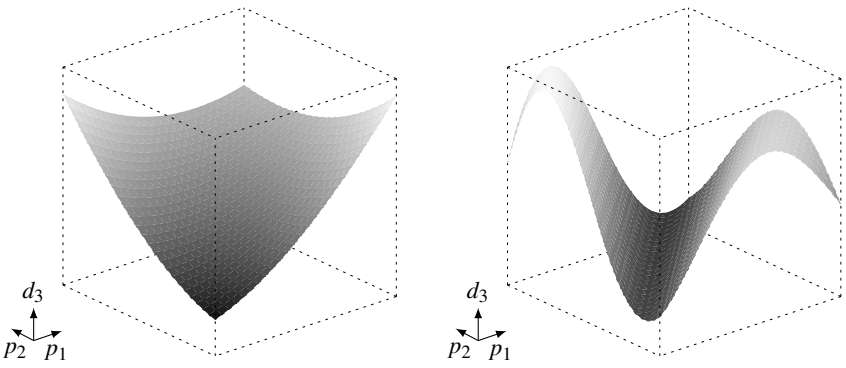


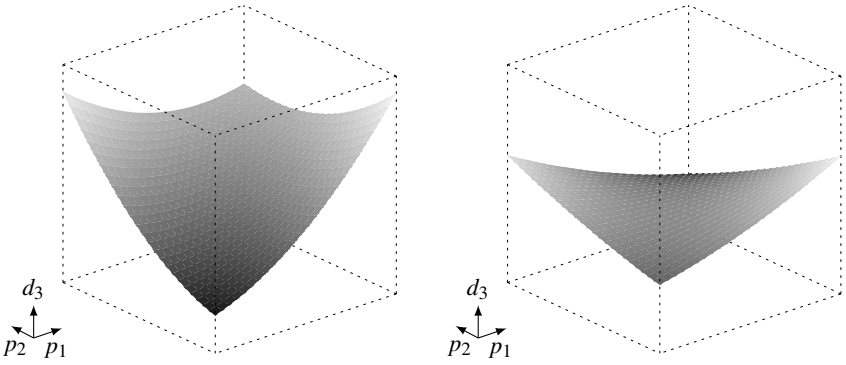
Figure 9: Bounds on the transient response at membership level 0.8, computed using the discussed algorithms.

Table 1: Comparison of the computational cost of the solution of the transient dynamic finite element problem using different optimisation algorithms.

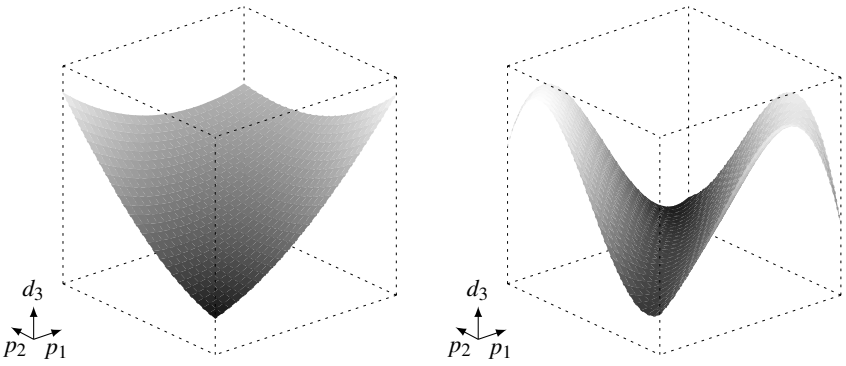
Optimisation algorithm	Function evaluations	
	Goal function	Approximation
MCS		
Without database	207689	
With database	47475	
FMINCON		
Without database	13195	
With database	3363	
Linear regression based algorithm		
2nd order response surface	13	174915
4th order response surface	25	180715
Kriging based algorithm	30	183815



(a) Exact solution.



(b) Second order linear regression response surfaces.



(c) Fourth order linear regression response surfaces.

Figure 10: Surface plots of the displacement at $t = 0.25$s (left) and at $t = 1.85$s (right) as a function of the uncertain model parameters $p1$ and $p2$.

