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On the Approximation Methods for the Solution of a Coefficient Inverse Problem for a Transport-like Equation

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Abstract: We present the solvability of a two space dimensional coefficient inverse problem for a transport-like equation and investigate the approximate solution of this problem with the use of centered difference formulas and a symbolic approximation method. Since this inverse problem is overdetermined, which is the main difficulty in studying of its solvability, it is replaced by a related determined one by using some extension of the class of unknown functions.

Keywords: Coefficient Inverse Problem, Transport-like Equation, Finite Differences, Symbolic Approximation.

1 Introduction and the Statement of the Problem

Let $\Omega = \{(x,\varphi) : x \in D \subset \mathbb{R}^2,\ \varphi \in (0,2\pi),\ \partial D \in C^3\}$ be a bounded domain and in the domain Ω, consider the transport-like equation

$$u_{x_1}\cos\varphi + u_{x_2}\sin\varphi + u_\varphi K(x,\varphi) - \mu(x)u = 0, \tag{1}$$

where $K(x,\varphi) = f_2(x)\cos\varphi - f_1(x)\sin\varphi$ (for the explanation of why the function K is taken in this form, see [Amirov, Yildiz, and Ustaoglu (2009)] and the references therein). We study the solvability and approximation methods for the solution of the following coefficient inverse problem, where the data for the solution of equation (1) are specified on a part of the boundary of the domain $\Omega : \Gamma_1 = \partial D \times (0,2\pi)$.

Problem 1 *Given the function K, determine a pair of functions (u,μ) from the equation (1), provided that $u(x,\varphi) > 0$, $u(x,\varphi)$ is 2π-periodic in φ and the trace of $u(x,\varphi)$ is known on Γ_1, i.e. $u|_{\Gamma_1} = u_0$.*

We construct two solution algorithms for the approximate solution of this problem with the use of the centered difference formulas and a symbolic approximation

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method. Computational feasibility of these algorithms is presented by some numerical experiments. The solvability and numerical solution of Problem 1 have not been investigated previously, and this work presents two approximation methods to solve this coefficient inverse problem. The studies on the numerical solution methods of inverse problems for differential equations is important and there is an interest to develop new feasible methods (see, e.g. [Beilina and Klibanov (2008); Huang and Shih (2007); Ling and Atluri (2006); Ling and Takeuchi (2008); Liu (2008, 2009); Liu and Atluri (2008); Marin (2008); Marin et al (2008)]). Moreover, inverse problems for transport equations are of importance in applications and used in many measurement problems, objectives of which are to determine the conditions on the boundary or the scattering and absorption properties or the measure of the medium, and have variety of applications in theory of nuclear reactors, geophysical imaging and medical imaging such as tomography, etc. (see, e.g. [Anikonov, Kovtanyuk, and Prokhorov (2002); Anikonov (2001); Case and Zweifel (1967); Isakov (2006); Li et al (2009); Natterer (1986); Stefanov and Uhlmann (2003)]). The transport equations are used in the study of problems involving the propagation of particles within a medium or vacuum and governs diffusion processes such as scattering of light, near infrared lights, which has important applications in imaging, etc. (see, e.g. [Anikonov, Kovtanyuk, and Prokhorov (2002); Huang et al (2009); Mai-Cao and Tran-Cong (2008); Natterer (1986)]).

Here, we have to note that the coefficient μ in equation (1) depends only on x and hence Problem 1 is overdetermined (the term overdeterminacy is explained in Section 1.1). This fact is the main difficulty in studying of its solvability, so we propose a way to overcome this difficulty for Problem 1 (see Section 1.1 and 2).

Since $u > 0$ in Ω, we divide the equation (1) by $u(x,\varphi)$ and introduce a new unknown function $\upsilon = \ln u$, so we obtain the following first order differential equation in Ω

$$L\upsilon \equiv \upsilon_{x_1}\cos\varphi + \upsilon_{x_2}\sin\varphi + \upsilon_{\varphi}K(x,\varphi) = \mu(x). \tag{2}$$

Therefore, we can reduce Problem 1 to the following inverse problem of finding the right-hand side of the equation (2).

Problem 2 *Given the function K, determine a pair of functions (υ,μ) from the equation (2), provided that $\upsilon(x,\varphi)$ is 2π-periodic in φ and the trace of $\upsilon(x,\varphi)$ is known on Γ_1, i.e. $\upsilon|_{\Gamma_1} = \upsilon_0$, where $\upsilon_0 = \ln u_0$.*

Remark 1 *Problem 2 is related to a certain problem of integral geometry (see [Amirov, Yildiz, and Ustaoglu (2009)]). Assume that a family of regular curves is given by curvature such that curvature of the curve passing from each point x*

