 CMES, vol.47, no.2, pp.167-190, 2009

A Novel Method for Solving the Cauchy Problem of Laplace Equation Using the Fictitious Time Integration Method

Chih-Chang Chi[1], **Weichung Yeih**[1,2] **and Chein-Shan Liu**[3]

Abstract: In this study, a novel method for solving the Cauchy problem of Laplace equation is developed. Through the fictitious time integration method (FTIM), the finding of the root of the resulting linear equations can be transformed into for finding the fixed point of a system of first order ordinary differential equations, in which a fictitious time variable is introduced. In such a sense, the inverse of ill-posed leading matrix is not necessary for the FTIM. This method uses the residual of each equation to control the evolution of unknowns in the fictitious time, and it is different from the conventional iteration method where an artificial iteration rule is required. Comparing to the Tikhonov's regularization method, the FTIM does not need to seek for the optimal regularization parameter, and it also needs not to seek for the inverse of the leading coefficient matrix in each step. Numerical results are given to show the validity of the current approach and it can be seen that this method can obtain reasonable results with or without noise. It shows a better noise resistance than the Tikhonov's regularization method.

Keywords: Inverse Cauchy problem, Laplace equation, Boundary element method, Fictitious Time Integration Method, Tikhonov's regularization method

1 Introduction

Inverse problems attract attentions in decades and many research works have been done. In the following, we list some of them: Marin, Power, Bowtell, Sanchez, Becker, Glover and Jones (2008); Huang and Shih (2007); Marin (2008); Noroozi, Sewell and Vinney (2006); Ling and Takeuchi (2008); Mera, Elliott and Ingham

[1] Computation and Simulation Center, National Taiwan Ocean University, Keelung 20224, Taiwan, ROC. Email of the corresponding author: d88520002@mail.ntou.edu.tw

[2] Department of Harbor and River Engineering, National Taiwan Ocean University, Keelung 20224, Taiwan, ROC

[3] Department of Civil Engineering, National Taiwan University, Taipei 10671, Taiwan, ROC.

(2006); Mustata, Harris, Elliott, Lesnic and Ingham (2000); Harris, Mustata, Elliott, Ingham and Lesnic (2008).

The Cauchy problem is an inverse problem which is well-known as a highly ill-posed problem since the solution does not continuously depend on boundary data. It means that a small disturbance in the boundary data may result in an enormous error in numerical solution. This problem appears in many applications, for example, in the cardiography, the nondestructive testing, etc. Stable and efficient numerical methods are highly desired. The Holmgren theorem guarantees uniqueness of the Cauchy continuation problem, if the data are perfectly known on a dense subset of the boundary. This is however never the case in practice, and causes instability.

Many research works have been conducted to seek for a stable and efficient method to deal with the ill-posed behaviors of Cauchy problem. We cannot list all of them, and the following mentioned methods are some well-known and popular methods. The most popular one is the Tikhonov's regularization method [Tikhonov and Arsenin (1977)], which transforms the original incorrectly posed problem into a correctly posed problem by a minimization of the L_2-norm of the solution subjected to the constraint equations. To determine the optimal regularization parameter, Hansen (1992) proposed the so-called L-curve concept, in which the optimal parameter is to seek for the best balance between the distortion of the original equations and the norm of solution. Another well known method is the truncated singular value decomposition method [Chang, Yeih and Shieh (2001)]. This method discards some small singular values below the threshold, such that the amplification of error of data for these small singular values would not appear. Other methods have been proposed to regularize the Cauchy problem, for example, an energy-minimizing approach [Andrieux , Baranger and Abda (2006)], methods using quasi-reversibility [Klibanov and Santosa (1991); Bourgeois (2005)], and methods of alternating Dirichlet and Neumann problems, with regularizing properties [Kozlov, Mazya and Fomin (1992); Belgacem and Fekih (2005)], etc. For more references, the following listed papers are some related inverse problems appeared recently: Ling and Takeuchi (2008), Cheng, Hon, Wei and Yamamoto (2001), Hào and Lesnic (2000), Berntsson and Eld'en (2001), Hon and Wei (2001), Marin, Elliot, Ingham and Lesnic (2002), Marin and Lesnic (2002), Aliev and Hosseini (2002), Mera, Elliot, Inghalm and Lesnic (2000), Leitao (2000), Engl and Leitao (2001), Cheng and Cabral (2005), Liu (2008a, 2008b), Li (2004), Yeih, Koya and Mura (1993), and Koya, Yeih and Mura (1993).

In this paper, a new approach for solving the Cauchy problem of the Laplace equation using an iterative procedure, namely the fictitious time integration method (FTIM), is proposed. The FTIM is first proposed by Liu and Atluri (2008a), in which the large scale nonlinear algebraic equations are treated. Later, the FTIM is used to solve the mixed complementarity problems with applications to non-linear optimization [Liu and Atluri (2008b)], to solve the discretized inverse Sturm-Liouville problems [Liu and Atluri (2008c)], to solve the two-dimensional quasilinear elliptic boundary value problems [Liu (2008c)], to solve the non-linear obstacle problems with the aid of an NCP-function [Liu (2008d)], to solve the Burgers equation [Liu (2009a)], to deal with the m-point boundary value problem [Liu (2009b)], and to treat the Fredholm integral equation of the first-kind and perform numerical differentiation of noisy data [Liu and Atluri (2009)]. Recently, a new modified time-like function is proposed to accelerate the numerical convergence of FTIM [Ku, Yeih, Liu and Chi (2009)]. Among these efforts, the inverse Sturm-Liouvill problem, the Fredholm integral equation of the first-kind and numerical differentiation of noisy data are known as ill-posed problems. The FTIM has shown its powerful noise resistance for dealing with the ill-posed behaviors. Stimulated by the previous literature, we propose to use the FTIM to deal with the Cauchy problem for the Laplace equation. The discretization method for spatial domain is the boundary element method, in which only boundary discretization is required. We will study the performance of FTIM on this problem and compare it to Tikhonov's regularization method, which might be the most popular technique in dealing with the Cauchy problem.

2 BEM implementation and Cauchy problem for the Laplace equation

2.1 The boundary element method for the Laplace equation

By using the fundamental solution of the two-dimensional Laplace equation and Green's identity, the following boundary integral equation for representing the field quantity in the domain is obtained [Hong and Chen (1988)]:

$$2\pi u(\mathbf{x}) = \int_B T(\mathbf{s},\mathbf{x})u(\mathbf{s})dB(\mathbf{s}) - \int_B U(\mathbf{s},\mathbf{x})\frac{\partial u(\mathbf{s})}{\partial n_s}dB(\mathbf{s}),\ \mathbf{x}\in\Omega, \tag{1}$$

where Ω denotes the interested domain and its boundary is B,

$$U(\mathbf{s},\mathbf{x}) \equiv \frac{1}{2\pi}\ln(r), r \equiv \sqrt{(\mathbf{x}-\mathbf{s})\cdot(\mathbf{x}-\mathbf{s})},$$

satisfying

$$\nabla^2 U = \delta(\mathbf{s}-\mathbf{x}),$$

and

$$T(\mathbf{s},\mathbf{x}) = \frac{\partial U(\mathbf{s},\mathbf{x})}{\partial n_s}. \tag{2}$$

By tracing the field point $\mathbf{x}$ to the boundary, the boundary integral equations for representing the field quantity on the boundary can be obtained:

$$\alpha u(\mathbf{x}) = C.V.P. \int_B T(\mathbf{s},\mathbf{x}) u(\mathbf{s}) dB(\mathbf{s}) - R.P.V. \int_B U(\mathbf{s},\mathbf{x}) \frac{\partial u(\mathbf{s})}{\partial n_s} dB(\mathbf{s}), \tag{3}$$

where $C.V.P.$: Cauchy principal value, $R.V.P.$: Riemann principal value, αdepends on the solid angel and $\alpha = \pi$ if $\mathbf{x}$ locates at a point on the smooth boundary in the two-dimensional case.