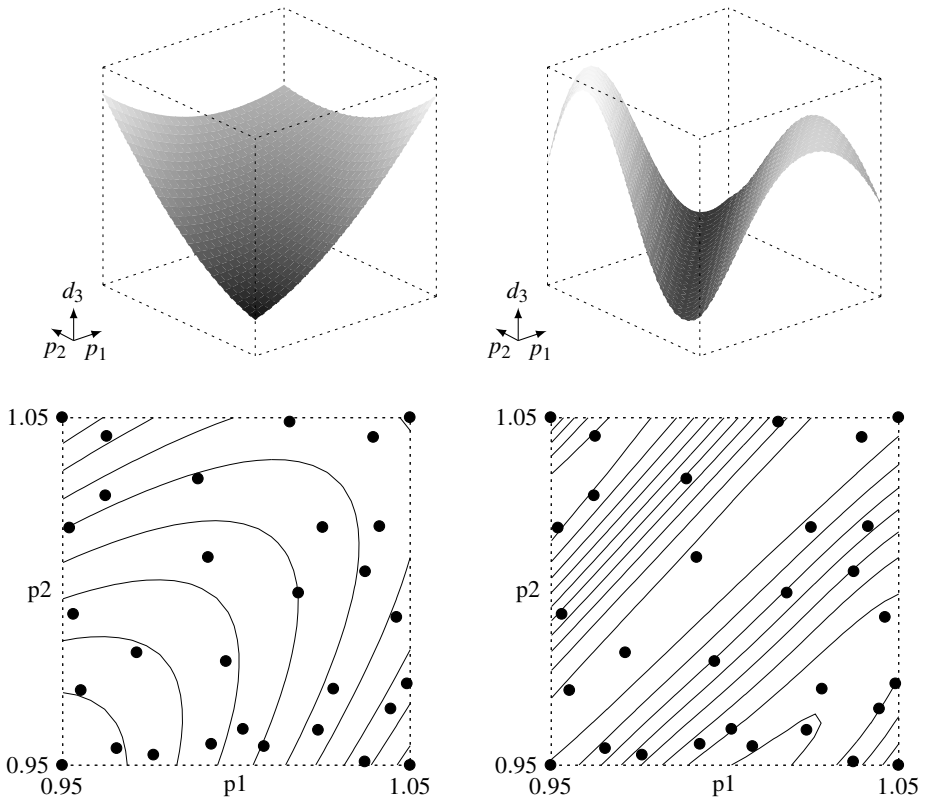


Figure 11: Surface and contour plots of Kriging approximations of the displacement at $t = 0.25$s (left) and at $t = 1.85$s (right) as a function of the uncertain model parameters $p1$ and $p2$. The response points used to construct the Kriging approximations are marked on the contour plots. Note that the same response points are used for both objective function approximations.

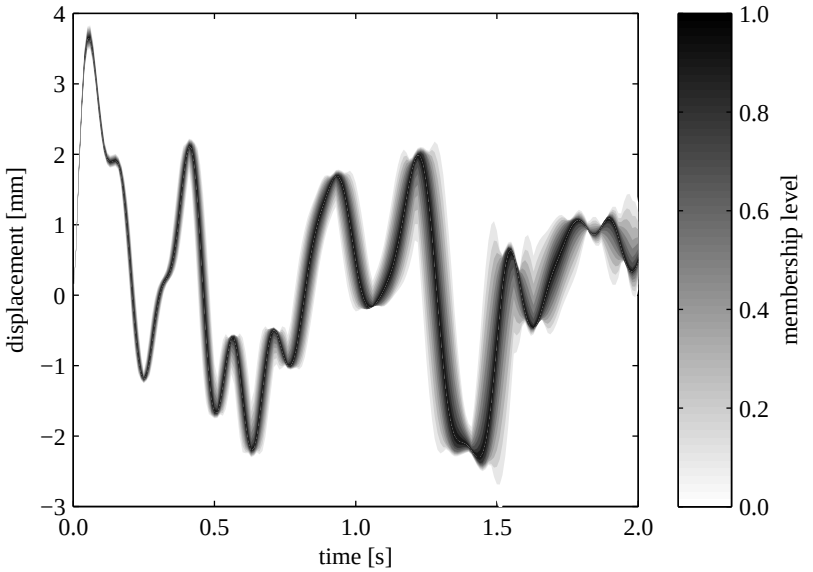


Figure 12: Fuzzy transient response computed using the linear regression based optimisation algorithm and a full quadratic model.