$\in D$, *in any direction* $\Phi = (\cos\varphi, \sin\varphi)$ $(\varphi \in (0, 2\pi))$ *is* $K(x,\varphi) = f_2(x)\cos\varphi - f_1(x)\sin\varphi$ *and for any point* $x \in D$ *and any direction* $\Phi = (\cos\varphi, \sin\varphi)$*, there exists a unique sufficiently smooth curve passing through the point x in the direction* Φ *with endpoints on the boundary of D. Suppose the lengths of these curves in D are upper-bounded by the same constant and denote the family of these curves by* $\{\Gamma\}$*. Then the problem of integral geometry (IGP) is formulated as follows: "Find a function* $\mu(x)$ *in a domain D from the integrals of* $\mu(x)$ *along the curves of the family* $\{\Gamma\}$*."*

The uniqueness of the solution of a problem of integral geometry by reducing it to an equivalent inverse problem for the differential equation was firstly proved in [Lavrent'ev and Anikonov (1967)]. Reduction of an integral geometry problem for general curve class to an inverse problem for transport equation and the solvability of this problem was investigated by Amirov in [Amirov (1986)] and the solvability of the IGP is proved via solvability of Problem 2 in [Amirov, Yildiz, and Ustaoglu (2009)]. Historically, the Radon transform (see [Radon (1917)]) is assumed to be the basis of the integral geometry problems and particularly in the second half of the last century, theory of these problems is developed by several researchers (see [Amirov (2001)] for a reference list) and from the practical point of view, problems of integral geometry have many important applications, especially in geophysics, astronomy and medicine (see, e.g. [Amirov (2001); Lavrent'ev, Romanov, and Shishatskii (1986)]). In particular, the reconstruction of a function from its line or plane integrals is the main problem in the computerized tomography and some of the applications related with the computerized tomography can be seen in problems of seismology, flaw detection, microscopy, X-ray tomography, etc. (see, e.g. [Natterer (1986)]). Moreover, integral geometry problems are closely interrelated with the inverse problems for kinetic equations which are also important both from theoretical and practical points of view (see, e.g. [Amirov, Golgeleyen, and Rahmanova (2009); Yildiz (2009)]).

1.1 Overdeterminacy

As it was indicated above dependence of the unknown function μ only on the variable x (which is the classical case in integral geometry) leads Problem 1 to be an overdetermined one. In fact, the underlying operator of the IGP is compact and its inverse operator is not bounded. Hence, proving the general existence results for the IGP and Problem 2, and therefore for Problem 1, is impossible and we need some special conditions on the data υ_0 (u_0) for the existence of the solution (for example υ_0 (u_0) must has some quasianalytic character (see, e.g. Chapter 6, Section 17 in [Courant and Hilbert (1962)] and Chapter 6, Section 1 in [Lavrent'ev, Romanov, and Shishatskii (1986)])), so here the term "overdeterminacy" is used in this sense.

It is worth noting that in the theory of inverse problems, usually "overdeterminacy" means that the number of free variables in the data exceeds the number of free variables in the unknown coefficient or right hand side of the equation ($\mu(x)$), and this is not the case for our problems.

Because of the overdeterminacy, in establishing the solvability of the above problems, the initial data can not be arbitrary; they should satisfy some nontrivial "solvability conditions" (see p. 4 in [Amirov (2001)] and Theorem 1.4 on p. 18 in [Romanov (1974)]). It should be noted that the set of functions υ_0 (u_0) for which IGP (Problem 1) is solvable is not everywhere dense in any of the spaces $L_2(\Gamma_1)$, $C^m(\Gamma_1)$ and $H^m(\Gamma_1)$. Furthermore, since the data in problems of integral geometry are of quasianalytic character, in particular, this implies that it is impossible to avoid overdeterminacy of the problem by specifying the data on a part of the boundary rather than on the whole boundary. Even though finding the solvability conditions for the mentioned overdetermined problems was possible, since the real data in practice usually have some errors and thus fall out of the data class for which the existence of a solution is established, these conditions would not be satisfactory from the practical point of view.

Let us propose the procedure of the method for establishing the solvability of Problem 1 (this method of investigating the solvability of overdetermined inverse problems was firstly proposed by Amirov in [Amirov (1986)] (see also [Amirov (2001)])). The overdetermined Problem 1 was reduced to Problem 2 above and on using some extension of the class of unknown functions μ, the latter is replaced by the determined Problem 3 (see Section 2). This is achieved by assuming the unknown function μ depends not only upon the space variable x, but also upon the direction φ in a specific way, i.e. $\mu(x,\varphi)$ satisfies a certain differential equation $\left(\hat{L}\mu = 0\right)$ where Problem 2 with the function $\mu(x,\varphi)$ becomes a determined one and the sufficiently smooth functions μ depending only on x satisfy this equation. Since this equation is not uniquely determined, the class of unknown functions μ extends so that Problem 2 becomes a determined problem for the new class and all sufficiently smooth functions in x belong to the class. With the use of this method, the construction of the equation $\hat{L}\mu = 0$ (which is one of the crucial part of this method) and some space in which the problem is uniquely solvable are given in Section 2. It should be noted that $\mu(x,\varphi)$ cannot be arbitrarily dependent upon φ, because in the opposite case the problem would be underdetermined.

