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Vibrations of In-Plane Non-Constant Inward and Outward Rotating Beams

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Abstract: In this study, the mathematical model of a non-constant rotating beam is established. It is an in-plane moving mass problem. Due to the effect of non-constant rotation, this model is composed of a governing differential equation with time-dependent coefficients and forcing term and three homogenous boundary conditions and one non-homogeneous boundary condition with time-dependent coefficients and forcing term. It is basically different to the system with constant rotation speed [Lin, 2008] and the linear moving beam system [Lin, 2009]. Obviously, a moving mass problem with time-dependent coefficients and forcing term is very complicated. A new solution method is here developed to derive the analytical solution for two systems with harmonic inward and outward rotations. First, using the balanced method the system with time-dependent coefficients is transformed into one with time-independent coefficients. Further, one derives the exact general solution of the transformed system which is composed of sixteen fundamental solutions and two particular ones. The exact fundamental solutions are derived by the Frobenius method. The effects of several parameters of a moving beam on the frequency spectrum are significant and investigated. It is found that increasing the excitation frequency greatly increases the tip amplitude. However, in a convention, there is no relation between the excitation frequency and the tip amplitude. Moreover, although the inertial force will induces the vibration during the root moving, the effect of tip mass on suppressing vibration is effective.

Keywords: non-constant rotation; rotating beam; analytical solution; vibration

1 Introduction

In engineering, the moving mass problem is important. The conveyed mass may be solid or fluid [Lin *et al.*, 2008[a]]. There are four kinds of moving solid mass problems. The first is the dynamic behavior of beam structures, such as bridges on

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railways, subjected to moving loads or masses. Mostly, a uniform beam is simply supported and carried a moving load [Fryba 1996; Nikkhoo et al., 2007]. The second is the vibration characteristics of a rotating shaft subjected to a moving load or mass [Gu and Cheng, 2004]. This model can simulate dynamic behavior of a ball screw and a nut moving along it, which are the key components of a feed drive system for a machine tool. The third is the axially moving beams problem. The belt drives, high-speed magnetic tapes and fiber winding are its typical examples [Lee and Jang, 2007]. The fourth is the transverse moving beam problem. It can be used to simulate a manipulator arm, a moving scanning probe and a transversely moving spindle [Gu and Piedboeuf, 2003; Brusa1 et al., 2009]. This mathematical model is different to the previous ones. The linear moving beam problem has been investigated by Lin [2009]. The model of the linear transverse moving beam is composed of a governing differential equation and a time-dependent boundary condition due to the tip mass inertia force. Moreover, the end static electric force [Lin, 2007] is used to suppress the overshoot. The dynamic positioning of a long-distance moving beam is investigated. Some literatures [Mindlin-Goodman, 1950; Lee and Lin, 1996 and 1998; Lin, 1998; Lin, 2002] investigated the system with the time-dependent boundary condition. Lee *et al.* (2008) studied the large static deflection of a beam with nonlinear boundary conditions. However, these studies are neither for a long-distance moving beam or the dynamic positioning. Further, the in-plane curvedly moving beam problem is studied here.

The model of the in-plane curvedly moving beam is composed of a governing differential equation with time and position-dependent coefficients and a forcing term and three homogenous boundary conditions and one non-homogeneous boundary condition with time-dependent coefficients and forcing term. It is well known that for the rotating beam model with constant rotation the coefficients of a governing differential equation and four boundary conditions are independent to time variable and these equations are homogeneous. Therefore, the model of the in-plane curvedly moving beam is basically different to the rotating beam with constant rotation speed [Lin, 2008; Lin *et al.*, 2008[b]] and the linear moving beam system [Lin, 2009]. So far, litter literature investigated the non-constant rotating beam problem. Yang and Tsao [1997] investigated the instability of a pretwisted blade under nonconstant rotating speed by using the perturbation method. The rotating speed is $\Omega = \Omega_0 + \varepsilon \cos(\Omega_p t)$ where Ω_0 is the constant rotating speed and ε is a small perturbation magnitude. Obviously, it is a limiting case of a nonconstant rotating beam.

In this study, the mathematical model of an in-plane curvedly moving beam is established. The analytical solution for this system is derived. The effects of several geometry and material parameters on the frequency spectrums of two beams with inward and outward rotation are investigated.

2 Governing equations and associated conditions

It is well known that the non-constant rotation of beam will induces the beam vibration. The geometries and coordinates of two beams with inward and outward rotation are shown in Figure 1. The vibration induces the error of dynamic positioning. Moreover, the harmonic excitation will result in the resonant phenomenon. In this study, the dynamic behaviors of non-constant inward and outward rotating beams are investigated. The comparison of their performances is made. For clarity, the mathematical model and solution method for the outward rotating case is derived firstly. Those for the inward rotating case are presented in Appendix.

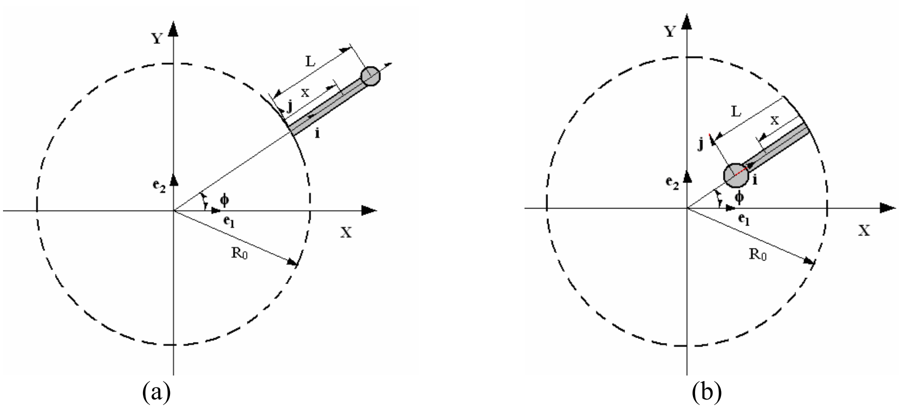


Figure 1: Geometry and coordinate system of a moving beam with a tip mass.

The governing differential equation of the outward rotating system is derived as follows:

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 W}{\partial x^2}\right]-\frac{\partial}{\partial x}\left[N\frac{\partial W}{\partial x}\right]+C\frac{\partial W}{\partial t}+m(x)\frac{\partial^2 W}{\partial t^2}-\left(\frac{d\varphi}{dt}\right)^2 m(x)W=p(x,t) \tag{1}$$

where $p(x,t) = -m(x)(x+R_0)d^2\varphi/dt^2$. C is the damping coefficient. The mass per unit length $m = \rho A_0$ where ρ and A_0 are the density and cross-sectional area of the beam. $W(x,t)$ is the flexural displacement, E is the Young's modulus. x is the coordinate along the beam, t is time and L is the length of the beam. I denotes the area moment of inertia. N is the centrifugal force. ϕ is the angle of rotation.