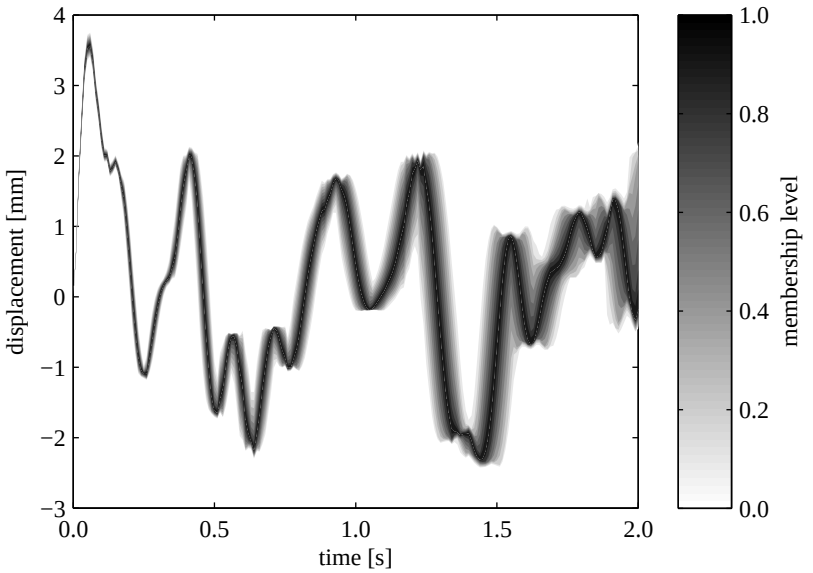


Figure 13: Fuzzy transient response computed using the linear regression based optimisation algorithm and a full fourth order model.

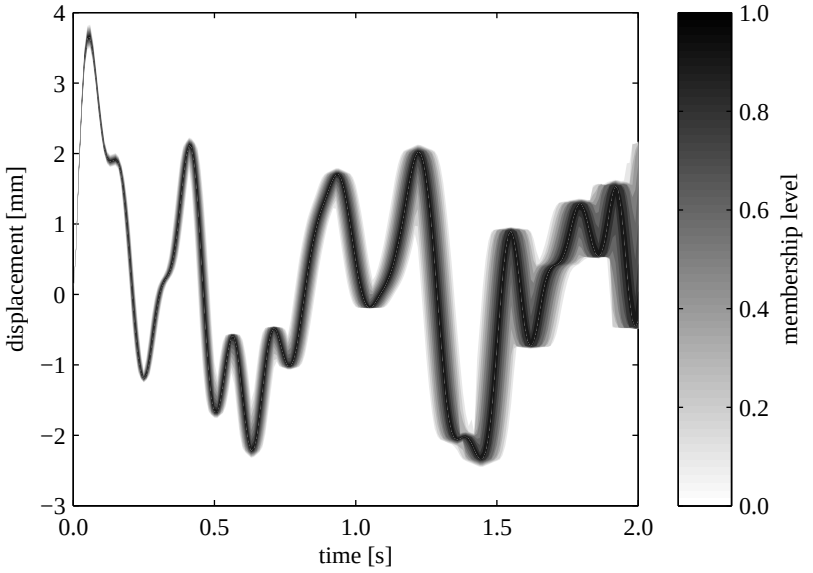


Figure 14: Fuzzy transient response computed using the Kriging response surface based optimisation algorithm.

methods still require 47475 respectively 3363 objective function evaluations. All response surface based methods on the other hand prove to be computationally cheap, requiring 13 to 30 function evaluations. Considering the flexibility regarding the shape of the response, the accuracy of the results and the computational cost, the Kriging based optimisation method is clearly the best choice for this problem.

5.2 Harmonic dynamic analysis of a benchmark model

In this section, the two-dimensional reference model illustrated in Fig. 15 is subject to a harmonic analysis, using the Kriging based optimisation approach described in section 4.