1.2 Some Definitions

Let us denote the set of real-valued functions $\upsilon(x,\varphi)$ that are 2π-periodic in φ and three times continuously differentiable on Ω with respect to all arguments by $C^3_\pi(\Omega)$. Here, 2π-periodicity of the function $\upsilon \in C^3_\pi(\Omega)$ with respect to argument

φ in the domain Ω means that $D^{\alpha}\upsilon(x,0) = D^{\alpha}\upsilon(x,2\pi)$, where $D^{\alpha} = D_{x_1}^{\alpha_1}D_{x_2}^{\alpha_2}D_{\varphi}^{\alpha_3}$ and $\alpha_i \geq 0$ are integers such that $0 \leq \alpha_1+\alpha_2+\alpha_3 \leq 3$. Let us introduce the scalar product

$$(\upsilon,z)_{1,2} = \int_{\Omega}\left(\upsilon z+\upsilon_{x_1}z_{x_1}+\upsilon_{x_2}z_{x_2}+\upsilon_{\varphi}z_{\varphi}+\upsilon_{x_1\varphi}z_{x_1\varphi}+\upsilon_{x_2\varphi}z_{x_2\varphi}+\upsilon_{\varphi\varphi}z_{\varphi\varphi}\right)d\Omega$$

in $C_{\pi}^{3}(\Omega)$, where $d\Omega = dx_1dx_2d\varphi$. Set $\|\upsilon\|_{1,2} = [(\upsilon,\upsilon)_{1,2}]^{1/2}$.

Let $H_{1,2}^{\pi}(\Omega)$ and $H_{m}^{\pi}(\Omega)$ be the completions of $C_{\pi}^{3}(\Omega)$ with respect to the norms $\|\cdot\|_{1,2}$ and $\|\cdot\|_{H^m(\Omega)}$ $(m = 1,2,3)$ respectively (for detailed information about the space H^m, see, e.g. [Lions and Magenes (1972); Mikhailov (1978)]). Let $C_{\pi 0}^{3} = \{\psi : \psi|_{\Gamma_1} = 0,\ \psi \in C_{\pi}^{3}(\Omega)\}$ and the spaces $\mathring{H}_{1,2}^{\pi}(\Omega)$ and $\mathring{H}_{m}^{\pi}(\Omega)$ be the completions of $C_{\pi 0}^{3}$ with respect to the norm $\|\cdot\|_{1,2}$ and $\|\cdot\|_{H^m(\Omega)}$ $(m = 1,2,3)$. Let us select a set $\{w_1,w_2,w_3,...\} \subset C_{\pi 0}^{3}$ which is complete and orthonormal in $L_2(\Omega)$. We may assume that the linear span of the set $\{w_i\}_{i=1}^{\infty}$ is everywhere dense in $\mathring{H}_{1,2}^{\pi}(\Omega)$. Indeed, since the space $\mathring{H}_{1,2}^{\pi}(\Omega)\cap\mathring{H}_1(\Omega)$ is seperable, there exists a countable set $\{\varphi_i\}_{i=1}^{\infty} \subset C_{\pi 0}^{3}$ which is everywhere dense in $\mathring{H}_{1,2}^{\pi}(\Omega)$ and this set up can be extended to a set which is everywhere dense in $L_2(\Omega)$, if necessary. The set $\{w_i\}_{i=1}^{\infty}$ is obtained by orthonormalizing the latter in $L_2(\Omega)$.

Let $\Gamma''(A)$ be the set of all functions $\upsilon(x,\varphi) \in L_2(\Omega)$ such that for any $\upsilon \in \Gamma''(A)$ there exists $y \in L_2(\Omega)$ such that $(\upsilon,A^*\eta)_{L_2(\Omega)} = (y,\eta)_{L_2(\Omega)}$ holds for every $\eta \in C_0^{\infty}(\Omega)$, where A^* is the differential expression conjugate to A in the sense of Lagrange, A is a differential expression of third order ($A = \hat{L}L$, see the following section) and $C_0^{\infty}(\Omega)$ is the set of all functions defined in Ω which have continuous partial derivatives of order up to all $k < \infty$, whose supports are compact subsets of Ω (see, e.g. [Lions and Magenes (1972)]). So $A\upsilon = y$ in the generalized functions sense. Take a subset $\Gamma(A) \subset \Gamma''(A)$ such that for any $\upsilon \in \Gamma(A)$ there exists a sequence $\{\upsilon_k\} \subset C_{\pi 0}^{3}$ such that $\upsilon_k \to \upsilon$ weakly in $L_2(\Omega)$ and $(A\upsilon_k,\upsilon_k)_{L_2(\Omega)} \to (A\upsilon,\upsilon)_{L_2(\Omega)}$ as $k \to \infty$. It can be seen that the inclusions $\mathring{H}_{3}^{\pi}(\Omega) \subset \Gamma''(A)\cap\mathring{H}_{1,2}^{\pi}(\Omega) \subset \Gamma(A) \subset L_2(\Omega)$ hold.

