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Vibration Analysis of Arbitrarily Shaped Membranes

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Abstract: In this paper a new numerical technique for problems of free vibrations of arbitrary shaped non-homogeneous membranes: $\nabla^2 w + k^2 q(\mathbf{x}) w = 0,\ \mathbf{x} \in \Omega \subset \mathscr{R}^2,\ B[w] = 0,\ \mathbf{x} \in \partial\Omega$ is presented. Homogeneous membranes of a complex form are considered as a particular case. The method is based on mathematically modeling of physical response of a system to excitation over a range of frequencies. The response amplitudes are then used to determine the resonant frequencies. Applying the method, one gets a sequence of boundary value problems (BVPs) depending on the spectral parameter k. The eigenvalues are sought as positions of the maxima of some norm of the solution. In the particular case of a homogeneous membrane the method of fundamental solutions (MFS) is proposed as an effective solver of such BVPs in domains of a complex geometry. For non-homogeneous membranes the combination of the finite difference method and conformal mapping is used as a solver of the BVPs. The results of the numerical experiments justifying the method are presented.

Keywords: Free vibration, eigenvalue problem, membrane, irregular domain, non-homogeneous membrane, conformal mapping, nonlinear eigenvalue problem.

1 Introduction

In the paper we deal with the problem of free vibrations of arbitrary shaped membranes. Homogeneous membranes and membranes with continuously varying properties are both under consideration. As a result, we deal with the following 2D eigenvalue problem:

$$\nabla^2 w + k^2 q(\mathbf{x}) w = 0,\ \mathbf{x} \in \Omega \subset \mathscr{R}^2,\ B[w] = 0,\ \mathbf{x} \in \partial\Omega. \quad (1)$$

Here, Ω is a simply or multiply connected domain with boundary $\partial\Omega$ and the density function $q > 0$ is smooth enough in Ω. The boundary operator $B[...]$ specifies

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the boundary conditions. The problem of free vibration is to find such real k for which there exist non-null functions w verifying (1). The corresponding resonant frequencies of the membrane are $\omega_i = k_i\sqrt{T/\rho}$, where T is the uniform tension per unit length in the membrane and ρ is the mass per unit area. The similar eigenvalue problems arise in many branches of engineering science, e.g. in the analysis of a hollow conducting waveguide.

In the particular case of the homogeneous membrane $q =$ const. of a simple geometry such as a rectangle, a circle or ellipse, exact solutions for the transverse vibration are available Strutt and Rayleigh (1945); Kinsler, Frey, Coppens, and Sanders (1982). However, for a membrane of a complex geometry, even in this case, only numerical solutions may be possible. Therefore, in resent years a lot of numerical techniques have been developed for free vibration analysis of membranes in many engineering applications. A general review of the dynamic aspects of membranes can be found in the review paper by Mazumdar (1975).

Nagaya (1978) proposed an analytical method to obtain higher order modes of arbitrarily shaped membranes by using the Fourier expansion method. Analytical solutions of the free vibration problems of arbitrarily shaped membranes have been investigated by Kang and Lee Kang and Lee (2000, 2004) using nondimensional dynamic influence functions. Radial basis function-based differential quadrature method was used for free vibration analysis of arbitrary shaped membrane by Wu, Shu, and Wang (2007).

An analysis of the free vibration of circular and annular membranes has been presented by Laura, Bambill, and Gutierrez (1997). The method of discrete singular convolution (DSC) (Wei (1999)) has been used recently for the vibration analysis of structures. DSC method has emerged as a new approach for numerical solutions of differential equations. This new method has a potential approach for computer realization as a wavelet collocation scheme (Wei (2000)). The use of the discrete singular convolution method for vibration analysis of beams, plates and shells (Wei (2001a,c,b); Wei, Zhao, and Xiang (2002)) has been proven to be quite satisfactory. The DSC method together with transforming an irregular physical domain into a rectangular domain is proposed by Civalek (2008). Free vibration analysis of plates and shells has been investigated by the same author in (Civalek (2007e,d,a,b,c, 2006)) and by Zhao, Wei, and Xiang (2002).

As it is mentioned above the problems like (1) with $q =$ const. also arise in the analysis of arbitrarily-shaped hollow conducting electromagnetic waveguide. To handle such problems the generalized differential quadrature method has been developed and applied for waveguide analysis by Shu, Wu, and Wang (2005).

In the past ten years, the meshless methods have gained a lot of interest on the part of investigators in the field of the numerical analysis of PDEs. A number of meshless methods has been proposed by this time. A more detailed information on this subject can be found in Zhu, Zhang, and Atluri (1998, 1999); Atluri and Zhu (2000); Atluri and Shen (2005, 2002); Atluri, Cho, and Kim (1999); Atluri, Han, and Rajendran (2004); Atluri, Liu, and Han (2006a,b); Atluri and Zhu (1998), Soric, Li, and Atluri (2004).

The method of fundamental solutions (MFS) also belongs to the group meshless methods and is very convenient in application to the problem of free vibrations. This technique is described by Karageorghis (2001a); Chen and I.L. Chen (2005) in more details. In the second paper there is a complete bibliography on the subject considered.

Passing on to the general case $q(\mathbf{x})$, it should be noted that the literature on the vibration of non-homogeneous membranes is not extensive. Masad (1996) solved the problem mentioned above by the finite difference method and the perturbation method. Laura, Rossi, and Gutierrez (1997) solved the same problem by the optimized Galerkin-Kantrovitch approach and the differential quadrature method. A closed form of the exact solution of non-homogeneous membrane with the density function which varies linearly with respect to an edge $q = c + dx$ is found by Wang (1998). An exact solution of non-homogeneous annular membrane with $q = c/r^2$ is also reported here. The fundamental frequencies of the circular membrane with the density which is a sinusoidal function of the radius are studied by Wang (1999). Four numerical techniques 1) the differential quadrature method; 2) the finite element technique; 3) the optimized and/or improved Rayleigh quotient method and 4) the Stodola Vianello iterative method are compared by Gutierrez, Laura, Bambil, Jederlinic, and Hodges (1998) in application to the problem of free vibration of non-homogeneous annular membrane. The three density functions are considered: $q = 1 + \alpha r^{\gamma}$, $\gamma = 1/2,\ 1,\ 3/2$. However, only the first two axisymmetric vibration modes are calculated here. In Jabareen and Eisenberge (2001) exact solutions for both the axisymmetric and antisymmetric modes of circular and annular membranes with any polynomial variation of the density are given using a power series solution. Ho and Chen (2000) introduced a hybrid method composed of differential transforms and the Kantorovitch method to solve the above-referenced problems. Filipich and Rosales Filipich and Rosales (2007) have studied the vibrations of membranes with a discontinuous density profile. Recently Amore (2008, 2009) has put forward a new numerical technique to study the vibrations of inhomogeneous membranes. This method utilizes a special kind of basis function - the Little Sinc functions (LSF) to obtain a discretization of a finite region of the two-dimensional plane. To deal with non-rectangular domains, Amoro suggests to combine his method with the conformal mapping technique.