The reference model is a two-dimensional beam model with 27 DOFs. It has 10 resonant frequencies between 20 Hz and 60 Hz. The model is originally designed for the validation of model updating procedures [Caesar, Eckert, and Hoppe (1995)]. It is an excellent test for the optimisation procedure because a single function evaluation is very fast (a modal analysis takes about 1 second on a 3 GHz Pentium IV system), but the model is extremely vulnerable to eigenfrequency crossover, resulting in objective functions that are very difficult to optimise.

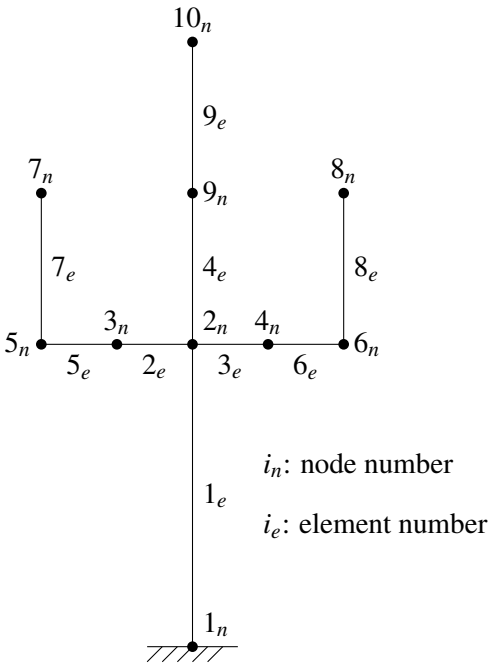


Figure 15: Two-dimensional reference model.

Table 2: Uncertainties in the reference model.

Property	Value	
length element 9	$(0.45/0.5/0.55)$	m
length elements $(3+6)$	$(0.49/0.50/0.59)$	m
length elements $(2+5)$	$(0.41/0.50/0.51)$	m
lumped mass node 3	$(150/180/200)$	kg
lumped mass node 9	$(320/360/400)$	kg
area element 6	$(4/4.5682/5)\cdot 10^{-4}$	m

Table 3: Comparison of the computational cost of the harmonic analyses of the two-dimensional reference model in terms of the number of objective function evaluations.

Method	Function evaluations
Transformation method	321
Linear regression based optimisation method	69
Kriging based optimisation method	60
Monte Carlo method	501

The model is subject to 6 uncertainties: the length of element 9, the length of elements 3 and 6, the length of elements 2 and 5, the lumped mass at node 3, the lumped mass at node 9 and the cross-sectional area of element 6. All uncertainties are fuzzy numbers with a triangular shape function. The exact values are specified in Tab. 2.

A fuzzy FRF, composed of five interval FRFs at the support and at membership levels 0.2, 0.4, 0.6 and 0.8 and a deterministic FRF at membership level 1.0, is calculated between the vertical DOFs of nodes 3 and 4. These interval FRFs are computed from 15 Hz to 55 Hz in 0.25 Hz increments, taking into account the first ten modes.

The harmonic analysis requires 1610 optimisations (the maximisation and minimisation of the response at 161 frequencies, repeated at five membership levels). These optimisations are performed using a linear regression based optimisation method and using the Kriging based optimisation method. Due to the difficult behaviour of the model, the computational cost of a reference solution using a global optimiser or using a local optimiser, as computed for the application in section 5.1, proved to be computationally infeasible. Instead, a transformation method analysis and a small Monte Carlo analysis of 501 uniformly distributed samples are performed to verify the results, although it should be noted that this reference solution is not guaranteed to be conservative, because most objective functions are non-monotonous. Tab. 3 gives the number of function evaluations required for each of the solution strategies. A single function evaluation takes about 1 s on a 3 GHz Pentium IV system.

Fig. 16 shows the interval FRFs at the support, computed using the transformation method (triangles), the linear regression based optimisation method (plus signs) and the Kriging based optimisation method (circles). The interval spanned by the Monte Carlo and vertex samples is shaded. The linear regression based optimisation method performs well up to about 33 Hz. At higher frequencies, it is much less accurate. The explanation for this is equivalent to the explanation of the gradu-