The associated boundary conditions are

At $x=0$:

$$W=0, \tag{2}$$

$$\frac{\partial W}{\partial x}=0 \tag{3}$$

At $x=L$:

$$\frac{\partial^2 W}{\partial x^2}=0, \tag{4}$$

$$-\frac{\partial}{\partial x}\left(EI\frac{\partial^2 W}{\partial x^2}\right)+N\frac{\partial W}{\partial x}+M_{tip}\frac{\partial^2 W}{\partial t^2}=-M_{tip}\left((L+R_0)\frac{d^2\varphi}{dt^2}-w(L,t)\left(\frac{d\varphi}{dt}\right)^2\right) \tag{5}$$

where M_{tip} is the tip mass. R_0 is the root radius. The centrifugal force is

$$N(x)-N(L)=\int_L^x M\left[-(R_0+x)\left(\frac{d\varphi}{dt}\right)^2-w(x,t)\frac{d^2\varphi}{dt^2}-2\frac{\partial w}{\partial t}\frac{d\varphi}{dt}\right]dx \tag{6}$$

Considering a small displacement, the centrifugal force becomes

$$N_x(x)=-\left(\frac{d\varphi}{dt}\right)^2\left[\int_L^x m(R_0+x)\,dx-M_{tip}(R_0+L)\right] \tag{7}$$

It should be noted that the coefficients are time-dependent. So far, no literature is devoted to derive the solution and study its physical phenomenon. However, if the rotating speed $d\phi/dt$ is constant, the coefficients are independent to the time variable and the system becomes the same as that given by Lin [1999].

In this study, the harmonic rotating movement is considered. The rotating angle is written as

$$\varphi=A\sin\Omega t, \tag{8}$$

where A and Ω is the amplitude and frequency of excitation. Without the loss of generality, a uniform beam with a tip mass is considered here. In terms of the following dimensionless quantities

$$c=\frac{CL^2}{\sqrt{EI\rho A_0}},\quad m_{tip}=\frac{M_{tip}}{\rho A_0 L},\quad r_0=\frac{R_0}{L}$$

$$w(\xi,\tau) = \frac{W(x,t)}{L}, \quad \xi = \frac{x}{L}, \quad \tau = \frac{t}{L^2}\sqrt{\frac{EI}{\rho A_0}}, \tag{9}$$

$$\omega = \Omega L^2 \sqrt{\frac{\rho A_0}{EI}},$$

the dimensionless governing equation is

$$\frac{\partial^4 w}{\partial \xi^4} - (A\omega \cos \omega\tau)^2 \frac{\partial}{\partial \xi}\left[n_x \frac{\partial w}{\partial \xi}\right] + c\frac{\partial w}{\partial \tau} + \frac{\partial^2 w}{\partial \tau^2} - (A\omega \cos \omega\tau)^2 w = (\xi + r_0)A\omega^2 \sin \omega\tau \tag{10}$$

where $n_x = r_0(1-\xi) + \frac{1}{2}\left(1-\xi^2\right) + m_{tip}(r_0+1)$.
The dimensionless boundary conditions are
At $\xi = 0$:

$$w = 0, \tag{11}$$

$$\frac{\partial w}{\partial \xi} = 0, \tag{12}$$

At $\xi = 1$:

$$\frac{\partial^2 w}{\partial \xi^2} = 0, \tag{13}$$

$$-\frac{\partial^3 w}{\partial \xi^3} + m_{tip}\left[(A\omega \cos \omega\tau)^2 \left[(1+r_0)\frac{\partial w}{\partial \xi} + w(1,\tau)\right] + \frac{\partial^2 w}{\partial \tau^2}\right] = m_{tip}(1+r_0)A\omega^2 \sin \omega\tau \tag{14}$$

3 Solution method

3.1 Transformed system

The solutions of the system is assumed to be

$$w(\xi,\tau) = \bar{W}(\xi)\sin(\varpi\tau - \varphi) = W_c(\xi)\cos \varpi\tau + W_s(\xi)\sin \varpi\tau, \tag{15}$$

where $\bar{W}(\xi) = \sqrt{W_c^2 + W_s^2}$, and $\phi(\xi) = \tan^{-1}(-W_c/W_s)$.

Substituting Eq. (15) into Eq. (10), one can obtain

$$\left[\frac{d^4W_c}{d\xi^4}+c\omega W_s(\xi)-\omega^2W_c(\xi)\right]\cos\omega\tau+\left[\frac{d^4W_s}{d\xi^4}-c\omega W_c(\xi)-\omega^2W_s(\xi)\right]\sin\omega\tau$$
$$-A^2\omega^2\left[\frac{d}{d\xi}\left[n_x\frac{dW_c}{d\xi}\right]+W_c(\xi)\right]\cos^3\omega\tau$$
$$-A^2\omega^2\left[\frac{d}{d\xi}\left[n_x\frac{dW_s}{d\xi}\right]+W_s(\xi)\right]\cos^2\omega\tau\sin\omega\tau=(\xi+r_0)A\omega^2\sin\omega\tau \quad (16)$$

Multiplying Eq. (16) by $\cos\omega\tau$ and integrating it from 0 to the period T, $2\pi/\omega$, Eq. (16) becomes

$$\frac{d^4W_c}{d\xi^4}-\frac{3A^2\omega^2}{4}\frac{d}{d\xi}\left[n_x\frac{dW_c}{d\xi}\right]-\omega^2\left(1+\frac{3A^2}{4}\right)W_c+c\omega W_s(\xi)=0 \quad (17)$$

where $n_x=r_0(1-\xi)+\frac{1}{2}\left(1-\xi^2\right)+m_{tip}(r_0+1)$.
Similarly, multiplying Eq. (16) by $\sin\omega\tau$ and integrating it from 0 to the period T, $2\pi/\omega$, Eq. (16) becomes

$$\frac{d^4W_s}{d\xi^4}-\frac{A^2\omega^2}{4}\frac{d}{d\xi}\left[n_x\frac{dW_s}{d\xi}\right]-\omega^2\left(1+\frac{A^2}{4}\right)W_s-c\omega W_c=(\xi+r_0)A\omega^2 \quad (18)$$