The goal of the paper is to present a universal technique for both homogeneous and inhomogeneous membrane of an arbitrary form.

2 Main algorithm. Regularizing procedures

The method uses the direct-searching scheme when the natural frequencies are determined as the extremum of some function $F(k)$ of the spectral parameter. Usually the determinant $\det[A(k)]$ of the linear system which approximates the initial problem (1) is used as the function $F(k)$ (see, e.g., Karageorghis (2001b); Chen, Chen, and Lee (2005)). The technique of the presented paper is based on the following quite trivial statement. Let $w_e(\mathbf{x})$ be a smooth enough function defined in the solution domain below named as the *exciting field*. If the *response field* w_r is a solution of the boundary value problem

$$\nabla^2 w_r + k^2 q(\mathbf{x}) w_r = -\nabla^2 w_e - k^2 q(\mathbf{x}) w_e, \tag{2}$$

$$B[w_r] = -B[w_e], \tag{3}$$

then, the sum $w(\mathbf{x},k) = w_r + w_e$ satisfies the initial problem (1). Let $F(k)$ be some norm of the solution w. This function of k has extremum at the eigenvalues and, under some conditions described below, can be used for their determining.

Note that we can take *any* smooth enough function as the *exciting field* w_e. On the other hand, w_r depends on this choice because it should satisfy the BVP (2), (3).

So, generally, we do not impose any conditions on w_e. However, when $q = \text{const.}$ (homogeneous membrane), the exciting field can be chosen in such a way that the right hand side of (2) is equal to zero: $\nabla^2 w_e + k^2 q w_e = 0$. It can be taken in a simple analytic form, e.g., in the form of a traveling wave $w_e = \exp\left[ikq^{1/2}(\cos\theta x + \sin\theta y)\right]$. Here $0 \le \theta \le 2\pi$ is the angle of incidence. Note that in this case the *response field* w_r satisfies the homogeneous equation too.

$$\nabla^2 w_r + k^2 q w_r = 0$$

This PDE has the known fundamental solutions $\Phi(\mathbf{x}-\zeta,k) = H_0^{(1)}(kq^{1/2}|\mathbf{x}-\zeta|)$, where $H_0^{(1)}$ is the Hankel function. This admits of applying the method of fundamental solutions (MFS). Recently this technique has been applied for solving problems of free vibrations of beams, membranes and plates in Reutskiy (2005, 2006, 2007a,b) and for analyzing arbitrarily-shaped waveguide in Reutskiy (2008b). The same method was applied to nonlinear, and generalized Sturm-Liouville problems in Reutskiy (2008a). In Section 3, we present the algorithm in application to the problems of free vibrations of homogeneous membranes. Here we also perform comparison of results obtained by the present method and the data obtained by Kang and Lee (2000, 2004),Wu, Shu, and Wang (2007) and Civalek (2008).

Considering the general case of a non-homogeneous membrane we have no fundamental solutions and should use a volume method to solve the BVP (2), (3). When the solution domain Ω has a simple form of a square, the BVP can be solved most effectively. In Section 4, we show that the initial problem (1) in the irregular domain Ω can be transformed to problem with a new density function q_1 in the square domain. Here we demonstrate the applying of the conformal mapping (analytic and numerical) for this goal.

With each fixed k we get the solution $w_r(\mathbf{x},k)$ and so, $w(\mathbf{x},k) = w_r(\mathbf{x},k) + w_e(\mathbf{x})$. We introduce the norm

$$F(k) = \sqrt{\frac{1}{N_t}\sum_{i=1}^{N_t}\left[w(\mathbf{x}_i,k)\right]^2}, \tag{4}$$

where $\mathbf{x}_i, i = 1,...,N_t$ are the test points placed inside the solution domain. However, as it shown in Reutskiy (2005, 2006, 2007a,b), the method described above produces non smooth functions $F(k)$. To get a smooth response curve $F(k)$ the following two regularizing procedures are presented.

Using the $\varepsilon-$procedure we solve the following BVP

$$\begin{aligned} \nabla^2 w_r + k_\varepsilon^2 q(\mathbf{x}) w_r &= -\nabla^2 w_e - k^2 q(\mathbf{x}) w_e, \\ B[w_r] &= -B[w_e], \end{aligned} \tag{5}$$

where $k_\varepsilon^2 = k^2 + i\varepsilon$ and $\varepsilon > 0$ is a small parameter.

The $k-$procedure leads to the problem

$$\begin{aligned} \nabla^2 w_r + k^2 q(\mathbf{x}) w_r &= -\nabla^2 w_e - (k+\Delta k)^2 q(\mathbf{x}) w_e, \\ B[w_r] &= -B[w_e], \end{aligned} \tag{6}$$

with some regularizing parameter Δk. This technique is explained with more details in the papers listed above.

Having a smooth response curve, we apply the following simple algorithm. First, we localize these maxima of $F(k)$ on the intervals $[a_i, b_i]$. Next, we solve the univariate optimization problem inside each one. In particular, we apply Brent's method based on a combination of parabolic interpolation and bisection of the function near the extremum(see Press, Teukolsky, Vetterling, and Flannery (2002)).

3 Homogeneous membrane

Without the loss of generality we can take $q = 1$. So, instead of (2), (3) we get BVP

$$\nabla^2 w_r + k^2 w_r = -\nabla^2 w_e - k^2 w_e, \tag{7}$$

$$B[w_r] = -B[w_e]. \tag{8}$$

When the exciting field w_e is chosen in such a way that the right hand side of (7) is equal to zero:

$$\nabla^2 w_e + k^2 w_e = 0, \tag{9}$$

then the response field w_r also satisfies the homogeneous Helmholtz equation

$$\nabla^2 w_r + k^2 w_r = 0, \tag{10}$$

which can be solved by a boundary method. Note that we can take *any* solution of (9) as the *exciting field*, e.g. we can take it in the form of a traveling field or as a field of a point source. On the other hand, w_r depends on this choice because it should satisfy the boundary condition (8).