When the boundary is discretized into N constant elements, the linear algebraic equations can be obtained as follows:

$$[\mathbf{U}]_{N\times N} \{\mathbf{t}\}_{N\times 1} = [\mathbf{T}]_{N\times N} \{\mathbf{u}\}_{N\times 1}. \tag{4}$$

It is known that once the boundary quantities on the boundary are entirely known by using Eq. (3), the field quantity inside the domain can be obtained by using Eq. (1).

2.2 The Cauchy problem

In this section, we will discuss the Cauchy problem. Here, we define the sub-boundaries Γ_i, Γ_j, Γ_k and Γ_l, which depend on the boundary data given. Γ_i denotes the sub-boundary with given Neumann data and Γ_j denotes the sub-boundary without Neumann data. For convenience, we further denote $\{t_i\}$ as given Neumann data on Γ_i and $\{t_j\}$ as unknown Neumann data on Γ_j. Γ_k denotes the sub-boundary with given Dirichlet data and Γ_l denotes the sub-boundary without Dirichlet data. Similarly, we use the notation $\{u_k\}$ to denote the given Dirichlet data on Γ_k and $\{u_l\}$ denotes the unknown Dirichlet data on Γ_l. Eq. (4) for the Cauchy problem is then written as

$$\begin{bmatrix} \mathrm{U}_{ii} & \mathrm{U}_{ij} \\ \mathrm{U}_{ji} & \mathrm{U}_{jj} \end{bmatrix}_{N\times N} \begin{Bmatrix} t_i \\ t_j \end{Bmatrix}_{N\times 1} = \begin{bmatrix} \mathrm{T}_{kk} & \mathrm{T}_{kl} \\ \mathrm{T}_{lk} & \mathrm{T}_{ll} \end{bmatrix}_{N\times N} \begin{Bmatrix} u_k \\ u_l \end{Bmatrix}_{N\times 1}. \tag{5}$$

Rearranging the known data and unknown quantities in the equation, one can obtain

$$\{\mathbf{F}(\mathbf{x})\} = [\mathbf{A}]\{\mathbf{x}*\} - \{\mathbf{b}\} = \{\mathbf{0}\}, \tag{6}$$

where

$$[\mathbf{A}] = \begin{bmatrix} \mathrm{U}_{ij} & -\mathrm{T}_{kl} \\ \mathrm{U}_{jj} & -\mathrm{T}_{ll} \end{bmatrix}, \{\mathbf{x}*\} = \begin{Bmatrix} t_j \\ u_l \end{Bmatrix}, \{\mathbf{b}\} = \begin{bmatrix} -\mathrm{U}_{ii} & \mathrm{T}_{kk} \\ -\mathrm{U}_{ji} & \mathrm{T}_{lk} \end{bmatrix} \begin{Bmatrix} t_i \\ u_k \end{Bmatrix}. \tag{7}$$

For the well-posed boundary value problem, we give one boundary condition at each boundary point. When the potential function is given on the boundary, it is known as a Dirichlet boundary value problem. It means that $\Gamma_i = \emptyset$, $\Gamma_j = \mathrm{U}$, $\Gamma_k = \mathrm{U}$, $\Gamma_l = \emptyset$ where $\emptyset$ denotes the null sets and U denotes the universal set. When the normal derivative is given on the boundary, it is known as the Neumann boundary value problem. It means that $\Gamma_i = \mathrm{U}$, $\Gamma_j = \emptyset$, $\Gamma_k = \emptyset$ and $\Gamma_l = \mathrm{U}$. In general, the mixed type boundary value problem is given for every boundary point, that is, $\alpha u + \beta \frac{\partial u}{\partial n} = \bar{g}, \alpha^2 + \beta^2 \neq 0$ given on the boundary. The above-mentioned boundary value problem is correctly posed in the Hadamard sense [Hadamard (1923)], such that one will not encounter the numerical instability. The problem we are dealing in this paper is different from the well-posed boundary value problem. On part of the boundary, the potential and its normal derivative are overprescribed, but on the remaining boundary no information is known. It means that there exist some boundary points without any information, i.e., $\Gamma_j \cap \Gamma_l \neq \emptyset$. Unfortunately, the Cauchy problem is well known for its ill-posedness. In numerical calculation, some efforts should be done to stabilize the procedure such that one can approximate the solution steadily. In the previous literature review, we have mentioned many techniques. In this paper, we propose to use the FTIM to deal with the ill-posed Cauchy boundary value problem.

3 A fictitious time integration evolution

A novel time integration method, FTIM, has been proposed by Liu and Atluri (2008a).

Let us consider the following algebraic equations:

$$\mathbf{F}(\mathbf{x}) = \mathbf{0}. \tag{8}$$

Liu and Atluri (2008a) transform the problem in eq.(8) into a dynamic system written as

$$\dot{\mathbf{x}} = -\frac{\nu}{1+\tau}\mathbf{F}(\mathbf{x}) \tag{9}$$

where τ is the fictitious time variable and ν is a parameter to control the convergence.

The roots of $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ are fixed points of the above equation. We should stress that the factor $-\nu/(1+\tau)$ before $\mathbf{F}(\mathbf{x})$ is important. Actually, in a previous published literature $\dot{\mathbf{x}} = -\mathbf{F}(\mathbf{x})$ [Ramm (2007)] has been proposed but it sometimes fails as the trajectory of $\mathbf{x}$ goes to infinity when the initial guess is not appropriate. The term $-\nu/(1+\tau)$ plays as a controller to help convergence and is a stabilizer of the trajectory of $\mathbf{x}$. However, one should carefully choose the parameter ν such that the solution can be reached before $-\nu/(1+\tau)$ tends to zero; otherwise, one may find the fictitious fixed points of the system of ODEs because of $-\nu/(1+\tau)$=0, but not the $\mathbf{x}$ for $\mathbf{F}(\mathbf{x}) = \mathbf{0}$.

Eq. (9) can be used to solve any algebraic equation, linear or nonlinear. For solving the linear algebraic equations, the FTIM does not really compute the inverse of the leading coefficient matrix but transform the algebraic equations into a system of ODEs. The ill-posed behavior of the linear algebraic equations system now does not appear, even the condition number of the leading coefficient matrix is high, but how many iteration steps (or evolution time) one should use. Theoretically speaking, the system in Eq. (9), even ill-posed, can be solved when the infinite fictitious time (or infinite iterations) is allowed [Ramm (2007)]. A good iteration (or evolution) procedure should have a reasonable steps (or evolution time) to reach the acceptable result even when the system is ill-posed.

The FTIM is a system of ODEs such that one needs to perform numerical integration. For example, one can use the Euler scheme, or the Runge-Kutta method. In this paper, we use the group preserving scheme (GPS) developed by Liu (2001) for the integration of Eq. (9). GPS has a beautiful mathematical structure such that some properties of the system can be preserved. Especially when the equations to be solved are nonlinear, GPS shows its superiority over some well-known integration methods, such as the Euler scheme [Press, Teukolsky, Vetterling, and Flannery (2007)], the Runge-Kutta method [Press, Teukolsky, Vetterling, and Flannery (2007)], and the symplectic group integration method [Hairer, Lubich and Wanner (2006)].

4 Numerical examples

In this section, we will demonstrate the validity for using the FTIM in conjunction with BEM to solve the Cauchy problem of the Laplace equation. We will discuss the convergence, fictitious time for evolution and numerical stability under given random noise, and we compare our results with those obtained from the conventional Tikhonov's regularization method, and show the superiority of the current approach. For the Tikhonov's regularization method, we use L-curve concept to determine the optimal regularization parameter.

4.1 Reconstruction of boundary data for a unit circle

As shown in Fig. 1, a unit circular domain is considered. The exact solution is designed as $u = r\sin\theta$, where r denotes the radial distance between the observation point and the origin, and θ is the angle between the position vector of the observation point and the x-axis. Then the distribution of the potential and its normal derivative on the boundary are $u = \sin\theta$ and $t = \sin\theta$, respectively. We study the Cauchy problems for giving Cauchy data on half of the boundary and no data given on the remaining boundary. It has been mentioned in a previous published literature [Chang, Yeih and Shieh (2001)] that if the data are given more diversely, the result of boundary data reconstruction will be more accurate. In this case, we will validate this statement once more by studying two different Cauchy data as shown in Table 1 and compare their results. In these two cases, we use 36 and 72 elements on boundary respectively and the convergence criterion is given by $\varepsilon = 10^{-5}$. The parameters used for FTIM are: ν=20.0, and $\Delta\tau$=0.1. The details of initial guesses are given in Table 1.