2 The Determined Problem and the Solvability Result

With the use of the proposed method in Section 1.1, the second order differential expression $\hat{L}$, which is defined in Ω, can be constructed as

$$\hat{L}\upsilon = \frac{\partial^2\upsilon}{\partial l\partial\varphi} = \frac{\partial}{\partial l}\upsilon_{\varphi}, \tag{3}$$

where $\frac{\partial}{\partial l} = (\sin\varphi)\left(\frac{\partial}{\partial x_1}+f_2\frac{\partial}{\partial\varphi}-f_1\right)-(\cos\varphi)\left(\frac{\partial}{\partial x_2}-f_1\frac{\partial}{\partial\varphi}-f_2\right)$. Let $A = \hat{L}L$.

Remark 2 *Suppose that, a priori the function υ_0^e is known, which represents the exact data of Problem 2 related to a function μ depending only on x. Then, utilizing υ_0^e, a solution $\tilde{\mu}$ to IGP can be constructed. If the problem has a unique solution then $\tilde{\mu}$ and $\mu(x)$ coincide. At the same time, knowing the approximate data υ_0^a with $\left\|\upsilon_0^e - \upsilon_0^a\right\|_{H^3(\Gamma_1)} \leq \varepsilon$, an approximate solution $\mu^a(x,\varphi)$ can be constructed such that $\|\mu - \mu^a\|_{L_2(\Omega)} \leq C\varepsilon$. Note that, if μ depends only on x and υ_0^a does not satisfy the "solvability conditions", the solution μ^a depending only x does not exist. Here the data are specified on Γ_1 and $C > 0$ is independent of υ_0^e and υ_0^a. In other words, a regularising procedure is constructed for Problem 2.*

Let us replace the equation (2) by the following one (where μ depends also on the variable φ)

$$L\upsilon = \mu(x,\varphi), \tag{4}$$

and consider the following determined problem.

Problem 3 *Given the function K, determine a pair of functions (υ,μ) from the equation (4), provided that $\upsilon(x,\varphi)$ is 2π-periodic in φ, $\upsilon|_{\Gamma_1} = \upsilon_0$, and the condition $\hat{L}\mu = 0$ holds.*

Here the equation $\hat{L}\mu = 0$ is satisfied in the generalized functions sense, i.e. for each $\eta \in C_0^\infty(\Omega)$, $(\mu,(\hat{L})^*\eta)_{L_2(\Omega)} = 0$, where $(\hat{L})^*$ is conjugate to $\hat{L}$ in the sense of Lagrange. If $\upsilon_0 \in C^3(\Gamma_1)$ and $\partial D \in C^3$, then one can obtain homogenous data on Γ_1 instead of nonhomogeneous data by considering a new unknown function $\bar{\upsilon} = \upsilon - G$, where G is the function from the set $C_\pi^3(\bar{\Omega})$ such that $G|_{\Gamma_1} = \upsilon_0$ (see, e.g. Theorem 2 on p. 130 in [Mikhailov (1978)]). If we denote $\bar{\upsilon}$ again by υ for the simplicity, then the equation (4) is reduced to

$$L\upsilon = \mu(x,\varphi) + F, \tag{5}$$

where $F = -LG$. So, Problem 3 can be reduced to the following one (see p. 20 in [Amirov (2001)]).

Problem 4 *Given the functions K and F, determine a pair of functions (υ,μ) from the equation (5), provided that $\upsilon(x,\varphi)$ is 2π-periodic in φ, $\upsilon|_{\Gamma_1} = 0$, and the condition $\hat{L}\mu = 0$ holds.*

Theorem 1 (Amirov, Yildiz, and Ustaoglu (2009)) *If $f_1(x)$, $f_2(x) \in C^3(\bar{D})$, and for all $x \in \bar{D}$, $f_{1x_1} + f_{2x_2} > 0$ and $F \in H_2^\pi(\Omega)$ then Problem 4 has a unique solution*

(υ,μ), such that $\upsilon \in \Gamma(A) \cap \mathring{H}_1^{\pi}(\Omega)$, $\mu \in L_2(\Omega)$, and the inequality $\|\upsilon\|_{\mathring{H}_1^{\pi}(\Omega)} + \|\mu\|_{L_2(\Omega)} \le C(\|F\|_{L_2(\Omega)} + \|F_\varphi\|_{L_2(\Omega)})$ holds, where $C > 0$ depends on f_1, f_2 and the Lebesgue measure of D and $\bar{D}$ is the closure of D.

Considering its relation with Problem 4, we investigate and present two approximation methods for the solution of Problem 1.

3 Finite Difference Approximation and Symbolic Approximation

Let us consider the following auxiliary Dirichlet type boundary value problem.

Problem 5 *Find a function υ which satisfies the third order partial differential equation*

$$A\upsilon = \mathscr{F}, \tag{6}$$

provided that $\upsilon(x,\varphi)$ is 2π-periodic in φ and $\upsilon|_{\Gamma_1} = 0$, where $A\upsilon \equiv \hat{L}L\upsilon$ and $\mathscr{F} = \hat{L}F$.