Substituting Eq. (15) into the boundary conditions (11-14), one obtains
At $\xi=0$:

$$W_c(0)=0 \quad (19a)$$

$$W_s(0)=0, \quad (19b)$$

$$\frac{dW_c(0)}{d\xi}=0, \quad (20a)$$

$$\frac{dW_s(0)}{d\xi}=0. \quad (20b)$$

At $\xi=1$:

$$\frac{d^2W_c(1)}{d\xi^2}=0, \quad (21a)$$

$$\frac{d^2W_s(1)}{d\xi^2}=0, \quad (21b)$$

$$-\left(\frac{d^3W_c}{d\xi^3}\cos\omega\tau+\frac{d^3W_s}{d\xi^3}\sin\omega\tau\right)-m_{tip}\omega^2\left(W_c\cos\omega\tau+W_s\sin\omega\tau\right)$$
$$+m_{tip}\left(1+r_0\right)\left(\frac{dW_c}{d\xi}\cos\omega\tau+\frac{dW_s}{d\xi}\sin\omega\tau\right)\left(A\omega\cos\omega\tau\right)^2$$
$$=-m_{tip}\left(-(1+r_0)A\omega^2\sin\omega\tau-\left(W_c\cos\omega\tau+W_s\sin\omega\tau\right)\left(A\omega\cos\omega\tau\right)^2\right) \quad (22)$$

Moreover, multiplying Eq. (22) by $\cos\omega\tau$ and integrating it from 0 to the period T, $2\pi/\omega$, Eq. (22) becomes

$$\frac{d^3W_c\left(1\right)}{d\xi^3}+\frac{3}{4}m_{tip}\left(1+r_0\right)A^2\omega^2\frac{dW_c}{d\xi}+\left(1+\frac{3}{4}A^2\right)m_{tip}\omega^2W_c\left(1\right)=0 \quad (23a)$$

Multiplying Eq. (22) by $\sin\omega\tau$ and integrating it from 0 to the period T, $2\pi/\omega$, Eq. (22) becomes

$$\frac{d^3W_s}{d\xi^3}+\left(1+\frac{1}{4}A^2\right)m_{tip}\omega^2W_s=-(1+r_0)A\omega^2m_{tip} \quad (23b)$$

So far, the transformed system is composed of the two coupled governing equations (17-18) and the eight boundary conditions (19-21, 23).

3.2 General solution

The general solution of the transformed governing equations (17-18) includes the homogenous and particular solutions as follows:

$$\begin{bmatrix}W_c(\xi)\\W_s(\xi)\end{bmatrix}=\sum_{i=1}^{8}C_i\begin{bmatrix}\bar{W}_{c,i}\left(\xi\right)\\\bar{W}_{s,i}\left(\xi\right)\end{bmatrix}+\begin{bmatrix}W_{pc}(\xi)\\W_{ps}(\xi)\end{bmatrix}, \quad (24)$$

where the eight linearly independent fundamental solutions $[\bar{W}_{c,i}\left(\xi\right) \quad \bar{W}_{s,i}\left(\xi\right)]^T$, i = 1, 2, ..., 8, of Eqs. (17-18) are chosen such that they satisfy the following normalization conditions at the origin of the coordinated system:

$$\begin{bmatrix} \bar{W}_{c,1} & \bar{W}_{c,2} & \bar{W}_{c,3} & \bar{W}_{c,4} & \bar{W}_{c,5} & \bar{W}_{c,6} & \bar{W}_{c,7} & \bar{W}_{c,8} \\ \bar{W}'_{c,1} & \bar{W}'_{c,2} & \bar{W}'_{c,3} & \bar{W}'_{c,4} & \bar{W}'_{c,5} & \bar{W}'_{c,6} & \bar{W}'_{c,7} & \bar{W}'_{c,8} \\ \bar{W}''_{c,1} & \bar{W}''_{c,2} & \bar{W}''_{c,3} & \bar{W}''_{c,4} & \bar{W}''_{c,5} & \bar{W}''_{c,6} & \bar{W}''_{c,7} & \bar{W}''_{c,8} \\ \bar{W}'''_{c,1} & \bar{W}'''_{c,2} & \bar{W}'''_{c,3} & \bar{W}'''_{c,4} & \bar{W}'''_{c,5} & \bar{W}'''_{c,6} & \bar{W}'''_{c,7} & \bar{W}'''_{c,8} \\ \bar{W}_{s,1} & \bar{W}_{s,2} & \bar{W}_{s,3} & \bar{W}_{s,4} & \bar{W}_{s,5} & \bar{W}_{s,6} & \bar{W}_{s,7} & \bar{W}_{s,8} \\ \bar{W}'_{s,1} & \bar{W}'_{s,2} & \bar{W}'_{s,3} & \bar{W}'_{s,4} & \bar{W}'_{s,5} & \bar{W}'_{s,6} & \bar{W}'_{s,7} & \bar{W}'_{s,8} \\ \bar{W}''_{s,1} & \bar{W}''_{s,2} & \bar{W}''_{s,3} & \bar{W}''_{s,4} & \bar{W}''_{s,5} & \bar{W}''_{s,6} & \bar{W}''_{s,7} & \bar{W}''_{s,8} \\ \bar{W}'''_{s,1} & \bar{W}'''_{s,2} & \bar{W}'''_{s,3} & \bar{W}'''_{s,4} & \bar{W}'''_{s,5} & \bar{W}'''_{s,6} & \bar{W}'''_{s,7} & \bar{W}'''_{s,8} \end{bmatrix}_{\xi=0} = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}, \quad (25)$$

where primes indicate differentiation with respect to the dimensionless spatial variable ξ. The particular solution is assume to be

$$\begin{bmatrix} W_{pc}(\xi) \\ W_{ps}(\xi) \end{bmatrix} = \begin{bmatrix} \alpha_0 + \alpha_1 \xi \\ \beta_0 + \beta_1 \xi \end{bmatrix} \quad (26)$$

Substituting (26) into the transformed governing equations (17-18), the coefficients of the particular solutions are derived

$$\alpha_0 = \frac{-\left[r_0 A\omega c + \frac{A^3\omega^3 r_0 c}{4(\omega^2+c^2)}\left(4+\frac{3A^2}{4}\right)\right]}{\left[\omega^2\left(1+\frac{A^2}{4}\right)\left(1+\frac{3A^2}{4}\right)+c^2\right]}, \quad \alpha_1 = \frac{-A\omega c}{(\omega^2+c^2)},$$

$$\beta_0 = \frac{3A^3\omega^2 r_0}{4(\omega^2+c^2)} + \frac{\omega}{c}\left(1+\frac{3A^2}{4}\right)\alpha_0, \quad \beta_1 = \alpha_1 \omega / c = \frac{-A\omega^2}{(\omega^2+c^2)}. \quad (27)$$