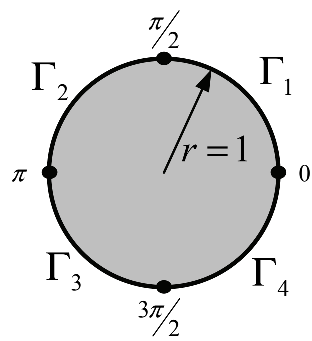


Figure 1: A circular region for Cauchy problem.

Table 1: Problem setups for recovering boundary data of a circular region.

		Γ_1	Γ_2	Γ_3	Γ_4	
Case1	u	$\sin\theta$	?/ Initial=1	$\sin\theta$	?/ Initial=-1	$\nu = 20, \Delta\tau = 0.1$
	t	$\sin\theta$	?/ Initial=1	$\sin\theta$	?/ Initial=-1	
Case2	u	$\sin\theta$	$\sin\theta$	?/ Initial=-1	?/ Initial=-1	$\nu = 20, \Delta\tau = 0.1$
	t	$\sin\theta$	$\sin\theta$	?/ Initial=-1	?/ Initial=-1	

? : unknown boundary condition

In the first case, the Cauchy data are given on the sub-boundaries $\theta = 0 \sim \pi/2$ and $\theta = \pi \sim 3\pi/2$. It means that the sub-boundaries without data are separated by the sub-boundaries with Cauchy data. The results for boundary potential and its normal derivative are illustrated in Figs. 2(a) and 2(b). It can be seen that for the FTIM, no matter 36 or 72 elements are used the reconstruction of boundary data are satisfactory. However, for the Tikhonov's regularization method even 72 elements are used the normal derivative of potential cannot be recovered very well. From Fig. 3, we can find that the convergence performance for the FTIM is very good. For both discretization meshes, within 1.5 seconds (fictitious time) the FTIM already almost reaches the result. It reaches the convergence criterion when the fictitious time is 9 and 16 seconds for 36 elements and 72 elements, respectively. In Figs. 4(a) and 4(b), we show the results for the recovery of boundary data when a maximum random noise of 0.05 is added. It can be found that the FTIM still can recover the boundary data rather well, while the Tikhonov's regularization method gives results apparently deviating from the exact solution, especially for the normal derivative of potential.

In the second case, the Cauchy data are given on the sub-boundary $\theta = 0 \sim \pi$ and no boundary data given on the remaining boundary. As mentioned previously, the recovery of boundary data is expected to be more difficult for this case. We study the performance of FTIM and Tikhonov's regularization method with or without noise in data. The maximum random noise level in this case is 0.05. The results of boundary potential and its derivative are given in Figs. 5(a) and 5(b). In comparison with the previous case, it can be seen that a worse recovery of the boundary data was found provided that the distribution of sub-boundary with Cauchy data given concentrates locally, which confirms the conclusion in the previous published literature. In addition, it can be found that the Tikhonov's method cannot yield reasonable result in this case while the result of FTIM is still acceptable. It once again shows the superior performance for the FTIM.

In a previous paper of Hào and Lesnic (2000), the conjugate gradient method (CGM), which is an iterative procedure, has been used to solve the similar Cauchy problem in this example. If one compares results for FTIM and CGM, it can be found that FTIM has a better performance and a better noise resistance than CGM.

4.2 Reconstruction of boundary data for a square domain

A square domain $\Omega = (0,L) \times (0,L)$as shown in Fig. 6 is considered, where L =1. The designed exact solution is $u(x,y) = \cos(x)\cosh(y) + \sin(x)\sinh(y)$. The Cauchy data then are given on the specified sub-boundaries according to the designed exact solution. Two cases are studied and the ways that we assign the Cauchy data and the details of initial guesses are given in Table 2. In both cases,

(a)

(b)

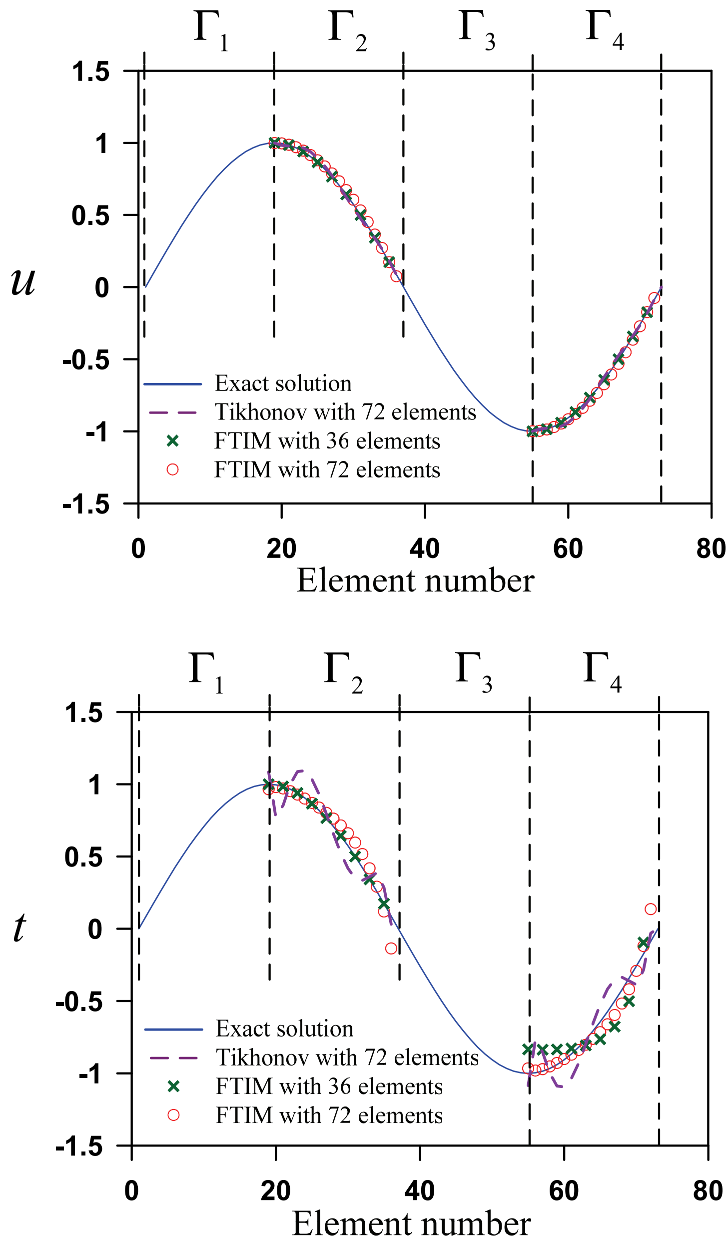


Figure 2: Recovery of boundary data for the circular region of case 1 using Cauchy data without noise: (a) potential; (b) normal derivative of potential.

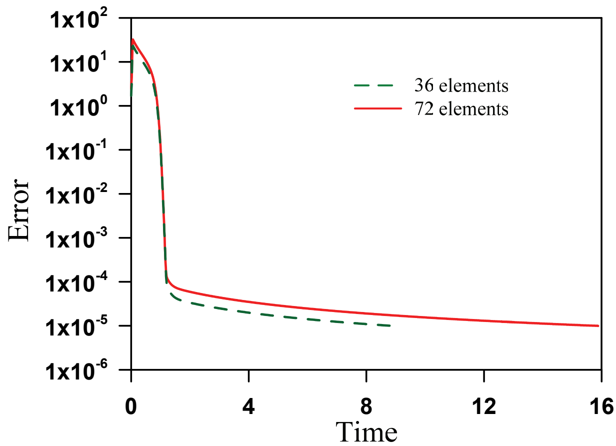


Figure 3: Convergence performance of FTIM for 36 and 72 constant elements for recovering boundary data for the circular region of case 1.