3.1 Finite Difference Approximation

We investigate finite difference approximation to the solution of Problem 1 on $\Omega = D \times (0,2\pi)$, where $D = (a,b) \times (c,d)$ and a, b, c, d are real numbers. By using the centered-difference formulas in (6), we obtain the difference equations;

$$\begin{aligned}
&-d_1^{(k)}\tilde{\upsilon}_{i-1,j-1,k-1} + d_6^{(k)}\tilde{\upsilon}_{i-1,j-1,k} + d_1^{(k)}\tilde{\upsilon}_{i-1,j-1,k+1} - (d_3^{(k)} + e_1^{(i,j,k)} - e_5^{(i,j,k)})\\
&.\tilde{\upsilon}_{i-1,j,k-1} + (d_4^{(k)} + 2e_1^{(i,j,k)} - e_7^{(i,j,k)})\tilde{\upsilon}_{i-1,j,k} + (d_3^{(k)} - e_1^{(i,j,k)} - e_5^{(i,j,k)})\tilde{\upsilon}_{i-1,j,k+1}\\
&+ d_1^{(k)}\tilde{\upsilon}_{i-1,j+1,k-1} - d_6^{(k)}\tilde{\upsilon}_{i-1,j+1,k} - d_1^{(k)}\tilde{\upsilon}_{i-1,j+1,k+1} - (d_2^{(k)} + e_2^{(i,j,k)} - e_6^{(i,j,k)})\\
&.\tilde{\upsilon}_{i,j-1,k-1} + (d_5^{(k)} + 2e_2^{(i,j,k)} - e_8^{(i,j,k)})\tilde{\upsilon}_{i,j-1,k} + (d_2^{(k)} - e_2^{(i,j,k)} - e_6^{(i,j,k)})\tilde{\upsilon}_{i,j-1,k+1}\\
&- e_3^{(i,j,k)}\tilde{\upsilon}_{i,j,k-2} + (2(d_2^{(k)} + d_3^{(k)} + e_3^{(i,j,k)}) + e_4^{(i,j,k)} - e_9^{(i,j,k)})\tilde{\upsilon}_{i,j,k-1} - 2(d_4^{(k)}\\
&+ d_5^{(k)} + e_4^{(i,j,k)})\tilde{\upsilon}_{i,j,k} - (2(d_2^{(k)} + d_3^{(k)} + e_3^{(i,j,k)}) - e_4^{(i,j,k)} - e_9^{(i,j,k)})\tilde{\upsilon}_{i,j,k+1}\\
&+ e_3^{(i,j,k)}\tilde{\upsilon}_{i,j,k+2} - (d_2^{(k)} - e_2^{(i,j,k)} + e_6^{(i,j,k)})\tilde{\upsilon}_{i,j+1,k-1} + (d_5^{(k)} - 2e_2^{(i,j,k)} + e_8^{(i,j,k)})\\
&.\tilde{\upsilon}_{i,j+1,k} + (d_2^{(k)} + e_2^{(i,j,k)} + e_6^{(i,j,k)})\tilde{\upsilon}_{i,j+1,k+1} + d_1^{(k)}\tilde{\upsilon}_{i+1,j-1,k-1} - d_6^{(k)}\tilde{\upsilon}_{i+1,j-1,k}\\
&- d_1^{(k)}\tilde{\upsilon}_{i+1,j-1,k+1} - (d_3^{(k)} - e_1^{(i,j,k)} + e_5^{(i,j,k)})\tilde{\upsilon}_{i+1,j,k-1} + (d_4^{(k)} - 2e_1^{(i,j,k)} + e_7^{(i,j,k)})\\
&.\tilde{\upsilon}_{i+1,j,k} + (d_3^{(k)} + e_1^{(i,j,k)} + e_5^{(i,j,k)})\tilde{\upsilon}_{i+1,j,k+1} - d_1^{(k)}\tilde{\upsilon}_{i+1,j+1,k-1} + d_6^{(k)}\tilde{\upsilon}_{i+1,j+1,k}\\
&+ d_1^{(k)}\tilde{\upsilon}_{i+1,j+1,k+1} = \mathscr{F}_{i,j,k},
\end{aligned} \tag{7}$$

where

$d_1^{(k)} = \frac{1}{8\Delta x_1 \Delta x_2 \Delta\varphi}(\sin^2(\varphi_k) - \cos^2(\varphi_k)),\ d_2^{(k)} = -\frac{1}{2(\Delta x_2)^2 \Delta\varphi}\cos(\varphi_k)\sin(\varphi_k),$

$d_3^{(k)} = \frac{1}{2(\Delta x_1)^2 \Delta\varphi}\cos(\varphi_k)\sin(\varphi_k),\ d_4^{(k)} = -\frac{1}{(\Delta x_1)^2}\sin^2(\varphi_k),$