Further, substituting the general solution into the transformed boundary conditions (19-21, 23), the corresponding coefficients of the homogeneous solutions are obtained

$$C_1 = -\alpha_0, \quad C_2 = -\alpha_1, \quad C_5 = -\beta_0, \quad C_6 = -\beta_1,$$

and

$$\begin{bmatrix} \bar{W}''_{c3}(1) & \bar{W}''_{c4}(1) & \bar{W}''_{c7}(1) & \bar{W}''_{c8}(1) \\ \bar{W}''_{s3}(1) & \bar{W}''_{s4}(1) & \bar{W}''_{s7}(1) & \bar{W}''_{s8}(1) \\ \eta_3 & \eta_4 & \eta_7 & \eta_8 \\ \mu_3 & \mu_4 & \mu_7 & \mu_8 \end{bmatrix} \begin{bmatrix} C_3 \\ C_4 \\ C_7 \\ C_8 \end{bmatrix} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{bmatrix}, \tag{28a}$$

where

$$\varepsilon_1 = -\left(C_1 \frac{d^2\bar{W}_{c1}(1)}{d\xi^2} + C_2 \frac{d^2\bar{W}_{c2}(1)}{d\xi^2} + C_5 \frac{d^2\bar{W}_{c5}(1)}{d\xi^2} + C_6 \frac{d^2\bar{W}_{c6}(1)}{d\xi^2} \right),$$

$$\varepsilon_2 = -\left(C_1 \frac{d^2\bar{W}_{s1}(1)}{d\xi^2} + C_2 \frac{d^2\bar{W}_{s2}(1)}{d\xi^2} + C_5 \frac{d^2\bar{W}_{s5}(1)}{d\xi^2} + C_6 \frac{d^2\bar{W}_{s6}(1)}{d\xi^2} \right),$$

$$\eta_i = \frac{d^3\bar{W}_{ci}(1)}{d\xi^3} - \frac{3}{4} m_{tip} (1+r_0) A^2 \omega^2 \frac{d\bar{W}_{ci}(1)}{d\xi} + \left(1 + \frac{3}{4}A^2\right) m_{tip} \omega^2 \bar{W}_{ci}(1),$$

$$\varepsilon_3 = -(C_1\eta_1 + C_2\eta_2 + C_5\eta_5 + C_6\eta_6) + \frac{3}{4} m_{tip} (1+r_0) A^2 \omega^2 \alpha_1 - \left(1 + \frac{3}{4}A^2\right) m_{tip} \omega^2 (\alpha_0 + \alpha_1),$$

$$\mu_i = \frac{d^3\bar{W}_{si}(1)}{d\xi^3} - \frac{1}{4} m_{tip} (1+r_0) (A\omega)^2 \frac{d\bar{W}_{si}(1)}{d\xi} + \left(1 + \frac{1}{4}A^2\right) m_{tip} \omega^2 \bar{W}_{si}(1),$$

$$\varepsilon_4 = -(C_1\mu_1 + C_2\mu_2 + C_5\mu_5 + C_6\mu_6) + \frac{1}{4} m_{tip} (1+r_0) (A\omega)^2 \beta_1 - \left(1 + \frac{1}{4}A^2\right) m_{tip} \omega^2 (\beta_0 + \beta_1) - (1+r_0) A\omega^2 m_{tip}. \tag{28b}$$

3.3 Exact Fundamental Solutions

The transformed governing equations (17-18) can be rewritten as

$$\frac{d^4W_c}{d\xi^4} + \tilde{A}(\xi)\frac{d^2W_c}{d\xi^2} + \tilde{B}(\xi)\frac{dW_c}{d\xi} + \tilde{C}(\xi)W_c + \tilde{D}(\xi)W_s = 0,\ \xi \in (0,\ 1) \tag{29}$$

$$\frac{d^4W_s}{d\xi^4}+\bar{A}(\xi)\frac{d^2W_s}{d\xi^2}+\bar{B}(\xi)\frac{dW_s}{d\xi}+\bar{C}(\xi)W_s+\bar{D}(\xi)W_s=0 \quad \xi\in(0,\ 1) \tag{30}$$

where the coefficients are expressed in the polynomial form as follows:

$$\tilde{A}=\sum_{k=0}^{n_3}\tilde{a}_k\xi^k=-\frac{3A^2\omega^2}{4}\left[\left(r_0+\frac{1}{2}+m_{tip}(r_0+1)\right)-r_0\xi-\frac{1}{2}\xi^2\right],$$

$$\tilde{B}=\sum_{k=0}^{n_4}\tilde{b}_k\xi^k=\frac{3A^2\omega^2}{4}(r_0+\xi),$$

$$\tilde{C}=\sum_{k=0}^{n_5}\tilde{c}_k\xi^k=-\omega^2\left(1+\frac{3A^2}{4}\right),$$

$$\tilde{D}=\sum_{k=0}^{n_9}\tilde{d}_k\xi^k=c\omega; \tag{31a}$$

$$\bar{A}=\sum_{k=0}^{n_{12}}\bar{a}_k\xi^k=-\frac{A^2\omega^2}{4}\left[\left(r_0+\frac{1}{2}+m_{tip}(r_0+1)\right)-r_0\xi-\frac{1}{2}\xi^2\right]$$

$$\bar{B}=\sum_{k=0}^{n_{13}}\bar{b}_k\xi^k=\frac{A^2\omega^2}{4}(r_0+\xi)$$

$$\bar{C}=\sum_{k=0}^{n_{14}}\bar{c}_k\xi^k=-\omega^2\left(1+\frac{A^2}{4}\right)$$

$$\bar{D}=\sum_{k=0}^{n_{18}}\bar{d}_k\xi^k=-c\omega. \tag{31b}$$

One can assume that the eight fundamental solutions of Eqs. (29-30) are in the form of

$$\begin{bmatrix}\bar{W}_{c,j}\\ \bar{W}_{s,j}\end{bmatrix}=\sum_{k=0}^{\infty}\begin{bmatrix}\alpha_{j,k}\xi^k\\ \beta_{j,k}\xi^k\end{bmatrix},\quad j=1,2,\ldots,8 \tag{32}$$

and

For $\bar{W}_{c,1}$: $\alpha_{1,0}=1$, $\alpha_{1,1}=\alpha_{1,2}=\alpha_{1,3}=0$,
For $\bar{W}_{c,2}$: $\alpha_{2,1}=1$, $\alpha_{2,0}=\alpha_{12}=\alpha_{2,3}=0$,
For $\bar{W}_{c,3}$: $\alpha_{3,2}=1/2$, $\alpha_{3,0}=\alpha_{3,1}=\alpha_{3,3}=0$,
For $\bar{W}_{c,4}$: $\alpha_{4,3}=1/6$, $\alpha_{1,1}=\alpha_{1,2}=\alpha_{1,3}=0$,
For $\bar{W}_{c,5}$: $\alpha_{5,0}=\alpha_{5,1}=\alpha_{5,2}=\alpha_{5,3}=0$,
For $\bar{W}_{c,6}$: $\alpha_{6,0}=\alpha_{6,1}=\alpha_{6,2}=\alpha_{6,3}=0$,