40 and 80 elements are used and the convergence criterion is given as $\varepsilon = 10^{-4}$.

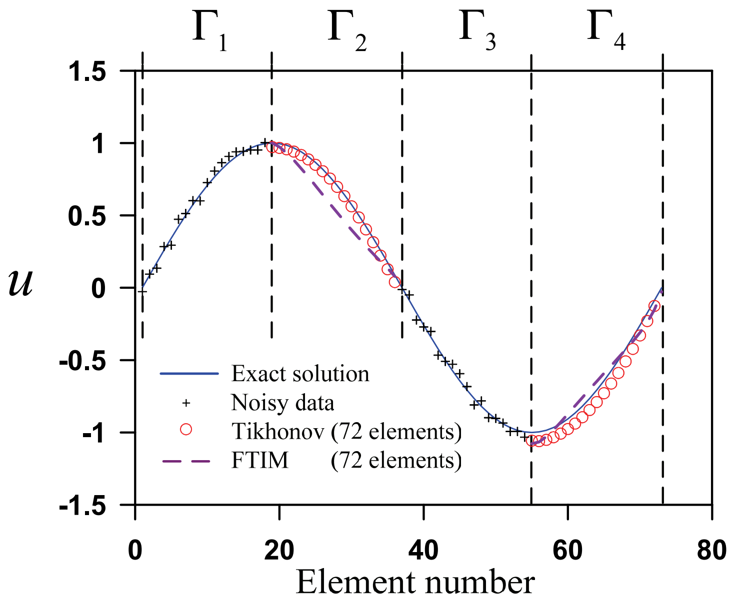
Table 2: Problem setups for recovering boundary data of a square region.

		Γ_1	Γ_2	Γ_3	Γ_4	
Case1	u	$u(\Gamma_1)$	?/ Initial=1	$u(\Gamma_3)$	?/ Initial=1	$\nu = 1, \Delta\tau = 0.1$
	t	$t(\Gamma_1)$	?/ Initial=-1	$t(\Gamma_3)$	?/ Initial=-1	
Case2	u	$u(\Gamma_1)$	$u(\Gamma_2)$	?/ Initial=1	?/ Initial=1	$\nu = 1, \Delta\tau = 0.1$
	t	$t(\Gamma_1)$	$t(\Gamma_2)$	?/ Initial=1	?/ Initial=-1	

? : unknown boundary condition

For the first case, the results for boundary data recovery are illustrated in Figs. 7(a) and 7(b). It can be found that the FTIM can yield good result for both 40 and 80 elements. On the other hand, even 80 elements are used the Tikhonov's regularization method still cannot recover the boundary data very well, especially for the normal derivative of the potential near the corners. In Fig. 8, the convergence speed of the FTIM is shown. Within 20 seconds (fictitious time), the FTIM gives result very near the exact one. FTIM reaches the convergence criterion when the fictitious time is 80 seconds and 100 seconds for 40 elements and 80 elements, respectively. When a random noise of 0.05 is added in the Cauchy data, the performances of both methods are shown in Figs. 9(a) and 9(b). It can be seen that FTIM has a better noise resistance than the Tikhonov's regularization method.

(a)



(b)

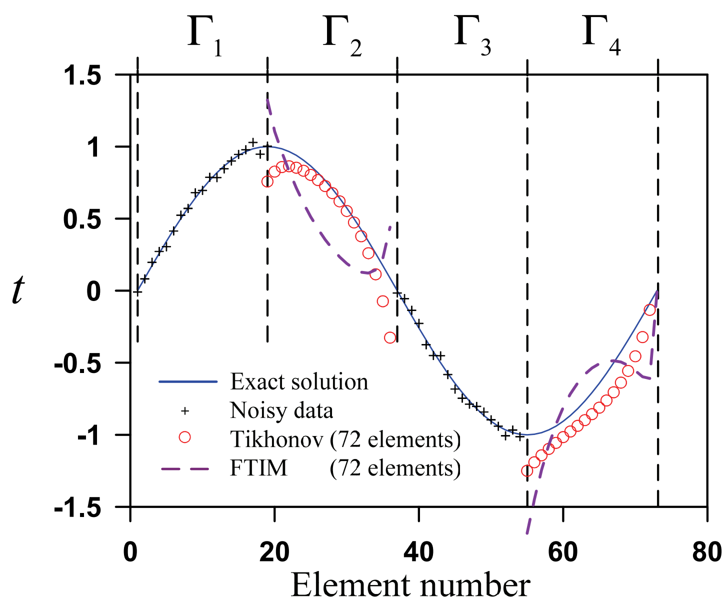
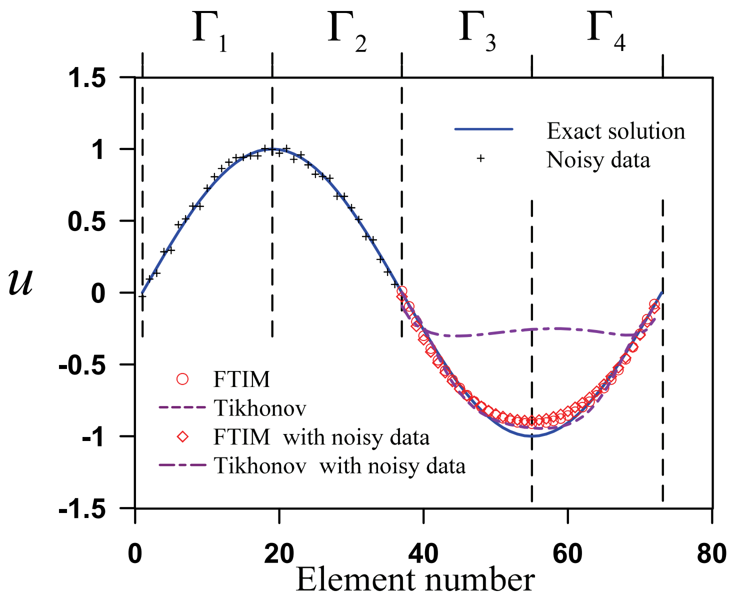


Figure 4: Recovery of boundary data for the circular region of case 1 using Cauchy data with noise level of 0.05: (a) potential; (b) normal derivative of potential.

(a)



(b)

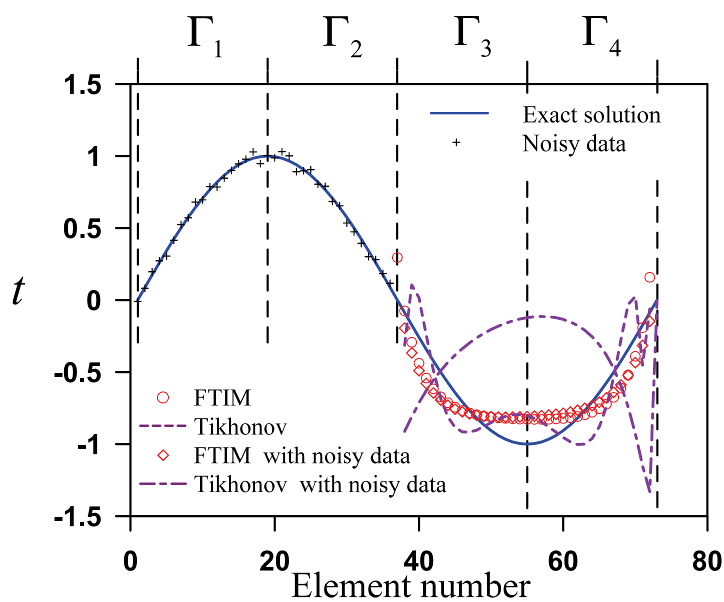


Figure 5: Recovery of boundary data for the circular region of case 2 using Cauchy data without noise and with noise level of 0.05: (a) potential; (b) normal derivative of potential.