The 2D Helmholtz equation has the known fundamental solutions

$$\Phi(\mathbf{x} - \zeta, k) = H_0^{(1)}(k|\mathbf{x} - \zeta|), \tag{11}$$

where $H_0^{(1)}$ is the Hankel function. This admits of applying very effective meshless numerical techniques

Let Ω be a simply connected membranes with a smooth and piecewise-smooth boundary $\partial\Omega$ without singular points. We apply the MFS and look for the solution of the Helmholtz equation (10) in the form of the linear combination

$$w_r(\mathbf{x}) = \sum_{n=1}^{N} q_n \Phi(\mathbf{x} - \zeta_n, k), \tag{12}$$

Here q_n are free parameters of the problem and the source points ζ_n are placed outside the solution domain Ω. The free parameters are obtained from the boundary condition (8) as a solution of the collocation problem

$$\begin{aligned} B[w_r(\mathbf{x}_i)] &= \sum_{n=1}^{N} q_n B[\Phi(\mathbf{x}_i - \zeta_n, k)] = \\ &= -B[w_e(\mathbf{x}_i, k)], \mathbf{x}_i \in \partial\Omega. \end{aligned} \tag{13}$$

The collocation points $\mathbf{x}_i$ are uniformly distributed on the boundary. The number of the collocation points is taken twice as large as the number of unknowns N and the resulting linear system is solved by the procedure of the least squares. Then, having the solution $w_r(\mathbf{x})$ and so, $w(\mathbf{x}) = w_r(\mathbf{x}) + w_e(\mathbf{x})$, we compute the function $F(k)$ like (4). Varying k, we get the response curve and calculate the eigenvalues as positions of maxima. We take the exciting field in the form of the travelling wave

$$w_e(\mathbf{x},k) = \exp\left[ik\left(x\cos\upsilon + y\sin\upsilon\right)\right], \tag{14}$$

which satisfies (9) for any angle of incidence υ.

Using the $\varepsilon-$procedure (5), we replace the matrix terms $\Phi(\mathbf{x}_i-\zeta_n,k)$ with $\Phi(\mathbf{x}_i-\zeta_n,k_\varepsilon)$, $k_\varepsilon = \sqrt{k^2+ik\varepsilon}$. When the $k-$procedure (6) is used for smoothing the response curve $F(k)$, the exciting field $w_e(\mathbf{x}_i,k)$ in the right hand side of (13) is replaced by $w_e(\mathbf{x}_i, k+\Delta k)$.

Example 1. Trapezoidal membrane. Free vibration analysis of trapezoidal membrane (Fig. 1) is considered. The bases are $\overline{AB} = 2$, $\overline{CD} = 1$, the height $\overline{DE}$=1. The MFS source points are placed on the equidistant contour Γ. In Table 1 the results obtained by the present method are compared with the finite element solution Kang and Lee (2004) and with DSC solution Civalek (2008).

Table 1: Comparison of frequency values of the trapezoidal membrane ($a/b = 2.0; \beta = 70^\circ$; $\alpha = 60^\circ$) obtained by the present method, Kang and Lee (2004) (I) and Civalek (2008) (II).

i	$N = 61$	$N = 127$	$N = 257$	I	II
1	3.8107	3.81082	3.81081	3.81	3.82
2	5.2825	5.28243	5.28241	5.28	5.27
3	6.5722	6.57206	6.57209	6.57	6.56
4	7.0613	7.06114	7.06116	7.06	7.05
5	7.6055	7.60570	7.60575	7.60	7.60
6	8.7273	8.72743	8.72741	8.73	8.73
7	9.0311	9.03132	9.03135	9.03	-
8	9.7382	9.73783	9.73777	9.73	-
9	10.1111	10.11135	10.11140	-	-
10	10.4987	10.49895	10.49892	-	-

Example 2. Half-circle+triangle membrane. Let us consider the freely vibrations half-circle+triangle membrane to demonstrate the capability of the method presented in solving problems with arbitrarily shaped domains. The Dirichlet boundary condition is considered. The geometry of this membrane and the placement of

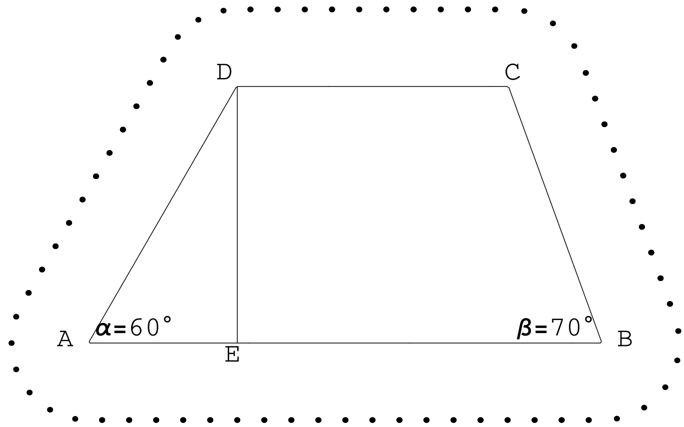

Figure 1: Trapezoidal membrane. Geometry and the placement of the MFS sources. $\overline{AB}=2$, $\overline{CD}=1$, height $\overline{DE}$=1.

the MFS sources are shown in Fig. 2. Table 2 shows the results for the first ten wave numbers that are computed by using three distributions of the MFS sources in the domain, i.e. $N = 63, 126$ and 254 sources. The $k-$procedure is used for smoothing with $\Delta k = 10^{-6}$. For this case there are no exact solutions. Only the numerical solutions obtained by Kang and Lee (2000) using the non-dimensional dynamic influence function and 24 boundary points, and the RBF results using 1697 nodal points (Wu et al. Wu, Shu, and Wang (2007)) are available for comparison. We can see that our results are in a very good agreement with Wu's and Kang's numerical results.

Example 3. Circular membrane with circular ridges. The next example is taken from Amore (2008). In Fig. 3 two regions of the plane are displayed: the left plot corresponds to a square $S = [-\pi/2, \pi/2] \times [-\pi/2, \pi/2]$ in the complex $z = x+iy$ plane; the right plot corresponds to a circular membrane with circular ridges in the $\zeta = u+iv$ plane. The function

$$\zeta = \tan\frac{1}{2}z, u(x,y) = \frac{\sin x}{\cos x + \cosh y}, \tag{15}$$

$$v(x,y) = \frac{\sinh y}{\cos x + \cosh y} \tag{16}$$

maps the first region into the second one. The placement of the MFS sources ζ_n is shown in Fig. 4. The $k-$procedure is used for smoothing with $\Delta k = 10^{-6}$. Some results of the computations are shown in Table 3. This problem also has no exact solutions and we compare our results with the computations presented in Amore

Table 2: Comparison of wave numbers of the half-circle+triangle membrane obtained by the present method, Wu, Shu, and Wang (2007) (I), and Kang and Lee (2000) (II).