For $\bar{W}_{c,7}$: $\alpha_{7,0} = \alpha_{7,1} = \alpha_{7,2} = \alpha_{7,3} = 0,$
For $\bar{W}_{c,8}$: $\alpha_{8,0} = \alpha_{8,1} = \alpha_{8,2} = \alpha_{8,3} = 0;$
For $\bar{W}_{s,1}$: $\beta_{1,0} = \beta_{1,1} = \beta_{1,2} = \beta_{1,3} = 0,$
For $\bar{W}_{s,2}$: $\beta_{2,0} = \beta_{2,1} = \beta_{2,2} = \beta_{2,3} = 0,$
For $\bar{W}_{s,3}$: $\beta_{3,0} = \beta_{3,1} = \beta_{3,2} = \beta_{3,3} = 0,$
For $\bar{W}_{s,4}$: $\beta_{4,0} = \beta_{4,1} = \beta_{4,2} = \beta_{4,3} = 0,$
For $\bar{W}_{s,5}$: $\beta_{5,0} = 1, \beta_{5,1} = \beta_{5,2} = \beta_{5,3} = 0,$
For $\bar{W}_{s,6}$: $\beta_{6,1} = 1, \beta_{6,0} = \beta_{6,2} = \beta_{6,3} = 0,$
For $\bar{W}_{s,7}$: $\beta_{7,2} = 1/2, \beta_{7,0} = \beta_{7,1} = \beta_{7,3} = 0,$
For $\bar{W}_{s,8}$: $\beta_{8,3} = 1/6, \beta_{8,0} = \beta_{8,1} = \beta_{8,2} = 0$ (33)

These eight fundamental solutions satisfy the normalization condition

$$\begin{bmatrix} \bar{W}_{c,1} & \bar{W}_{c,2} & \bar{W}_{c,3} & \bar{W}_{c,4} & \bar{W}_{c,5} & \bar{W}_{c,6} & \bar{W}_{c,7} & \bar{W}_{c,8} \\ \bar{W}'_{c,1} & \bar{W}'_{c,2} & \bar{W}'_{c,3} & \bar{W}'_{c,4} & \bar{W}'_{c,5} & \bar{W}'_{c,6} & \bar{W}'_{c,7} & \bar{W}'_{c,8} \\ \bar{W}''_{c,1} & \bar{W}''_{c,2} & \bar{W}''_{c,3} & \bar{W}''_{c,4} & \bar{W}''_{c,5} & \bar{W}''_{c,6} & \bar{W}''_{c,7} & \bar{W}''_{c,8} \\ \bar{W}'''_{c,1} & \bar{W}'''_{c,2} & \bar{W}'''_{c,3} & \bar{W}'''_{c,4} & \bar{W}'''_{c,5} & \bar{W}'''_{c,6} & \bar{W}'''_{c,7} & \bar{W}'''_{c,8} \\ \bar{W}_{s,1} & \bar{W}_{s,2} & \bar{W}_{s,3} & \bar{W}_{s,4} & \bar{W}_{s,5} & \bar{W}_{s,6} & \bar{W}_{s,7} & \bar{W}_{s,8} \\ \bar{W}'_{s,1} & \bar{W}'_{s,2} & \bar{W}'_{s,3} & \bar{W}'_{s,4} & \bar{W}'_{s,5} & \bar{W}'_{s,6} & \bar{W}'_{s,7} & \bar{W}'_{s,8} \\ \bar{W}''_{s,1} & \bar{W}''_{s,2} & \bar{W}''_{s,3} & \bar{W}''_{s,4} & \bar{W}''_{s,5} & \bar{W}''_{s,6} & \bar{W}''_{s,7} & \bar{W}''_{s,8} \\ \bar{W}'''_{s,1} & \bar{W}'''_{s,2} & \bar{W}'''_{s,3} & \bar{W}'''_{s,4} & \bar{W}'''_{s,5} & \bar{W}'''_{s,6} & \bar{W}'''_{s,7} & \bar{W}'''_{s,8} \end{bmatrix}_{\xi=0} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (34)$$

Upon substituting Eq. (32) into Eqs. (29-30) and collecting the coefficients of like powers of ξ, the following recurrence formula can be obtained:

$$\alpha_{j,m+4} = \frac{-1}{(m+4)(m+3)(m+2)(m+1)} \left[\tilde{d}_0 \beta_{j,m} + \sum_{k=0}^{m} \tilde{c}_k \alpha_{j,m-k} + \sum_{k=0}^{m} \tilde{b}_k (m-k+1) \alpha_{j,m-k+1} + \sum_{k=0}^{m} \tilde{a}_k (m-k+2)(m-k+1) \alpha_{j,m-k+2} \right], \quad (35)$$

$$\beta_{j,m+4} = \frac{-1}{(m+4)(m+3)(m+2)(m+1)}\left[\bar{d}_0\alpha_{j,m} + \sum_{k=0}^{m}\bar{c}_k\beta_{j,m-k} + \sum_{k=0}^{m}\bar{b}_k(m-k+1)\beta_{j,m-k+1} + \sum_{k=0}^{m}\bar{a}_k(m-k+2)(m-k+1)\beta_{j,m-k+2}\right]. \quad (36)$$

With these recurrence formulas, one can generate the eight exact normalized fundamental solutions of the differential equations (29-30).

4 Numerical results and discussion

The effects of several parameters on the frequency spectrums of two beams with inward and outward rotation are investigated here.