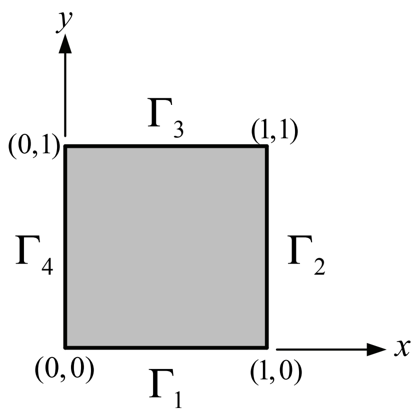


Figure 6: A square domain for Cauchy problem.

For the second case, the boundary Cauchy data are given in a worse scenario and a worse boundary data recovery is expected. The numerical results for both methods using 80 elements are given in Figs. 10(a) and 10(b). It can be found that FTIM outperforms the Tikhonov's regularization method, especially when the Cauchy data contain noise.

4.3 Recovery of boundary information for an annular region

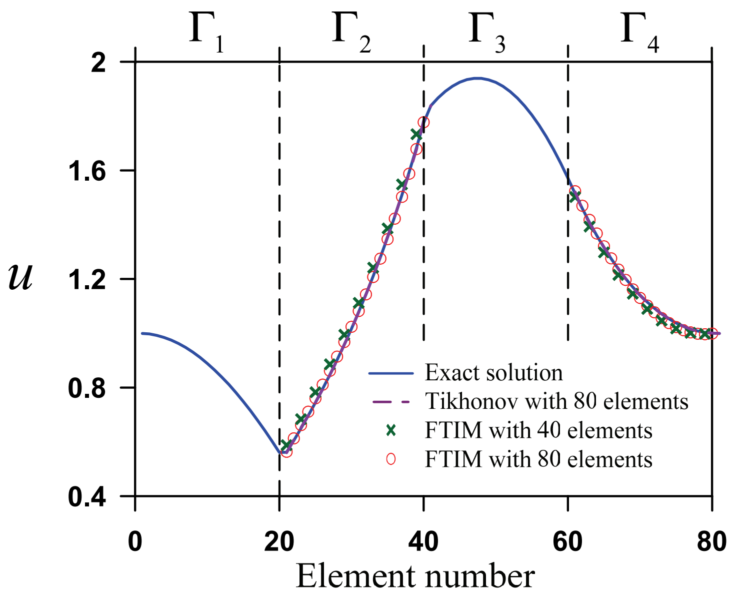
An annular case with outer radius of $r_1$and inner radius of r_2 ($r_1 = 1,\ r_2 = 0.75$) is sketched in Fig. 11. In this case, the designed exact solution is $u(x,y) = \cos(x)\cosh(y) + \sin(x)\sinh(y)$. The Cauchy data then are given on specified sub-boundaries according to the designed exact solution. Two cases are studied and the ways we assign Cauchy data and the details of initial guesses are given in Table 3. In both cases, 96 elements are used, and the convergence criterion is given as $\varepsilon = 10^{-6}$.

Table 3: Problem setups for recovering boundary data of an annular region.

		Γ_1	Γ_2	
Case1	u	$u(\Gamma_1)$	?/ Initial=1	$\nu = 10, \Delta\tau = 0.01$
	t	$t(\Gamma_1)$	?/ Initial=0	
Case2	u	?/ Initial=1	$u(\Gamma_2)$	$\nu = 10, \Delta\tau = 0.01$
	t	?/ Initial=0	$t(\Gamma_2)$	

? : unknown boundary condition

(a)



(b)

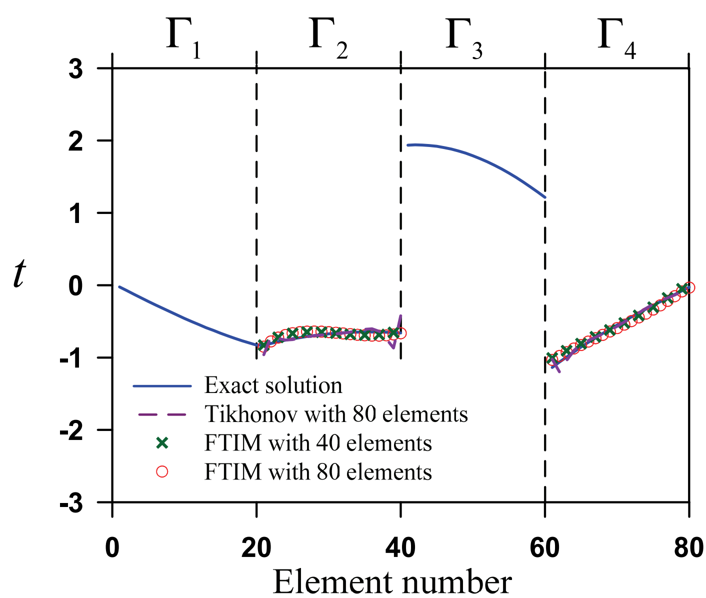


Figure 7: Recovery of boundary data for the square region of case 1 using Cauchy data without noise: (a) potential; (b) normal derivative of potential.

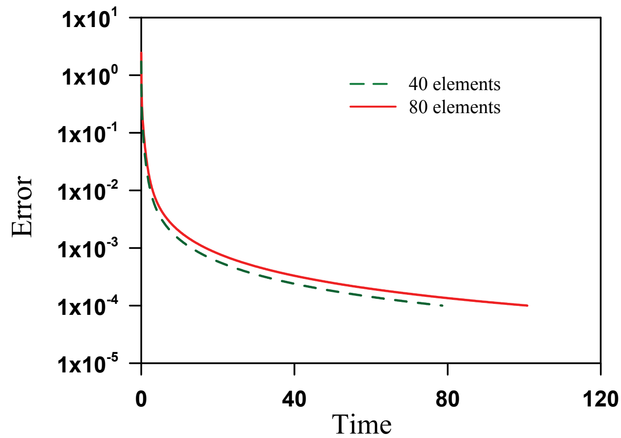


Figure 8: Convergence performance of FTIM for 40 and 80 constant elements for recovering boundary data for the circular region of case 1.

For the first case, a maximum random noise of 0.05 is added in the Cauchy data. The results for boundary data recovery are illustrated in Fig. 12(a) and 12(b). It can be found that the FTIM can yield good result, no matter Cauchy data contains noise or not.

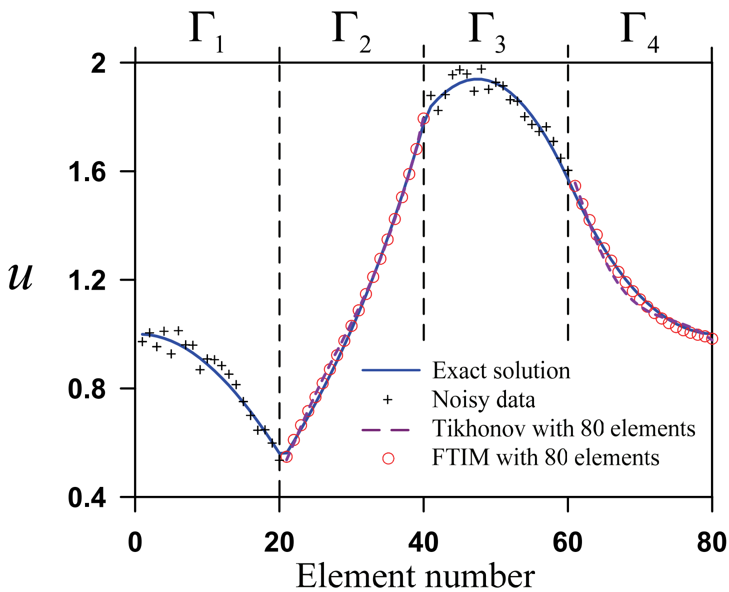
For the second case, a maximum random noise of 0.02 is added in the Cauchy data. The performances of FTIM can be found in Fig. 13(a) and 13(b). Although, to seek the missing information on the boundary of the outer radius by specifying Cauchy data on the boundary of the inner radius is more difficult than seeking information on the boundary of the inner radius by specifying Cauchy data on the boundary of the outer radius, it can be found that FTIM successfully recovers the boundary information well no matter data contains noise or not.