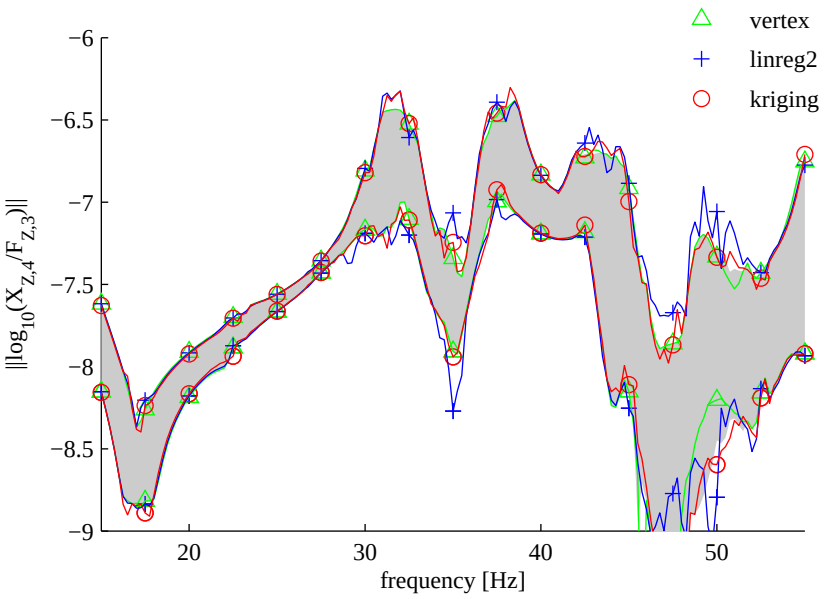


Figure 16: Comparison of the interval FRFs of the two-dimensional reference model at the support.

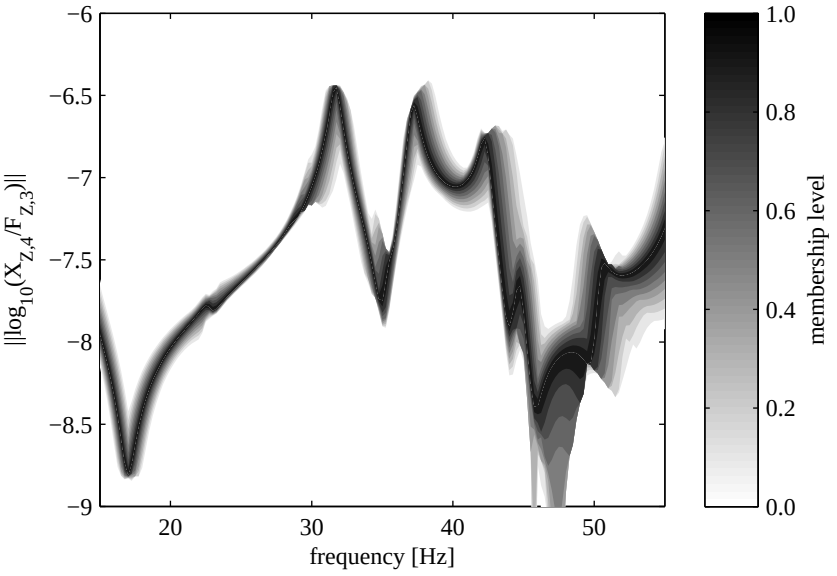


Figure 17: Fuzzy FRF of the two-dimensional reference model, computed using the transformation algorithm.