Figure 2 shows influence of the excitation frequency ω, the tip mass m_{tip} and the root radius r_0 on the tip amplitude $w(1)$of an inward rotating beam. For the case without the tip mass, m_{tip} = 0, the more the excitation frequency ω is, the more the amplitude of tip displacement $w(1)$. For the case with the root radius r_0 = 1, increasing the tip mass greatly decreases the amplitude of tip displacement. It is because the tip mass is exactly at the center of rotation. The effect of the moving acceleration on the tip mass is almost negligible. However, for the case with the root radius r_0 = 0.5, even for the tip mass of 1000, the amplitude of tip displacement is about 0.0005. Moreover, the amplitude is independent to the excitation frequency. Briefly, due to the inertial force the vibration occurs. However, the inertial force has also the effect of suppression of vibration.

Figure 3 shows influence of the excitation frequency ω, the tip mass m_{tip} and the root radius $r_0$on the amplitude of tip displacement of an outward rotating beam. For the case without the tip mass, m_{tip} = 0, the more the excitation frequency ω is, the more the tip amplitude $w(1)$. For the case with the root radius r_0 = 1, even for the tip mass of 1000, the amplitude of tip displacement is about 0.002. Moreover, the amplitude is independent to the excitation frequency. It is different to that of the inward rotating beam with the root radius r_0 = 1. Moreover, increasing the root radius obviously increases the tip amplitude $w(1)$.

Figures 4a and 4b show influence of the excitation frequency ω, the angle amplitude of excitation A and the root radius r_0 on the tip amplitude of an outward rotating beam. It is found that the more the excitation frequency ω, the angle amplitude of excitation A and the root radius r_0 are, greatly the more the tip amplitude $w(1)$.

Figure 5 shows influence of the excitation frequency ω and the angle amplitude of excitation A on the tip amplitude $w(1)$ of an inward rotating beam withr_0 = 1. It is found that increasing the angle amplitude of excitation slightly increases the tip amplitude. It is different to that of the case with $r_0 \neq 1$, shown in Figures 4.

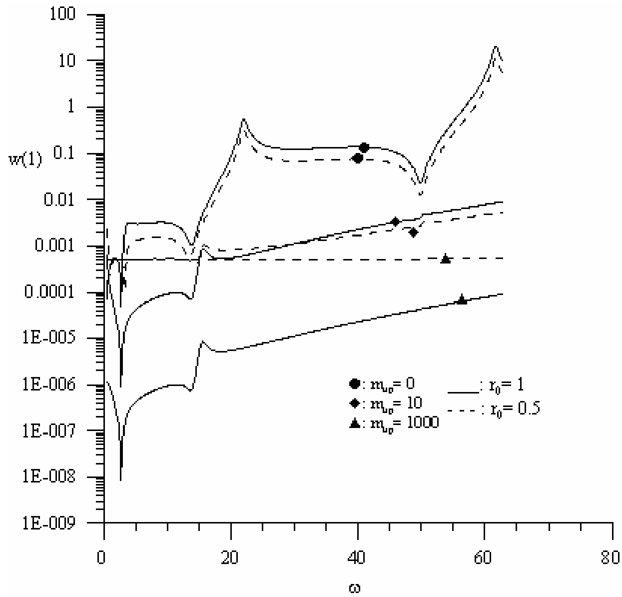


Figure 2: Influence of the excitation frequency ω, the tip mass m_{tip} and the root radius r_0 on the tip amplitude of an inward rotating beam.[A = 0.001, C = 1].

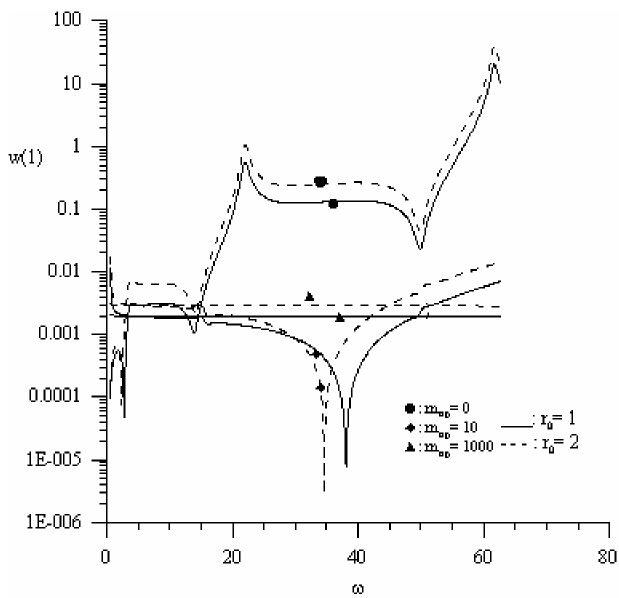
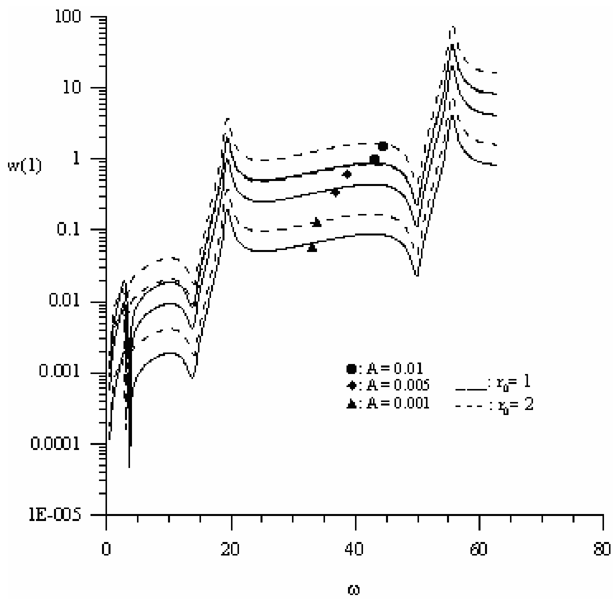
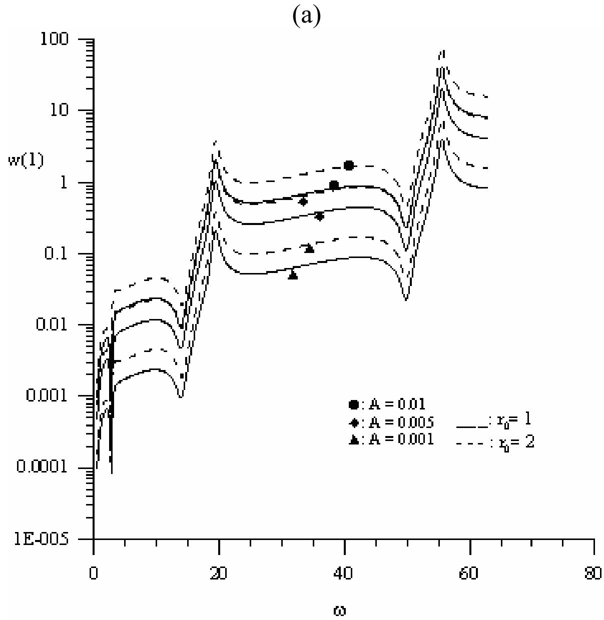


Figure 3: Influence of the excitation frequency ω, the tip mass m_{tip} and the root radius r_0 on the tip amplitude of an outward rotating beam. [A = 0.001, C = 1].