From the abovementioned three examples, we can conclude that FTIM has a very good noise resistance in solving inverse Cauchy problem. In the previous paper published by Liu and Atluri (2009), they have explained why the FTIM can have such a good noise resistance for ill-posed system. In their paper, it can be found that the FTIM is a better noise filter than the conventional Tikhonov's regularization method. Our results basically reconfirm the earlier claims of Liu and Atluri (2009).

5 Conclusions

The FTIM shows its superior performance for solving the Cauchy problem. It does not need to face the inverse of the ill-posed leading coefficient matrix, and uses

(a)



(b)

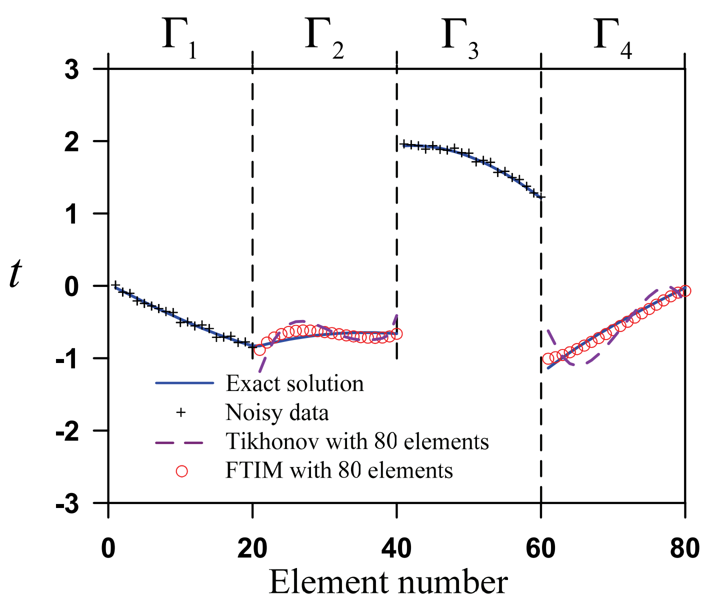


Figure 9: Recovery of boundary data for the square region of case 1 using Cauchy data with noise level of 0.05: (a) potential; (b) normal derivative of potential.

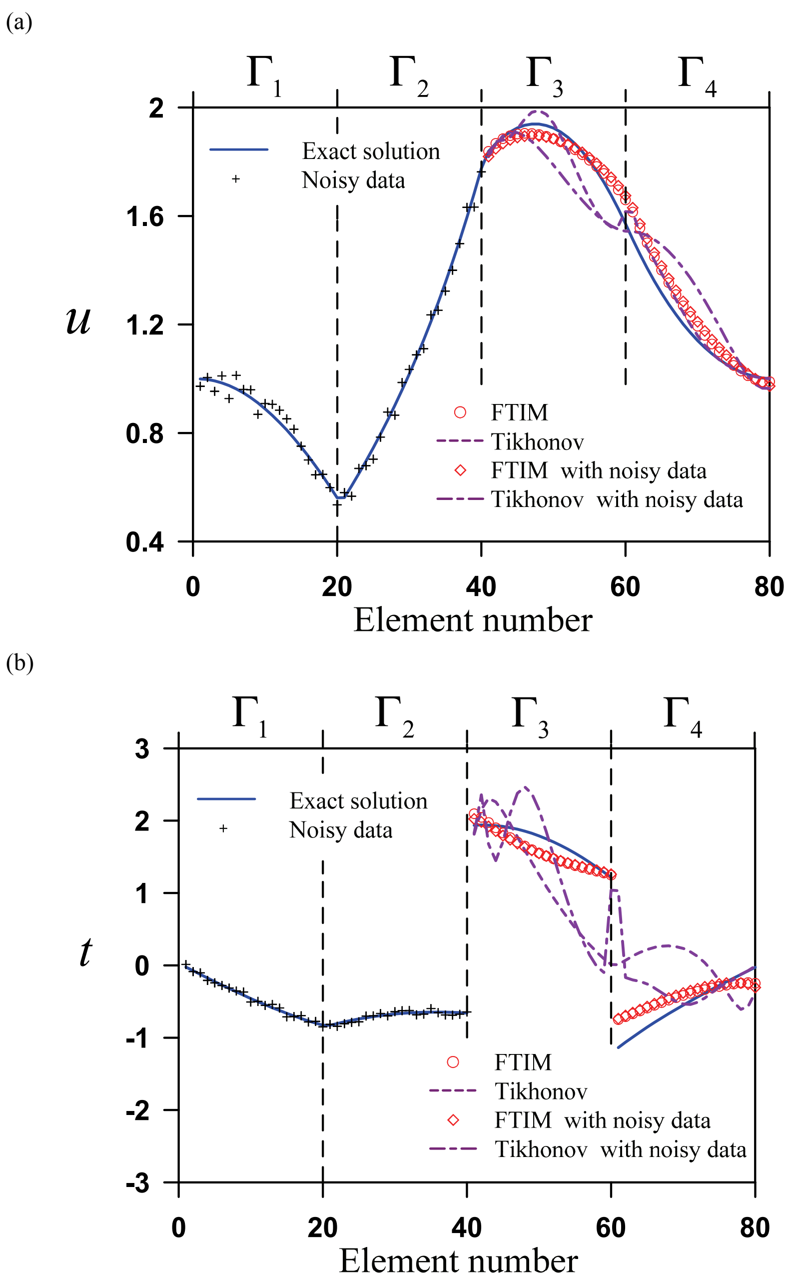


Figure 10: Recovery of boundary data for the square region of case 2 using Cauchy data without noise and with noise level of 0.05: (a) potential; (b) normal derivative of potential.

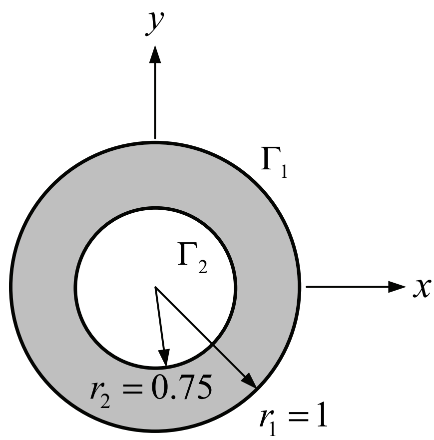


Figure 11: An annular region for Cauchy problem.

the residual of each equation to give the evolution power of unknowns. It does not use the regularization parameter such that the complex L-curve procedure is not necessary at all. In addition, FTIM is expected to have no difficulty in dealing with the problem having large scale matrix since it really does not solve the problem from the algebraic concept but from the dynamics of ODE. In comparison with the Tikhonov's regularization method, FTIM shows its superiority especially when data contain noise. Although we do not show how it performs when other discretization methods are adopted, we expect FTIM still can perform well since the development of FTIM does not have anything to do with the discretization procedure. Since FTIM can deal with linear algebraic equations and nonlinear algebraic equations as well, it is expected that FTIM can be used to solve the inverse Cauchy problem of nonlinear PDE as well.

References

Aliev, N.; Hosseini, S. M. (2002): Cauchy problem for the Navier-Stokes equation and its reduction to a non-linear system of second kind Fredholm integral equations. *Int. J. of Pure and Applied Mathematics*, vol. 3, pp. 317-324.

Andrieux, S.; Baranger, T. N.; Abda, A. B. (2006): Solving Cauchy problems by minimizing an energy-like functional. *Inverse Problems*, vol. 22, pp. 115–133.

Belgacem, F. B.; Fekih, H. E. (2005): On Cauchy's problem: I. a variational Steklov-Poincar'e theory. *Inverse Problems*, vol. 21, pp. 1915–1936.