$d_5^{(k)} = -\frac{1}{(\Delta x_2)^2}\cos^2(\varphi_k),\ d_6^{(k)} = \frac{1}{2\Delta x_1 \Delta x_2}\cos(\varphi_k)\sin(\varphi_k),$

$e_1^{(i,j,k)} = \frac{1}{2\Delta x_1 (\Delta\varphi)^2}(f_{1i,j}\cos(2\varphi_k) + f_{2i,j}\sin(2\varphi_k)),$

$e_2^{(i,j,k)} = -\frac{1}{2\Delta x_2 (\Delta\varphi)^2}(f_{2i,j}\cos(2\varphi_k) - f_{1i,j}\sin(2\varphi_k)),$

$e_3^{(i,j,k)} = \frac{1}{2\Delta\varphi^3}(f_{1i,j}\cos(\varphi_k) + f_{2i,j}\sin(\varphi_k))(f_{2i,j}\cos(\varphi_k) - f_{1i,j}\sin(\varphi_k)),$

$$\begin{aligned} e_4^{(i,j,k)} = \frac{1}{(\Delta\varphi)^2}\Big(&\frac{1}{2\Delta x_1}((f_{2i+1,j} - f_{2i-1,j})\cos(\varphi_k) - (f_{1i+1,j} - f_{1i-1,j})\sin(\varphi_k))\sin(\varphi_k) \\ &-\frac{1}{2\Delta x_2}((f_{2i,j+1} - f_{2i,j-1})\cos(\varphi_k) - (f_{1i,j+1} - f_{1i,j-1})\sin(\varphi_k))\cos(\varphi_k) \\ &-2(f_{2i,j}\sin(\varphi_k) + f_{1i,j}\cos(\varphi_k))^2 + (f_{2i,j}\cos(\varphi_k) - f_{1i,j}\sin(\varphi_k))^2\Big), \end{aligned}$$

$$\begin{aligned} e_5^{(i,j,k)} = \frac{1}{4\Delta x_1 \Delta\varphi}&((f_{2i,j}\cos(\varphi_k) - f_{1i,j}\sin(\varphi_k))\cos(\varphi_k) \\ &-3(f_{2i,j}\sin(\varphi_k) + f_{1i,j}\cos(\varphi_k))\sin(\varphi_k), \end{aligned}$$

$$\begin{aligned} e_6^{(i,j,k)} = \frac{1}{4\Delta x_2 \Delta\varphi}&((f_{2i,j}\cos(\varphi_k) - f_{1i,j}\sin(\varphi_k))\sin(\varphi_k) \\ &+3(f_{2i,j}\sin(\varphi_k) + f_{1i,j}\cos(\varphi_k))\cos(\varphi_k), \end{aligned}$$

$e_7^{(i,j,k)} = -\frac{1}{2\Delta x_1}(f_{1i,j}\cos(2\varphi_k) + f_{2i,j}\sin(2\varphi_k)),$

$e_8^{(i,j,k)} = \frac{1}{2\Delta x_2}(f_{2i,j}\cos(2\varphi_k) - f_{1i,j}\sin(2\varphi_k)),$

$$\begin{aligned} e_9^{(i,j,k)} = \frac{1}{2\Delta\varphi}\Big(&\frac{1}{2\Delta x_2}((f_{2i,j+1} - f_{2i,j-1})\sin(\varphi_k) + (f_{1i,j+1} - f_{1i,j-1})\cos(\varphi_k))\cos(\varphi_k) \\ &-\frac{1}{2\Delta x_1}((f_{2i+1,j} - f_{2i-1,j})\sin(\varphi_k) + (f_{1i+1,j} - f_{1i-1,j})\cos(\varphi_k))\sin(\varphi_k) \\ &-2(f_{1i,j}\cos(\varphi_k) + f_{2i,j}\sin(\varphi_k))(f_{2i,j}\cos(\varphi_k) - f_{1i,j}\sin(\varphi_k))), \end{aligned}$$

$(i = 1, 2, ..., I, \quad j = 1, 2, ..., J, \ k = 1, 2, ..., K).$

In the above equations, $I,\ J,\ K$ are positive integers, $\Delta x_1 = (b-a)/(I+1),\ \Delta x_2 = (d-c)/(J+1)$ and $\Delta\varphi = 2\pi/K$ are step sizes for $x_1,\ x_2$ and φ respectively. $\tilde{\upsilon}_{i,j,k}$ is the approximation to the solution $\upsilon(x_{1i}, x_{2j}, \varphi_k) = \upsilon(a + i\Delta x_1, c + j\Delta x_2, k\Delta\varphi\ )$ and $f_{1i,j} = f_1(x_{1i}, x_{2j})$, $f_{2i,j} = f_1(x_{1i}, x_{2j})$, $\mathscr{F}_{i,j,k} = \mathscr{F}(x_{1i}, x_{2j}, \varphi_k)$. Since $\upsilon(x, \varphi)$ is 2π-periodic, $\tilde{\upsilon}_{i,j,0} = \tilde{\upsilon}_{i,j,K}$ and $\tilde{\upsilon}_{i,j,K+1} = \tilde{\upsilon}_{i,j,1}$. The condition $\upsilon|_{\Gamma_1} = 0$ in Problem 5 is discretized as

$$\tilde{\upsilon}_{0,j,k} = \tilde{\upsilon}_{I+1,j,k} = \tilde{\upsilon}_{i,0,k} = \tilde{\upsilon}_{i,J+1,k} = 0,$$
$$(i = 0, 1, ..., I+1,\ j = 0, 1, ..., J+1,\ k = 1, 2, ..., K).$$