(a)



(b)

Figure 4: Influence of the excitation frequency ω, the amplitude of excitation angle A and the root radiusr_0 on the tip amplitude of (a) an outward rotating beam and (b) an inward rotating beam. [$m_{tip} = 0.1$, C = 1].

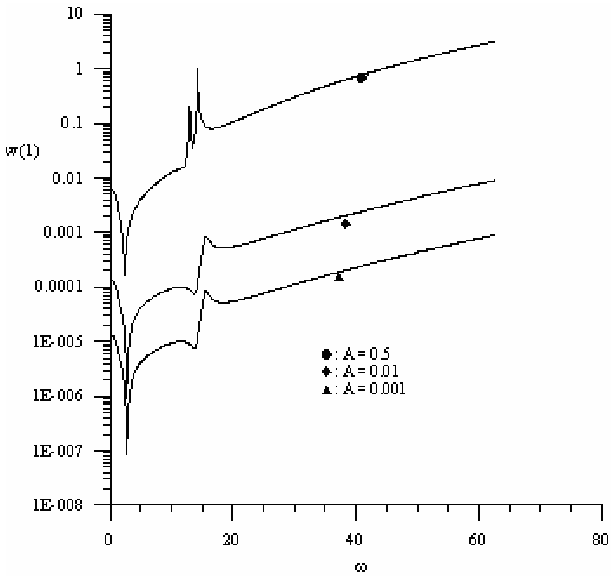


Figure 5: Influence of the excitation frequency ω and the amplitude of excitation angle A on the tip amplitude of an inward rotating beam. [r_0=1, $m_{tip} = 100$, C = 1].

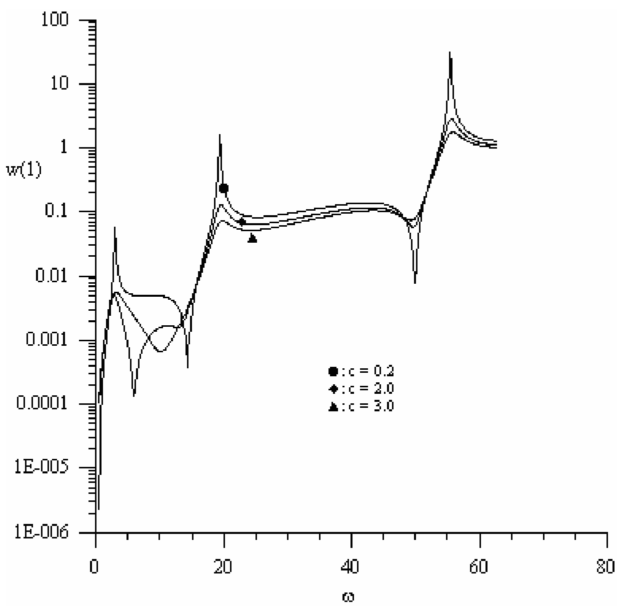


Figure 6: Influence of the excitation frequency ω and the damping constant c on the tip amplitude of an inward rotating beam. [r_0=1.5, $m_{tip} = 0.1$, A = 0.001].

The reason is that for the case with the root radius $r_0 = 1$, the effect of the moving acceleration on the tip mass is almost negligible.

Figure 6 shows influence of the excitation frequency ω and the damping coefficient c on the amplitude of tip displacement of an inward rotating beam. Increasing the damping constant c obviously decreases the tip amplitude at the resonance.

5 Conclusion

In this study, the mathematical models of the inward and outward rotating beam are established. Because the rotating speed is not constant, the coefficients of the equations are time and position-dependent. The analytical solution for this system is derived. Beside, the effects of several important parameters on the vibration of a moving beam are concluded as follows:

(1) The more the excitation frequency ω is, the more the tip amplitude $w(1)$.

(2) The more the angle amplitude A is, the more the tip amplitude $w(1)$ of the outward rotating case.

(3) The more the root radius r_0 is, the more the tip amplitude $w(1)$.

(4) Increasing the tip mass greatly decreases the amplitude of tip displacement especially for the inward rotating case with the root radius $r_0 = 1$.

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Appendix A: The system of inward rotating beam as shown in Figure 1b.

The transformed governing equations are

$$\frac{d^4W_c}{d\xi^4}-\frac{3A^2\omega^2}{4}\frac{d}{d\xi}\left[n_x\frac{dW_c}{d\xi}\right]-\omega^2\left(1+\frac{3A^2}{4}\right)W_c+c\omega W_s(\xi)=0 \tag{A1}$$

$$\frac{d^4W_s}{d\xi^4}-\frac{A^2\omega^2}{4}\frac{d}{d\xi}\left[n_x\frac{dW_s}{d\xi}\right]-\omega^2\left(1+\frac{A^2}{4}\right)W_s-c\omega W_c=(r_0-\xi)A\omega^2 \tag{A2}$$

where $n_x=r_0(\xi-1)-\frac{1}{2}\left(\xi^2-1\right)-m_{tip}(r_0-1)$.
The transformed boundary conditions are
At $\xi=0$:

$$W_c(0)=0, \tag{A3}$$

$$W_s(0)=0, \tag{A4}$$

$$\frac{dW_c(0)}{d\xi}=0, \tag{A5}$$

$$\frac{dW_s(0)}{d\xi}=0. \tag{A6}$$

At $\xi=1$:

$$\frac{d^2W_c(1)}{d\xi^2}=0, \tag{A7}$$

$$\frac{d^2W_s(1)}{d\xi^2}=0, \tag{A8}$$

$$\frac{d^3W_c(1)}{d\xi^3}+\frac{3}{4}m_{tip}(-1+r_0)A^2\omega^2\frac{dW_c}{d\xi}+\left(1+\frac{3}{4}A^2\right)m_{tip}\omega^2W_c(1)=0 \tag{A9}$$

$$\frac{d^3W_s}{d\xi^3}+\frac{1}{4}m_{tip}(-1+r_0)(A\omega)^2\frac{dW_s}{d\xi}+\left(1+\frac{1}{4}A^2\right)m_{tip}\omega^2W_s=-(r_0-1)A\omega^2m_{tip} \tag{A10}$$