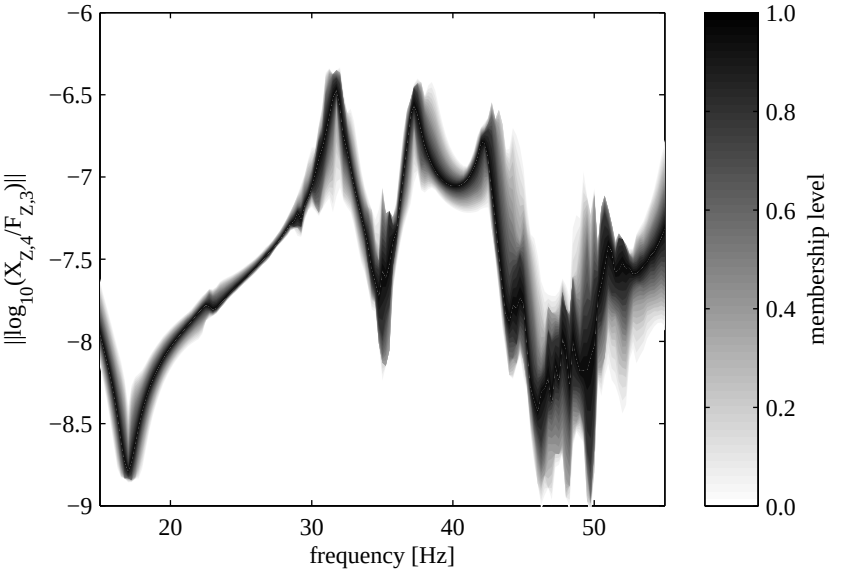


Figure 18: Fuzzy FRF of the two-dimensional reference model, computed using the linear regression based optimisation algorithm.

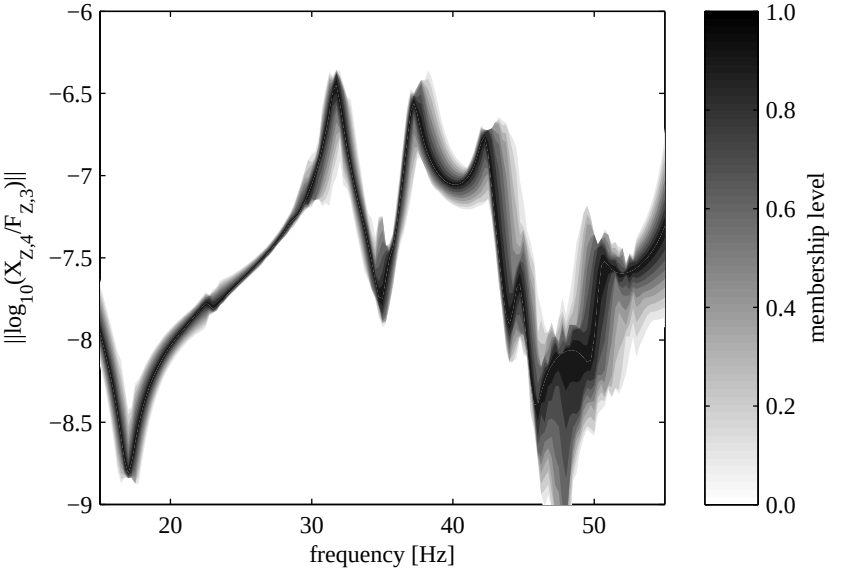


Figure 19: Fuzzy FRF of the two-dimensional reference model, computed using the Kriging response surface based optimisation algorithm.

ally poorer accuracy of the transient dynamic response of the example discussed in section 5.1. The optimisation in the frequency domain is very similar to the optimisation in the time domain and as the frequency increases, dispersion increases and more and more peaks and valleys enter the space defined by the uncertain parameters. As such, the dynamic response becomes more and more difficult to model, which first affects the linear regression based method with its less flexible response surface shape. The Kriging based optimisation method uses more flexible response surfaces and is not or at least much less affected by this problem and yields accurate results over the full frequency range. The figures at the other membership levels confirm these conclusions and are therefore not reproduced.

Fig. 17, 18 and 19 show the fuzzy FRFs computed with the transformation method, the linear regression based optimisation method and the Kriging based optimisation method. The same remarks given for the interval FRF above, apply.

Based on the number of function evaluations required, the Kriging based optimisation method would be the most appropriate optimisation method for the calculation of an interval or fuzzy frequency response function of this model.

6 Conclusions

Non-deterministic approaches are gaining momentum in the field of numerical modelling techniques. The ability to include non-deterministic properties is of great value for a design engineer. It enables realistic reliability assessment that incorporates the uncertain aspects of the design. Furthermore, the design can be optimised for robust behaviour under varying external influences. In this context, interval and fuzzy approaches are becoming increasingly popular for the analysis of numerical models that incorporate uncertainty in their description.

This paper presents a novel method for efficient and accurate interval and fuzzy numerical analysis. The method is based on the global optimisation approach and uses Kriging response surfaces and an adaptive response point selection algorithm to search for the minima and maxima of all objective functions at once.

The application of the method on a transient dynamic finite element analysis of a cable stayed bridge and on a harmonic dynamic finite element analysis of a benchmark model shows its superior efficiency compared to the classical local and global optimisation algorithms.

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