This system of linear equations can be written in the matrix form

$$\tilde{A}\tilde{\mathfrak{v}} = \mathbf{b}, \tag{8}$$

where the block tridiagonal matrix $\tilde{A}$ is in the following form

$$\tilde{A} = \begin{bmatrix} A_1^{(1)} & A_2^{(1)} & 0 & \cdots & 0 \\ A_3^{(2)} & A_1^{(2)} & A_2^{(2)} & \ddots & \vdots \\ 0 & A_3^{(3)} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & A_2^{(I-1)} \\ 0 & \cdots & 0 & A_3^{(I)} & A_1^{(I)} \end{bmatrix}_{IJK \times IJK},$$

here the sub-block matrices $A_1^{(i)}, A_2^{(i)}$ and $A_3^{(i)}$ $(i = 1, 2, ..., I)$ are also block tridiagonal and they can be represented as follows.

$$A_1^{(i)} = \begin{bmatrix} B_1 + C_1^{(i,1)} & B_2 + C_2^{(i,1)} & 0 & \cdots & 0 \\ B_2 - C_2^{(i,2)} & B_1 + C_1^{(i,2)} & B_2 + C_2^{(i,2)} & \ddots & \vdots \\ 0 & B_2 - C_2^{(i,3)} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & B_2 + C_2^{(i,J-1)} \\ 0 & \cdots & 0 & B_2 - C_2^{(i,J)} & B_1 + C_1^{(i,J)} \end{bmatrix}_{JK \times JK},$$

$$A_2^{(i)} = \begin{bmatrix} B_3 + C_3^{(i,1)} & B_4 & 0 & \cdots & 0 \\ -B_4 & B_3 + C_3^{(i,2)} & B_4 & \ddots & \vdots \\ 0 & -B_4 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & B_4 \\ 0 & \cdots & 0 & -B_4 & B_3 + C_3^{(i,J)} \end{bmatrix}_{JK \times JK},$$

$$A_3^{(i)} = \begin{bmatrix} B_3 - C_3^{(i,1)} & -B_4 & 0 & \cdots & 0 \\ B_4 & B_3 - C_3^{(i,2)} & -B_4 & \ddots & \vdots \\ 0 & B_4 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -B_4 \\ 0 & \cdots & 0 & B_4 & B_3 - C_3^{(i,J)} \end{bmatrix}_{JK \times JK},$$

where the nonzero entries in the matrices $B_1 = (b_{1m,n})$, $B_2 = (b_{2m,n})$, $B_3 = (b_{3m,n})$

and $B_4 = (b_{4m,n})$ are

$$\begin{aligned} b_{1_{k,k}} &= -2d_4^{(k)} - 2d_5^{(k)},\ b_{2_{k,k}} = d_5^{(k)},\ b_{3_{k,k}} = d_4^{(k)},\ b_{4_{k,k}} = d_6^{(k)}, \\ (k &= 1,...,K) \\ b_{1_{k,k+1}} &= -2d_2^{(k)} - 2d_3^{(k)},\ b_{2_{k,k+1}} = d_2^{(k)},\ b_{3_{k,k+1}} = d_3^{(k)},\ b_{4_{k,k+1}} = d_1^{(k)}, \\ (k &= 1,...,K-1) \\ b_{1_{k,k-1}} &= 2d_2^{(k)} + 2d_3^{(k)},\ b_{2_{k,k-1}} = -d_2^{(k)},\ b_{3_{k,k-1}} = -d_3^{(k)},\ b_{4_{k,k-1}} = -d_1^{(k)}, \\ (k &= 2,...,K) \end{aligned}$$

and $b_{1_{1,K}} = 2d_2^{(1)} + 2d_3^{(1)}, b_{1_{K,1}} = -2d_2^{(K)} - 2d_3^{(K)}, b_{2_{1,K}} = -d_2^{(1)}, b_{2_{K,1}} = d_2^{(K)}, b_{3_{1,K}} = -d_3^{(1)}, b_{3_{K,1}} = d_3^{(K)}, b_{4_{1,K}} = -d_1^{(1)}, b_{4_{K,1}} = d_1^{(K)}$. The nonzero entries in the matrices $C_1^{(i,j)} = (c_{1m,n}^{(i,j)})$, $C_2^{(i,j)} = (c_{2m,n}^{(i,j)})$ and $C_3^{(i,j)} = (c_{3m,n}^{(i,j)})$ $(i = 1,2,...,I, \quad j = 1,2,...,J)$ are