The solution of the equations (9-10) is

$$\begin{bmatrix} W_c(\xi) \\ W_s(\xi) \end{bmatrix} = \sum_{i=1}^{8} C_i \begin{bmatrix} \bar{W}_{c,i}(\xi) \\ \bar{W}_{s,i}(\xi) \end{bmatrix} + \begin{bmatrix} W_{pc}(\xi) \\ W_{ps}(\xi) \end{bmatrix}, \tag{A11}$$

where the particular solutions are

$$\begin{bmatrix} W_{pc}(\xi) \\ W_{ps}(\xi) \end{bmatrix} = \begin{bmatrix} \alpha_0 + \alpha_1 \xi \\ \beta_0 + \beta_1 \xi \end{bmatrix} \tag{A12}$$

in which

$$\alpha_0 = \frac{-\left[r_0 A \omega c - \frac{A^3 \omega^3 r_0 c}{4(\omega^2+c^2)} \left(4 + \frac{3A^2}{4} \right) \right]}{\left[\omega^2 \left(1 + \frac{A^2}{4} \right) \left(1 + \frac{3A^2}{4} \right) + c^2 \right]}, \quad \alpha_1 = \frac{-A\omega c}{(\omega^2 + c^2)},$$

$$\beta_0 = \frac{-3A^3 \omega^2 r_0}{4(\omega^2 + c^2)} + \frac{\omega}{c} \left(1 + \frac{3A^2}{4} \right) \alpha_0, \quad \beta_1 = \alpha_1 \omega / c = \frac{-A\omega^2}{(\omega^2 + c^2)}. \tag{A13}$$

The coefficients of homogeneous solutions are

$$C_1 = -\alpha_0, \quad C_2 = -\alpha_1, \quad C_5 = -\beta_0, \quad C_6 = -\beta_1.$$

$$\begin{bmatrix} \bar{W}''_{c3}(1) & \bar{W}''_{c4}(1) & \bar{W}''_{c7}(1) & \bar{W}''_{c8}(1) \\ \bar{W}''_{s3}(1) & \bar{W}''_{s4}(1) & \bar{W}''_{s7}(1) & \bar{W}''_{s8}(1) \\ \eta_3 & \eta_4 & \eta_7 & \eta_8 \\ \mu_3 & \mu_4 & \mu_7 & \mu_8 \end{bmatrix} \begin{bmatrix} C_3 \\ C_4 \\ C_7 \\ C_8 \end{bmatrix} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{bmatrix}, \tag{A14}$$

where

$$\varepsilon_1 = -\left(C_1 \frac{d^2 \bar{W}_{c1}(1)}{d\xi^2} + C_2 \frac{d^2 \bar{W}_{c2}(1)}{d\xi^2} + C_5 \frac{d^2 \bar{W}_{c5}(1)}{d\xi^2} + C_6 \frac{d^2 \bar{W}_{c6}(1)}{d\xi^2} \right),$$

$$\varepsilon_2 = -\left(C_1 \frac{d^2 \bar{W}_{s1}(1)}{d\xi^2} + C_2 \frac{d^2 \bar{W}_{s2}(1)}{d\xi^2} + C_5 \frac{d^2 \bar{W}_{s5}(1)}{d\xi^2} + C_6 \frac{d^2 \bar{W}_{s6}(1)}{d\xi^2} \right),$$

$$\eta_i = \frac{d^3 \bar{W}_{ci}(1)}{d\xi^3} + \frac{3}{4} m_{tip} (-1 + r_0) A^2 \omega^2 \frac{d\bar{W}_{ci}(1)}{d\xi} + \left(1 + \frac{3}{4} A^2 \right) m_{tip} \omega^2 \bar{W}_{ci}(1),$$

$$\varepsilon_3 = -(C_1 \eta_1 + C_2 \eta_2 + C_5 \eta_5 + C_6 \eta_6) - \left(1 + \frac{3}{4} A^2 \right) m_{tip} \omega^2 (\alpha_0 + \alpha_1)$$
$$- \frac{3}{4} m_{tip} (-1 + r_0) A^2 \omega^2 \alpha_1,$$

$$\mu_i = \frac{d^3\bar{W}_{si}(1)}{d\xi^3} + \frac{1}{4}m_{tip}\left(-1+r_0\right)\left(A\omega\right)^2\frac{d\bar{W}_{si}(1)}{d\xi} + \left(1+\frac{1}{4}A^2\right)m_{tip}\omega^2\bar{W}_{si}(1),$$

$$\begin{aligned}\varepsilon_4 = &-\left(C_1\mu_1+C_2\mu_2+C_5\mu_5+C_6\mu_6\right) - \frac{1}{4}m_{tip}\left(-1+r_0\right)\left(A\omega\right)^2\beta_1 \\ &-\left(1+\frac{1}{4}A^2\right)m_{tip}\omega^2\left(\beta_0+\beta_1\right) - (r_0-1)A\omega^2 m_{tip}.\end{aligned} \tag{A15}$$

The homogeneous differential equations of Eqs. (A1-A2) are

$$\begin{aligned}&\frac{d^4W_c}{d\xi^4} - \frac{3A^2\omega^2}{4}\left[\left(-r_0+\frac{1}{2}-m_{tip}(r_0-1)\right)+r_0\xi-\frac{1}{2}\xi^2\right]\frac{d^2W_c}{d\xi^2} \\ &\qquad -\frac{3A^2\omega^2}{4}\left(r_0-\xi\right)\frac{dW_c}{d\xi} - \omega^2\left(1+\frac{3A^2}{4}\right)W_c + c\omega W_s\left(\xi\right) = 0,\end{aligned} \tag{A16}$$

$$\begin{aligned}&\frac{d^4W_s}{d\xi^4} - \frac{A^2\omega^2}{4}\left[\left(-r_0+\frac{1}{2}-m_{tip}(r_0-1)\right)+r_0\xi-\frac{1}{2}\xi^2\right]\frac{d^2W_s}{d\xi^2} \\ &\qquad -\frac{A^2\omega^2}{4}\left(r_0-\xi\right)\frac{dW_s}{d\xi} - \omega^2\left(1+\frac{A^2}{4}\right)W_s - c\omega W_c = 0.\end{aligned} \tag{A17}$$

The exact fundamental solutions of (A16-A17) can be derived via the recurrence formula (35-36).