Berntsson, F.; Eld'en, L. (2001): Numerical solution of a Cauchy problem for the

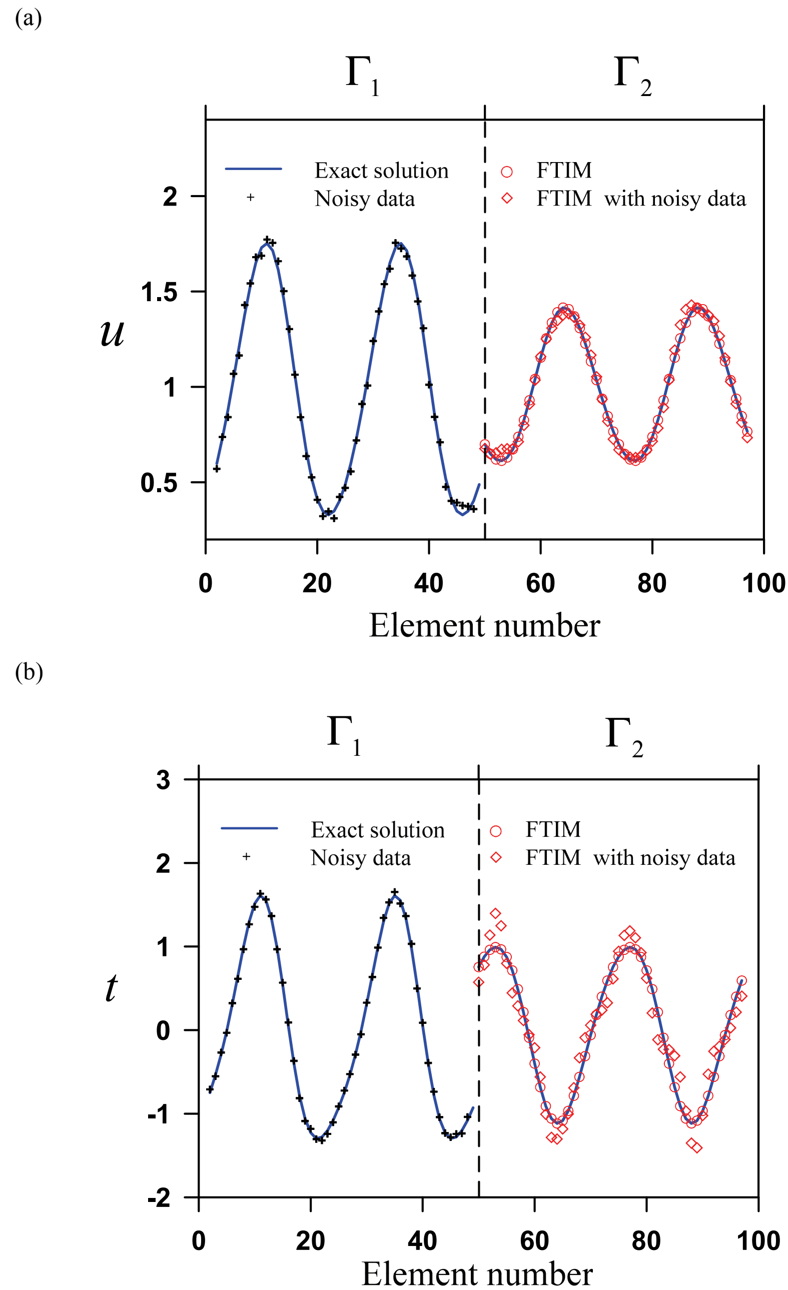


Figure 12: Recovery of boundary data for the annular region of case 1 using Cauchy data without noise and with noise level of 0.05: (a) potential; (b) normal derivative of potential.

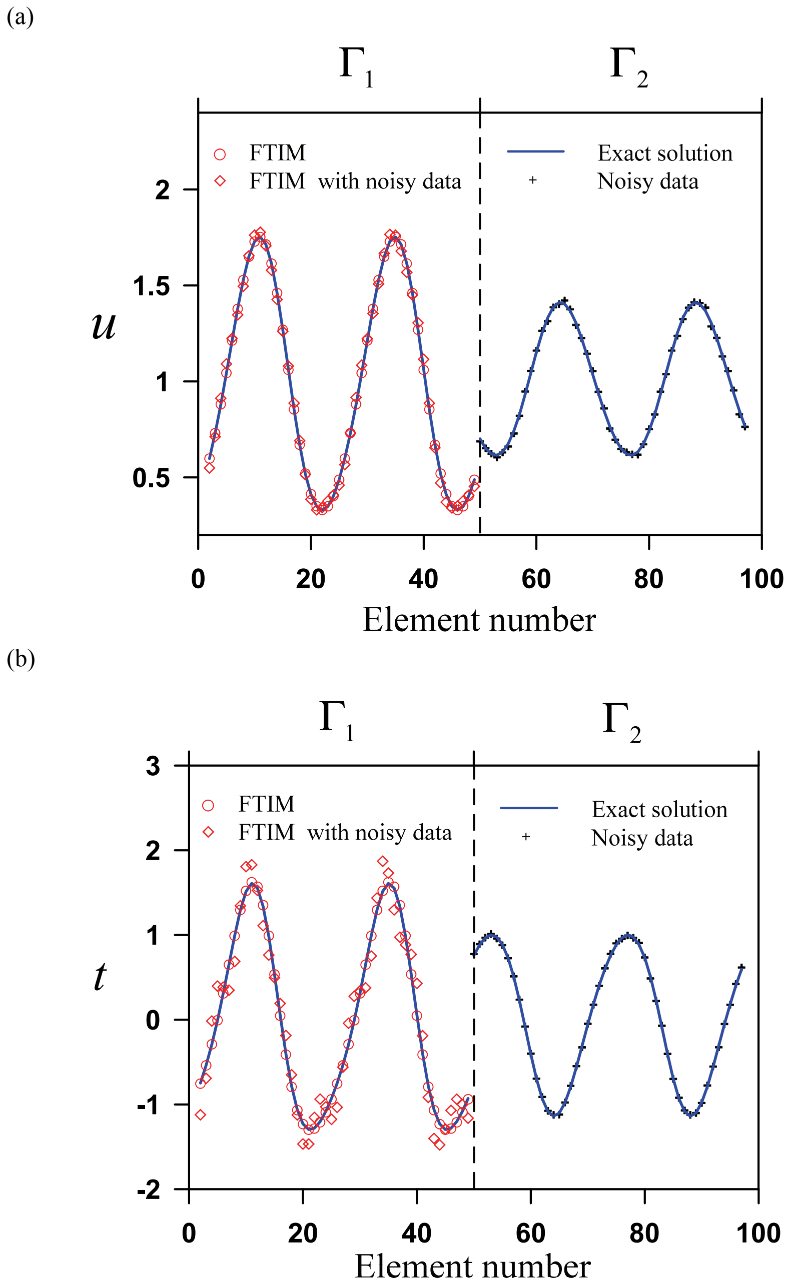


Figure 13: Recovery of boundary data for the annular region of case 2 using Cauchy data without noise and with noise level of 0.02: (a) potential; (b) normal derivative of potential.

Laplace equation. *Inverse Problems*, vol. 17, pp. 839–853.

Bourgeois, L. (2005): A mixed formulation of quasi-reversibility to solve the Cauchy problem for Laplace's equation. *Inverse Problems*, vol. 21, pp. 1087–1104.

Chang, J. R.; Yeih, W.; Shieh, M. H. (2001): On the modified Tikhonov's method for the Cauchy problem of the Laplace equation. *Journal of Marine Science and Technology*, vol. 9, pp. 113-121.

Cheng, A. H.-D.; Cabral, J. J. S. P. (2005): Direct solution of ill-posed boundary value problems by radial basis function collocation method. *Int. J. Numer. Meth. Engng.*, vol. 64, pp. 45–64.

Cheng, J.; Hon, Y. C.; Wei, T.; Yamamoto, M. (2001): Numerical computation of a Cauchy problem for Laplace's equation. *ZAMM Z. Angew. Math.Mech.*, vol. 81, pp. 665–674.

Engl, H. W.; Leitao, A. (2001): A main iterative regularization method for elliptic Cauchy problems. *Numer. Funct. Anal. Optim.*, vol. 22, pp. 861-884.