i	$N = 126$	$N = 254$	I	II
1	2.71056	2.710598	2.7106	2.7097
2	4.23179	4.231901	4.2310	4.2279
3	4.35788	4.357874	4.3579	4.3579
4	5.57261	5.572708	5.5728	5.5649
5	5.93389	5.933924	5.9339	5.9336
6	6.11804	6.117920	6.1180	6.1159
7	7.01321	7.013076	7.0134	6.9974
8	7.18785	7.187887	7.1880	7.1868
9	7.76253	7.762378	–	–
10	7.836460	7.836545	–	–

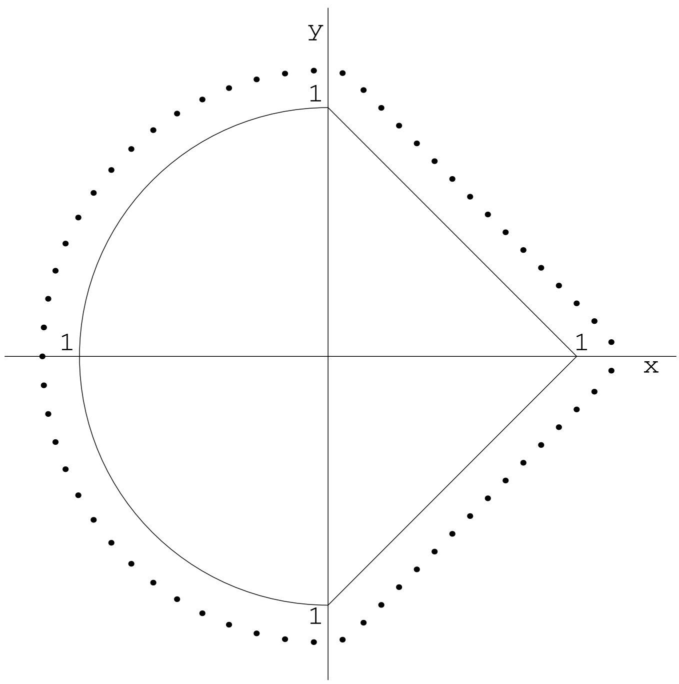

Figure 2: Half-circle+triangle membrane. Geometry and the placement of the MFS sources.

(2008).

Table 3: Comparison of wave numbers of the circular membrane with circular ridges obtained by the present method and Amore (2008).

i	$N=100$	N=200	Amore (2008)
1	7.5695769016	7.5695769016	7.569576902
2	15.2192802007	15.2192802007	–
3	22.1957603579	22.1957603579	–
4	29.1154334302	29.1154334233	29.11543343
5	29.1637723113	29.1637723043	-
6	44.8401269124	44.8401269228	44.84012692
7	46.1186093020	46.1186092926	–
8	46.3626601529	46.3626601361	–
9	51.2202116127	51.2202116265	–
10	66.4098459349	66.4098458917	-

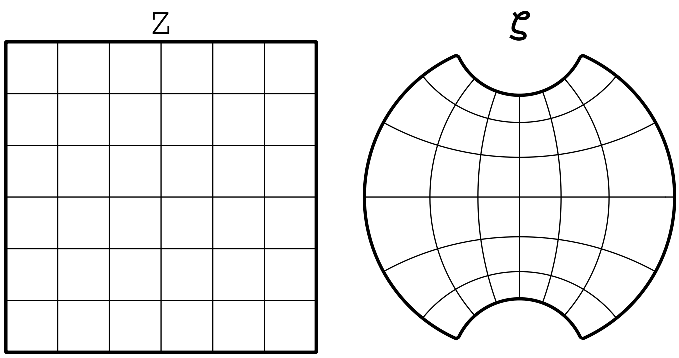

Figure 3: Circular membrane with circular ridges.

Note that in the case when the cross section Ω of the membrane has singularities like a small hole, a reentrant corner, or an abrupt change in the boundary conditions the MFS faces great difficulties when applied to such problems because it utilizes smooth basis functions. To extend the technique described above on to the case of the singular point instead of the linear combination (12) the response field w_r can be sought in the form

$$w_r = \sum_{n=1}^{N} q_n \Phi(\mathbf{x}-\zeta_n, k) + \sum_{s=1}^{S} \sum_{m=1}^{M} p_{s,m} \Psi_{s,m}(\mathbf{x}, k), \tag{17}$$

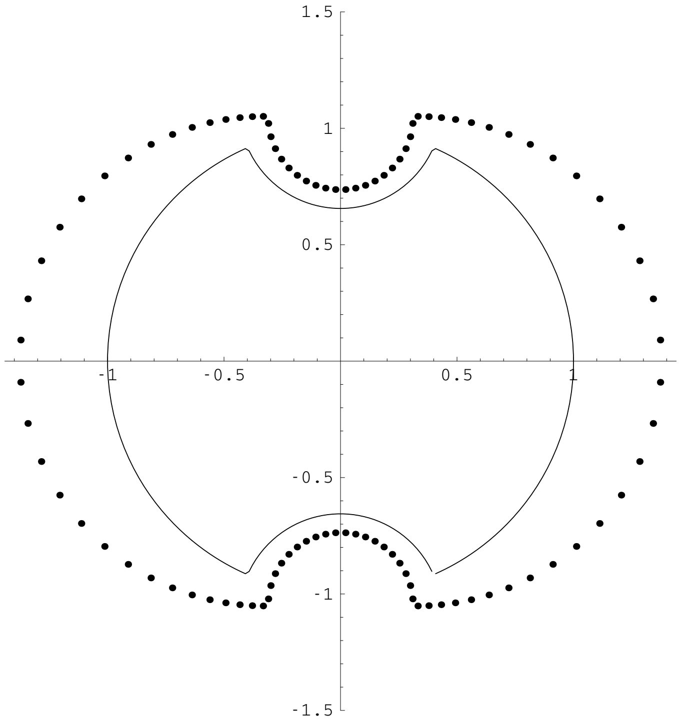

Figure 4: Circular membrane with circular ridges. The placement of the MFS sources.

where the functions $\Psi_{s,m}(\mathbf{x},k)$ correspond to the kind of the singularity and describe the local behavior of the eigenfunction near the singular points. This technique is described by Reutskiy (2006, 2009) with more details.

Example 4. "*Cracked beam" problem.* The eigenvalue problem for the rectangular membrane with the boundary conditions

$$\frac{\partial w}{\partial n} = 0, \text{ on AB} \cup \text{CD} \cup \text{DO and } w = 0 \text{ on OA} \cup \text{BC}$$

is shown in Fig 5. So, we have a problem with an abrupt change in the boundary conditions, which is studied by Li, Lu, Hu, Tsai, and Cheng (2006). We look for

Table 4: The eigenvalues of the "cracked beam" problem, I - results obtained by Li, Lu, Hu, Tsai, and Cheng (2006).

i	N/M=75/10	N/M=80/10	I
1	2.011697117	2.011697117	2.011697117212
2	3.293152635	3.293152635	3.293152635104
3	4.079864129	4.079864128	-
4	4.886314665	4.886314665	-
5	5.289378620	5.289378620	-
6	6.132689008	6.132689010	-
7	6.471915149	6.471915149	-
8	6.824620261	6.824620261	-
9	7.393971287	7.393971287	-
10	7.978125002	7.978125001	-

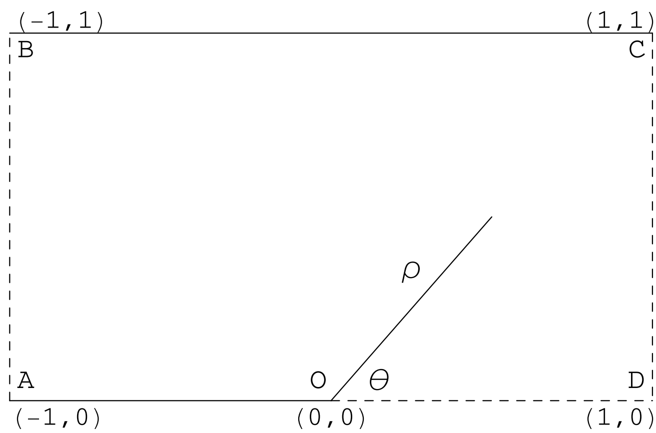

Figure 5: The cracked beam problem. The solid line corresponds to the Dirichlet boundary condition and the dashed line denotes Neumann's condition. Here (ρ,θ) is the local polar coordinate system with the origin at the singular point.

the MFS solution in the form:

$$\begin{aligned} w_r(\mathbf{x}) &= \sum_{n=1}^{N} q_n \Phi(\mathbf{x}-\zeta_n,k) + \\ &\quad + \sum_{j=1}^{M} p_j J_{j-1/2}(k\rho)\cos((j-1/2)\theta) \end{aligned}$$

with the Fourier–Bessel functions corresponding to the boundary conditions

$\partial \varphi_j / \partial n(\rho, 0) = \varphi_j(\rho, \pi) = 0.$

Here (ρ, θ) is the local polar coordinate system with the origin at the singular point. Some results of the calculations are presented in Table 4. Using the package Mathematica, the first two eigenvalues were calculated by Li, Lu, Hu, Tsai, and Cheng (2006) with 13 significant digits. They are shown in the last column of the table. One can see that the method presented gives the eigenvalues of the problem with 10 true digits.

4 Non-homogeneous membrane

Now we consider the general case of an inhomogeneous membrane. Taking into account (2), (3), we get the BVPs

$$\begin{aligned} \nabla^2 w_r + k_\varepsilon^2 q(\mathbf{x}) w_r &= -\nabla^2 w_e - k^2 q(\mathbf{x}) w_e, \\ B[w_r] &= -B[w_e] \end{aligned} \tag{18}$$

in the case of the $\varepsilon-$procedure and

$$\begin{aligned} \nabla^2 w_r + k^2 q(\mathbf{x}) w_r &= -\nabla^2 w_e - (k+\Delta k)^2 q(\mathbf{x}) w_e, \\ B[w_r] &= -B[w_e], \end{aligned} \tag{19}$$

when the $k-$procedure is applied for smoothing.

To solve these BVPs one should apply a volume method. In this section we apply the following FD scheme

$$\begin{aligned} 20w_{i,j} = 4(w_{i+1,j} + w_{i,j+1} + w_{i,j-1} + w_{i-1,j}) + \\ + w_{i+1,j+1} + w_{i+1,j-1} + w_{i-1,j+1} + w_{i-1,j-1} - 6h^2 g_{i,j} - \\ - 0.5h^2 (g_{i+1,j} + g_{i,j+1} + g_{i,j-1} + g_{i-1,j} - 4g_{i,j}) \end{aligned}$$

which approximates the equation $\nabla^2 w = g(x,y)$ with the fourth order Thom and Apelt (1961). Here we write (18) in the form: $\nabla^2 w_r = -k^2 q(\mathbf{x}) w_e - \nabla^2 w_e - k_\varepsilon^2 q(\mathbf{x}) w_r$ and denote the right hand side as $g(x,y)$. $h = 1/N$ is the mesh step; $w_{i,j} = w(x_i, y_j)$; $x_i = h(i-1)$, $y_j = h(j-1)$, $i, j = 1, ..., N+1$.

As a result, we write the system in the block tridiagonal form:

$$\widehat{\mathbf{A}}_j \mathbf{W}_{j+1} + \widehat{\mathbf{B}}_j \mathbf{W}_j + \widehat{\mathbf{C}}_j \mathbf{W}_{j-1} = \mathbf{F}_j,$$

where $\mathbf{W}_j = (w_{1,j}, w_{2,j}, ..., w_{N+1,j})^T$ are the vectors of the unknowns; $\mathbf{F}_j = (f_{1,j}, f_{2,j}, ..., f_{N+1,j})^T$ are the vectors of the right hand side;

$$\widehat{\mathbf{A}}_j,\ \widehat{\mathbf{B}}_j,\ \widehat{\mathbf{C}}_j$$

are $(N+1) \times (N+1)$ matrices. The system is solved by the sweep method.

The rest part of the algorithm is the same as the one described above. Having the nodal values $w_{r,i,j}(k)$ and $w_{e,i,j}(k)$, we compute the norm

$$F(k) = \sqrt{\frac{1}{N^2} \sum_{i,j=1}^{N} \left[w_{r,i,j}(k) + w_{e,i,j}(k) \right]^2}$$

and get the eigenvalues as positions of maxima with the help of Brent's procedure.

Example 5. In Table 5 we test a convergence of the eigenvalues of the membrane with the density function $q = 1 + 0.1\sin(\pi x)$. The number of mesh nodes varies from $N = 20$ to $N = 50$. The results are compared with the data placed in Amore (2009). This problem was considered in Reutskiy (2007b), but the 8 th eigenvalue is lost from the data presented there as it is indicated in Amore (2009).