Hadamard, J. (1923): Lectures on Cauchy Problem in Linear Partial Differential Equations. Yale University Press, New Heavens.

Hairer, E.; Lubich, C.; Wanner, G. (2006): *Geometric numerical integration: structure-preserving algorithms for ordinary differential equations,* Springer.

Hào, D. N.; Lesnic, D. (2000): The Cauchy problem for Laplace's equation via the conjugate gradient method. *IMA J. Appl. Math.*, vol. 65, pp. 199–217.

Hansen, P. C. (1992): Analysis of discrete ill-posed problems by means of the L-cureve. *SIAM Review*, vol. 34, pp. 561-580.

Harris, S. D.; Mustata, R.; Elliott, L.; Ingham, D. B.; **Lesnic, D.** (2008): Numerical identification of the hydraulic conductivity of composite anisotropic materials. *CMES: Computer Modeling in Engineering & Sciences*, vol. 25, pp. 69-79.

Hon, Y. C.; Wei, T. (2001): Backus-Gilbert algorithm for the Cauchy problem of the Laplace equation. *Inverse Problems*, vol. 17, pp. 261–271.

Hong, H. K.; Chen, J. T. (1988): Derivations of integral equations in elasticity. *ASCE, J. Eng. Mech.*, vol.114, pp.1028-1044.

Huang, C. H.; Shih, C. C. (2007): An inverse problem in estimating simultaneously the time-dependent applied force and moment of an Euler-Bernoulli beam. *CMES: Computer Modeling in Engineering & Sciences*, vol. 21, pp. 239-254.

Koya, T.; Yeih, W.-C.; Mura, T. (1993): An inverse problem in elasticity with partially overprescribed boundary conditions, part II: numerical details. *ASME, Journal of Applied Mechanics*, vol. 60, pp. 601-606.

Kozlov, V. A.; Mazya, V. G.; Fomin, A. V. (1992): An iterative method for solving the Cauchy problem for elliptic equations. *Comput. Maths. Math. Phys.*, vol. 31, pp. 45-52.

Klibanov, M. V.; Santosa, F. (1991): A computational quasi-reversibility method for Cauchy problems for Laplace's equation. *SIAM J. Appl. Math.*, vol. 51, pp.1653–1675.

Ku, C.-Y.; Yeih W.-C.; Liu, C.-S.; Chi, C.-C. (2009): Applications of the fictitious time integration method using a new time-like function. *CMES: Computer Modeling in Engineering & Sciences*, vol. 43, no. 2, pp.173-190.

Leitao, A. (2000): An iterative method for solving elliptic Cauchy problems. *Numer. Func. Anal. Optim.*, vol. 21, 715-742.

Li, J. (2004): A radial basis meshless method for solving inverse boundary value problem. *Communications in Numerical Methods in Engineering*, vol. 20, pp. 51-60.

Ling, L.; Takeuchi, T. (2008): Boundary control for inverse Cauchy problems of the Laplace equations. *CMES: Computer Modeling in Engineering & Sciences*, vol. 29, pp.45-54.

Liu, C.-S. (2001): Cone of non-linear dynamic system and group preserving schemes. *Int. J. Non-Linear Mech.*, vol. 36, pp. 1047-1068.

Liu, C.-S. (2008a): A modified collocation Trefftz method for the inverse Cauchy problem of Laplace equation. *Engineering Analysis with Boundary Elements*, vol. 32, pp. 778-785.

Liu, C.-S. (2008b): A highly accurate MCTM for inverse Cauchy problems of Laplace equation in arbitrary plane domains. *CMES: Computer Modeling in Engineering & Sciences*, vol. 35, pp. 91-111.

Liu, C.-S. (2008c): A fictitious time integration method for two-dimensional quasi-linear elliptic boundary value problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 33, pp.179-198.

Liu, C.-S. (2008d): A time-marching algorithm for solving non-linear obstacle problems with the aid of an NCP-function. *CMC: Computers, Materials & Continua*, vol. 8, pp. 53-66.

Liu, C.-S. (2009a): A fictitious time integration method for the Burgers equation. *CMC: Computers, Materials & Continua*, vol. 9, pp. 229-252.

Liu, C.-S. (2009b): A fictitious time integration method for solving *m*-point boundary value problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 39, pp. 125-154.

Liu, C.-S.; Atluri, S. N. (2008a): A novel time integration method for solving a large system of non-linear algebraic equation. *CMES: Computer Modeling in Engineering & Sciences*, vol. 31, pp. 71-83.

Liu, C.-S.; Atluri, S. N. (2008b): A fictitious time integration method (FTIM) for solving mixed complementarity problems with applications to non-linear optimization . *CMES: Computer Modeling in Engineering & Sciences*, vol. 34, pp. 155-178.

Liu, C.-S.; Atluri, S. N. (2008c): A novel fictitious time integration method for solving the discretized inverse Sturm-Liouville problems, for specified eigenvalues. *CMES: Computer Modeling in Engineering & Sciences*, vol. 36, pp. 261-286.

Liu, C.-S.; Atluri, S. N. (2009): A fictitious time integration method for the numerical solution of the Fredholm integral equation and for numerical differentiation of noisy data, and its relation to the filter theory. *CMES: Computer Modeling in Engineering & Sciences*, vol. 41, pp. 243-262.

Mera, N. S.; Elliott L.; Ingham, D. B. (2006): The detection of super-elliptical inclusions in infrared computerised axial tomography. *CMES: Computer Modeling in Engineering & Sciences*, vol. 15, pp. 107-114.

Mera, N. S.; Elliott, L.; Ingham, D. B.; Lesnic, D. (2000): An iterative boundary element method for the solution of a Cauchy steady state. *CMES: Computer Modeling in Engineering & Sciences*, vol. 1, pp. 101-106.

Marin, L. (2008): The method of fundamental solutions for inverse problems associated with the steady-state heat conduction in the presence of sources. *CMES: Computer Modeling in Engineering & Sciences*, vol. 30, pp 99-122.

Marin, L.; Elliot, L.; Ingham, D. B.; Lesnic, D. (2002): Boundary element regularization methods for solving the Cauchy problem in linear elasticity. *Inverse Problems in Engng*, vol. 10, pp. 335-357.

Marin, L; Lesnic, D. (2002): Boundary element solution for the Cauchy problem in linear elasticity using singular value decomposition. *Comput. Methods Appl. Mech. Engrg.*, vol. 191, pp. 3257-3270.

Marin, L.; Power, H.; Bowtell, R. W.; Sanchez, C. C.; Becker, A. A.; Glover, P.; Jones, A. (2008): Boundary element method for an inverse problem in magnetic resonance imaging gradient coils. *CMES: Computer Modeling in Engineering & Sciences*, vol. 23, pp. 149-174.

Mustata, R.; Harris, S. D.; Elliott, L.; Lesnic, D.; Ingham, D. B. (2000): An inverse boundary element method for determining the hydraulic conductivity in anisotropic rocks. *CMES: Computer Modeling in Engineering & Sciences*, vol. 1, pp. 107-116.

Noroozi, S.; Sewell, P.; Vinney, J. (2006): The application of a hybrid inverse boundary element problem engine for the solution of potential problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 14, pp. 171-180.

Press, W. H.; Teukolsky, S. A.; Vetterling, W. T.; Flannery, B. P. (2007): Numerical recipes: the art of scientific computing, 3^{RD} edition, Cambridge University Press.

Ramm, A. G. (2007): Dynamical system methods for solving operator equations, Mathematics in Science and Engineering,vol. 208 (series editor: Chu, C.K.), Elsevier, Amsterdam, Netherlands.

Tikhonov, A. N.; Arsenin, V. Y. (1977): Solutions of ill-posed problems. V. H. Winston & Sons, Washington, D.C.: John Wiley & Sons, New York. Translated from the Russian, Preface by translation editor Fritz John, Scripta Series in Mathematics.

Yeih, W.-C.; Koya, T.; Mura, T. (1993): An inverse problem in elasticity with partially overprescribed boundary conditions, part I: theoretical approach. *ASME Journal of Applied Mechanics*, vol. 60, pp. 595-600.