Table 5: First 15 eigenvalues of the membrane with the density function $q = 1 + 0.1\sin(\pi x)$ using the FD method with different meshes. The fourth column are the results of Amore (2009) (I).

i	20×20	40×40	80×80	I
1	4.265406	4.265404	4.265404	4.265402726
2	6.743833	6.743886	6.743889	6.743887484
3	6.797264	6.797324	6.797328	6.797319723
4	8.597764	8.597665	8.597659	8.597648785
5	9.535881	9.536548	9.536589	9.536589305
6	9.624124	9.624822	9.624865	9.624841722
7	10.959121	10.959152	10.959154	10.95914134
8	10.974103	10.974154	10.974156	10.97412691
9	12.429505	12.432728	12.432927	12.43293737
10	12.550903	12.554229	12.554434	12.55438511
11	12.914303	12.913522	12.913472	12.91343471
12	13.590046	13.591746	13.591850	–
13	13.615493	13.617263	13.617371	–
14	15.219785	15.219031	15.218981	–
15	15.225763	15.225050	15.225003	–

4.1 The use of analytic conformal mapping

The method described above is more convenient for problems with rectangular geometry only. One way to extend it to arbitrary shaped domains is the use of the conformal mapping. This approach was suggested by Amore (2008) without detailed description of the algorithm.

Let the conformal mapping $\zeta = \Psi(z)$ transform the unit square $S = [0,1] \times [0,1]$ onto an irregular domain Ω. Here $z = x + iy$ and $\zeta = \xi + i\eta$ are complex values, $(x,y) \in S$, $(\xi,\eta) \in \Omega$, $\Psi : S \to \Omega$ and the functions $\xi(x,y)$, $\eta(x,y)$ satisfy the conditions

$$\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} = 0, \frac{\partial \xi}{\partial y} - \frac{\partial \eta}{\partial x} = 0 \tag{20}$$

If the function $u(\xi,\eta)$ satisfies the Helmholtz equation in Ω

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + k^2 p(\xi,\eta)\, u = 0,\ (\xi,\eta) \in \Omega, \tag{21}$$

then the function $w(x,y) = u(\xi(x,y), \eta(x,y))$ satisfies the equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + k^2 q(x,y)\, w = 0,\ (x,y) \in S, \tag{22}$$

where

$$q(x,y) = p(\xi(x,y), \eta(x,y))\, J_{x,y}(\xi(x,y), \eta(x,y))$$

$$J_{x,y}(\xi(x,y), \eta(x,y)) = \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x} = |d\Psi(z)/dz|^2$$

Indeed, it can be easily proven that

$$\begin{aligned} \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} &= \left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \\ &= J_{\xi,\eta}(x,y)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \end{aligned}$$

when (20) is fulfilled (see Schinzinger and Laura (1991)). Using a well-known property of Jacobians,

$$J_{\xi,\eta}(x,y) = J_{x,y}^{-1}(\xi,\eta),$$

one gets (22). The difference between $J_{\xi,\eta}(x,y)$ and $J_{x,y}(\xi,\eta)$ is that to compute $J_{\xi,\eta}$ we need the inverse mapping $\Psi^{-1} : \Omega \to S$ and to compute $J_{x,y}$ we use the given

$\Psi : S \to \Omega$. It can be also shown that: 1) if we have the Dirichlet condition $u = 0$ on the part $\partial\Omega_1$ of the boundary $\partial\Omega$, then we get the same Dirichlet condition $w = 0$ on the part ∂S_1 of the boundary ∂S which corresponds to $\partial\Omega_1$ in the conformal mapping $\zeta = \Psi(z)$; 2) the same is true for the Neumann condition.

Applying the method presented to (22), one gets the sequence of the BVPs like (18) or (19) which can be solved by the FD scheme described above.

Example 6. Consider the same *circular membrane with circular ridges* of *Example 4* and displayed in Fig. 4. The conformal mapping $\zeta = \Psi(z) = \tan\frac{1}{2}z$ transforms the square $[-\pi/2, \pi/2] \times [-\pi/2, \pi/2]$ onto Ω and has the Jacobian

$$J_{x,y}(\xi(x,y), \eta(x,y)) = \frac{1}{(\cos x + \cosh y)^2}$$

So, we have the non-homogeneous Helmholtz equation $\nabla^2 w + k^2 q(x,y) w = 0$ with the membrane density $q(x,y) = J_{x,y}(\xi(x,y), \eta(x,y))$ and $-\pi/2 \le x, y \le \pi/2$. Using the transform

$$x_1 = (x + \pi/2)/\pi, y_1 = (y + \pi/2)/\pi,$$

one gets the equation

$$\nabla^2 w + k^2 q_1(x_1, y_1) w = 0$$

in the unit square $S = [0,1] \times [0,1]$. Here $q_1(x_1, y_1) = \pi^2 q(x(x_1), y(y_1))$. The boundary conditions are the same as the ones in the original problem

$$w = 0 \text{ on } \partial S.$$

The data placed in Table 6 are obtained with the help of the $k-$procedure and $\Delta k = 10^{-3}$.

Table 6: Wave numbers of the circular membrane with circular ridges. The use of conformal mapping.

i	$N = 10$	$N = 20$	$N = 40$	$N = 80$
1	7.578	7.5700	7.56960	7.569578
2	15.279	15.2221	15.21944	15.219289
3	22.214	22.1966	22.19581	22.195763
4	29.307	29.1226	29.11580	29.115455
5	29.317	29.1704	29.16414	29.163794

Example 7. Consider the homogeneous circular membrane as the next example. The mapping

$$\xi + i\eta = \zeta = \Psi(z) = \frac{\mathrm{sn}\left[a(z-0.5), m\right] - i}{\mathrm{sn}\left[a(z-0.5), m\right] + i}, \; z = x + iy \tag{23}$$

transforms the unit square $S = [0,1] \times [0,1]$ onto the disk $D = (\xi, \eta): \; \xi^2 + \eta^2 \leq 1$. So, instead the homogeneous equation $\nabla^2 u + k^2 u = 0$ in D one gets the inhomogeneous equation $\nabla^2 w + k^2 q(x,y) w = 0$ in the unit square. Here

$$q(x,y) = \left| \frac{d\Psi(z)}{dz} \right|^2 = \\ = \left| \frac{2ia \, \mathrm{cn}\left[a(z-0.5), m\right] \; \mathrm{dn}\left[a(z-0.5), m\right]}{\left[\mathrm{sn}\left[a(z-0.5), m\right] + i\right]^2} \right|^2, \tag{24}$$

$m = 17 - 12\sqrt{2} \simeq 0.0294372515$, $a = \mathbf{K}(1-m)$ see Appendix. The data placed in Table 7 are obtained using the $k-$procedure with $\Delta k = 10^{-3}$.

Table 7: Homogeneous circular membrane. The use of conformal mapping.

i	k_{ex}	$N=20$	$N=40$	$N=80$
1	2.4048254	2.4053760	2.4048570	2.4048274
2	3.8317060	3.8347858	3.8318678	3.8317154
3	5.1356225	5.1406596	5.1358615	5.1356368
4	5.5200782	5.5238757	5.5201981	5.5200840
5	6.3801618	6.3901421	6.3806282	6.3801878
6	7.0155869	7.0246098	7.0156355	7.0155684
7	7.5883427	7.6049516	7.5891050	7.5883837
8	8.4172440	8.4292240	8.4165991	8.4172270
9	8.6537275	8.6575946	8.6522827	8.6536285
10	8.7714834	8.7923123	8.7725234	8.7715387

Example 8. Consider the inhomogeneous circular membrane with the density function $p(\xi, \eta) = 1 + \rho^2 = 1 + \xi^2 + \eta^2$ and with the Dirichlet boundary conditions. Using the same conformal mapping (23), one gets the Helmholtz equation with the density function

$$q(x,y) = \left| \frac{d\Psi(z)}{dz} \right|^2 \left[1 + \left(\xi(x,y)^2 + \eta(x,y)^2 \right) \right],$$

where $|d\Psi(z)/dz|^2$ is given in (24). The data placed in Table 8 are obtained using the $k-$procedure with $\Delta k = 10^{-6}$. The exact solution is taken from Jabareen and Eisenberge (2001).

Table 8: Inhomogeneous circular membrane with the density function $p = 1+\rho^2$. Conformal mapping. I - the results obtained by Jabareen and Eisenberge (2001).

i	I	$N = 40$	$N = 80$	$N = 120$
1	2.1735840	2.1737	2.173589	2.173585
2	3.3052516	3.3057	3.305277	3.305257
3	4.3064737	4.3071	4.306511	4.306481
4	4.8416247	4.8422	4.841654	4.841630
5	5.2469850	5.2482	5.247051	5.246996

4.2 The use of numerical conformal mapping

Only the conformal transforms given in the closed analytical form are used in the previous subsection. However, an approximate numerical transform can be used for this goal (see, e.g. Trefethen (1980, 1989); Driscoll and Trefethen (2002); Driscoll (2005) and references presented there for more details). Here we use the *SCPACK* package (see Trefethen (1980, 1989)) which contains routines for computing the Schwarz-Christoffel transformation $w = \Phi(\zeta)$, a conformal map that maps the interior of the unit disk in the complex plane ζ onto the interior of a polygon P in the $w-$plane. Then the combined conformal mapping $w = \Phi(\Psi(z))$ maps the unit square $S = [0,1]\times[0,1]$ onto P. The transform $\Psi(z)$ is given in analytic form by (23). The complex derivative of the combined transform can be computed as the product

$$\frac{d\Phi(\Psi(z))}{dz} = \frac{d\Phi(\zeta)}{d\zeta}\frac{d\Psi(z)}{dz}.$$

Here $d\Phi(\zeta)/d\zeta$ is computet by *SCPACK*$'s$ routines and $d\Psi(z)/dz$ is given in the analytic form (see (24)).

In Fig. 6 we show the map of the unit square $[0,1]\times[0,1]$ onto the equilateral triangle with the width equal to 1. The algorithm is the same as the one with analytic conformal map described in the previous subsection. The first five eigenvalues computed with different N are presented in Table 9.

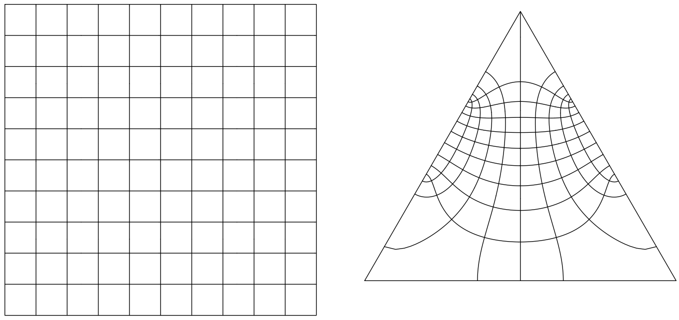

Figure 6: The conformal map obtained by the combination of the analytic map of the unit square onto the circle and the numerical conformal map of the circle onto the equilateral triangle.

Table 9: Equilateral triangle. The use of the numerical conformal map.

i	exact	$N = 50$	$N = 100$
1	7.2552	7.2558	7.2553
2	11.083	11.086	11.083
3	14.510	14.557	14.520
4	15.103	15.118	15.111
5	18.259	18.328	18.276

5 Concluding remarks

In this paper, a new numerical technique for the problem of free vibrations of inhomogeneous membranes with continuously varying properties is proposed. It can be regarded as a mathematical model of physical measurements, when a mechanical or acoustic system is excited by some source, and resonant frequencies can be determined using the growth of the amplitude of oscillations near these frequencies. It is convenient for determining some first eigenvalues of the system which are often of the most interest from the point of view of engineering applications.

Application of the method leads to the solution of a sequence of boundary value problems which depend on the spectral parameter k. Varying this parameter, one gets the eigenvalues as positions of maxima of the norm function $F(k)$. The growth of the amplitude of response near the eigenvalue is a sequence of the degeneracy of the matrix approximating the BVP under consideration. The key moment of the algorithm is the use of the special regularizing procedures which provide a smooth response curve and, as a sequence, provide a high precision in determining eigenvalues.

Note that the method described in the paper does not remove the need to discretize the problem and can be combined with many classical discretizations. In fact, it can be applied to the system of the algebraic equations which approximate the original problem. To illustrate this, let us consider the following simplest eigenvalue problem

$$\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mathbf{x} = \begin{bmatrix} \lambda - a & 0 \\ 0 & \lambda - b \end{bmatrix} \mathbf{x} = 0. \tag{25}$$

with the exact solution $\lambda_1 = a$, $\lambda_2 = b$.

In this situation the method presented in the paper is as follows. Let $\mathbf{y} = (y_1, y_2)^T \neq 0$ be an arbitrary $2-$vector. Consider the solution $\mathbf{x} = (x_1, x_2)^T$ of the system

$$\begin{bmatrix} \lambda - a & 0 \\ 0 & \lambda - b \end{bmatrix} \mathbf{x} = -\begin{bmatrix} \lambda - a & 0 \\ 0 & \lambda - b \end{bmatrix} \mathbf{y}. \tag{26}$$

The sum $\mathbf{z} = \mathbf{x} + \mathbf{y}$ satisfies (25). However, there is an unique solution $\mathbf{x} = -\mathbf{y}$ and $\mathbf{z} = \mathbf{0}$ for $\lambda \neq a, b$. So, we modernize (26) considering

$$\begin{bmatrix} \lambda + i\varepsilon - a & 0 \\ 0 & \lambda + i\varepsilon - b \end{bmatrix} \mathbf{x} = -\begin{bmatrix} \lambda - a & 0 \\ 0 & \lambda - b \end{bmatrix} \mathbf{y}, \tag{27}$$

where ε is a small parameter. System (27) has a unique solution for all real λ including $\lambda = a$ and $\lambda = b$.

$$(x_1, x_2)^T = -\left(\frac{\lambda - a}{\lambda + i\varepsilon - a} y_1, \frac{\lambda - b}{\lambda + i\varepsilon - b} y_2 \right)^T.$$

And the sum $\mathbf{z} = \mathbf{x} + \mathbf{y}$ is

$$\mathbf{z} = \left(\frac{i\varepsilon}{\lambda + i\varepsilon - a} y_1, \frac{i\varepsilon}{\lambda + i\varepsilon - b} y_2 \right)^T. \tag{28}$$

The norm of $\mathbf{z}$ is a function of λ

$$F(\lambda) = \varepsilon \sqrt{\frac{y_1^2}{(\lambda - a)^2 + \varepsilon^2} + \frac{y_2^2}{(\lambda - b)^2 + \varepsilon^2}}$$

The graphics shown in Fig. 7 correspond to $a = -1$, $b = 1$, $\mathbf{y} = (1, 1)^T$ and to the three values of the parameter ε: $\varepsilon = 10^{-1}$, $\varepsilon = 10^{-2}$, $\varepsilon = 10^{-3}$. The graph $F(\lambda)$ has maximums at the positions of the eigenvalues. The role of the parameter ε

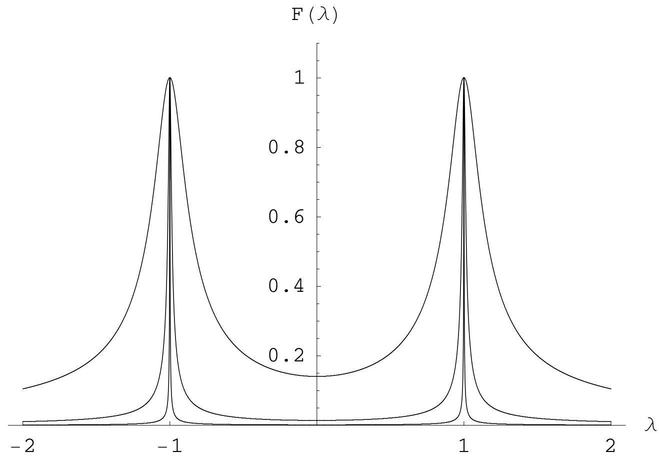

Figure 7: 2×2 linear eigenvalue problem. The function $F(\lambda)$ with different parameters ε.

is shown in Table 10. Here we place the absolute error in the calculations of the eigenvalue $\lambda = -1$ with different ε. The regularizing parameter ε coarsens the system. For a large ε the accuracy is not very high. When ε decreases, the peaks become sharper and more narrow and the accuracy in determining the eigenvalue increases. However, the graph of $F(\lambda)$ becomes delta shaped and the precision decreases when ε becomes very small. In many-dimensional case this leads to a dramatic increase of the errors.

Table 10: The absolute error in the calculations of the eigenvalue $\lambda = -1$ of the 2×2 problem with different values of the parameter ε.

ε	10^{-1}	10^{-2}	10^{-3}
e_a	1.2×10^{-5}	1.4×10^{-9}	2.7×10^{-13}
ε	10^{-4}	10^{-5}	10^{-10}
e_a	1.0×10^{-14}	7.0×10^{-12}	5.5×10^{-11}

Let us pay attention to the behaviour of the solution (28) near the solution, say $\lambda = a$. When $|\lambda - a| \to 0$, then $\mathbf{z} \to (y_1, 0)$, i.e. when the spectral parameter is close to the eigenvalue, then the sum $\mathbf{z} = \mathbf{x} + \mathbf{y}$ is close to the corresponding eigenvector.

Note that the same technique can be used for solving nonlinear eigenvalue problems (NEPs) $\mathbf{G}(\lambda)\mathbf{x} = 0$, where $\mathbf{G}(\lambda)$ is a $N \times N$ matrix-valued function of λ and λ and $\mathbf{x}$ are the sought eigenvalue and eigenvector, respectively (see Mehrmann and Voss (2004)). The problems considered in Section 3 belong to NEPs because the collocation matrices depend on the spectral parameter k in the nonlinear manner through the argument of the Bessel functions (see (11), (12), (13)). However, it is a particular and quite narrow group of NEPs with pure real spectra. The general NEPs with complex spectra can be solved in a similar way. This is the subject of further investigations.

6 Appendix

The elliptic integral

$$z = \int_0^{\zeta} \frac{dw}{\sqrt{(1-w^2)(1-mw^2)}}$$

transforms the half of the complex plane $Im(\zeta) \geq 0$ onto the rectangular $[-a/2, a/2] \times [0, b]$ in the complex plane z. Here

$$a = \int_0^1 \frac{dw}{\sqrt{(1-w^2)(1-mw^2)}} = 2\mathbf{K}(m), \; b = \mathbf{K}(1-m).$$

When m is a root of the equation

$$2\mathbf{K}(m) = \mathbf{K}(1-m),$$

the rectangular is the square $[-a/2, a/2] \times [0, a]$. From Table 17.3 of Abramowitz and Stegun (1964) it follows that $m = 0.0294372515$. Note that exact value $m = 17 - 12\sqrt{2}$ is given in Lavrentev and Shabat (1973).

So, the inverse mapping

$$\zeta = \mathrm{sn}(z, m)$$

transforms the square $[-a/2, a/2] \times [0, a]$ onto the upper half-plane $Im(\zeta) \geq 0$ and consequently

$$\zeta = \mathrm{sn}(a(z - 1/2), m)$$

maps the unit square $S = [0, 1] \times [0, 1]$ onto the upper half-plane $Im(\zeta) \geq 0$.

Thus the mapping

$$\zeta = \Psi(z) = \frac{\mathrm{sn}\left[a(z-0.5), m\right] - i}{\mathrm{sn}\left[a(z-0.5), m\right] + i}$$

transforms S onto the disk $D = \{(\xi, \eta) : \ |\zeta| = |\xi + i\eta| = 1\}$.
The derivative is

$$\frac{d\Psi(z)}{dz} = \frac{2ia\,\mathrm{cn}\left[a(z-0.5), m\right]\ \mathrm{dn}\left[a(z-0.5), m\right]}{\left[\mathrm{sn}\left[a(z-0.5), m\right] + i\right]^2}$$

see Abramowitz and Stegun (1964).

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