$$\begin{aligned} c_{1_{k,k}}^{(i,j)} &= -2e_4^{(i,j,k)},\ c_{2_{k,k}}^{(i,j)} = -2e_2^{(i,j,k)} + e_8^{(i,j,k)},\ c_{3_{k,k}}^{(i,j)} = -2e_1^{(i,j,k)} + e_7^{(i,j,k)}, \\ (k &= 1,...,K) \\ c_{1_{k,k+1}}^{(i,j)} &= -2e_3^{(i,j,k)} + e_4^{(i,j,k)} + e_9^{(i,j,k)},\ c_{2_{k,k+1}}^{(i,j)} = e_2^{(i,j,k)} + e_6^{(i,j,k)},\ c_{3_{k,k+1}}^{(i,j)} = e_1^{(i,j,k)} + e_5^{(i,j,k)}, \\ (k &= 1,...,K-1) \\ c_{1_{k,k-1}}^{(i,j)} &= 2e_3^{(i,j,k)} + e_4^{(i,j,k)} - e_9^{(i,j,k)},\ c_{2_{k,k-1}}^{(i,j)} = e_2^{(i,j,k)} - e_6^{(i,j,k)},\ c_{3_{k,k-1}}^{(i,j)} = e_1^{(i,j,k)} - e_5^{(i,j,k)}, \\ (k &= 2,...,K) \\ c_{1_{k,k+2}}^{(i,j)} &= e_3^{(i,j,k)},\ \ (k = 1,...,K-2),\ \ c_{1_{k,k-2}}^{(i,j)} = -e_3^{(i,j,k)},\ \ (k = 3,...,K) \end{aligned}$$

and $c_{1_{1,K-1}}^{(i,j)} = -e_3^{(i,j,1)}$, $c_{1_{2,K}}^{(i,j)} = -e_3^{(i,j,2)}$, $c_{1_{1,K}}^{(i,j)} = 2e_3^{(i,j,1)} + e_4^{(i,j,1)} - e_9^{(i,j,1)}$, $c_{1_{K-1,1}}^{(i,j)} = e_3^{(i,j,K-1)}$, $c_{1_{K,2}}^{(i,j)} = e_3^{(i,j,K)}$, $c_{1_{K,1}}^{(i,j)} = -2e_3^{(i,j,K)} + e_4^{(i,j,K)} + e_9^{(i,j,K)}$, $c_{2_{1,K}}^{(i,j)} = e_2^{(i,j,1)} - e_6^{(i,j,1)}$, $c_{2_{K,1}}^{(i,j)} = e_2^{(i,j,K)} + e_6^{(i,j,K)}$, $c_{3_{1,K}}^{(i,j)} = e_1^{(i,j,1)} - e_5^{(i,j,1)}$, $c_{3_{K,1}}^{(i,j)} = e_1^{(i,j,K)} + e_5^{(i,j,K)}$.

In equation (8), $\tilde{\upsilon}$ is the column matrix, each row of which consists of an unknown $\tilde{\upsilon}_{i,j,k}$ with the order

$$\begin{aligned} &\tilde{\upsilon}_{1,1,1}, \tilde{\upsilon}_{1,1,2}, ..., \tilde{\upsilon}_{1,1,K}, \tilde{\upsilon}_{1,2,1}, \tilde{\upsilon}_{1,2,2}, ..., \tilde{\upsilon}_{1,2,K}, ..., \tilde{\upsilon}_{1,J,1}, \tilde{\upsilon}_{1,J,2}, ..., \tilde{\upsilon}_{1,J,K}, \\ &\tilde{\upsilon}_{1,1,1}, \tilde{\upsilon}_{1,1,2}, ..., \tilde{\upsilon}_{1,1,K}, \tilde{\upsilon}_{1,2,1}, \tilde{\upsilon}_{1,2,2}, ..., \tilde{\upsilon}_{1,2,K}, ..., \tilde{\upsilon}_{1,J,1}, \tilde{\upsilon}_{1,J,2}, ..., \tilde{\upsilon}_{1,J,K}, \\ &\vdots \\ &\tilde{\upsilon}_{I,1,1}, \tilde{\upsilon}_{I,1,2}, ..., \tilde{\upsilon}_{I,1,K}, \tilde{\upsilon}_{I,2,1}, \tilde{\upsilon}_{I,2,2}, ..., \tilde{\upsilon}_{I,2,K}, ..., \tilde{\upsilon}_{I,J,1}, \tilde{\upsilon}_{I,J,2}, ..., \tilde{\upsilon}_{I,J,K} \end{aligned} \tag{9}$$

and $\mathbf{b}$ is the column matrix, which consists of the values $\mathscr{F}_{i,j,k}$.

By solving the matrix equation (8), we get the approximation $\tilde{\upsilon}_{i,j,k}$ at $I \times J \times K$ mesh points of Ω. So, the approximate values of u can be easily obtained by setting $\tilde{u}_{i,j,k} = \exp(\tilde{\upsilon}_{i,j,k} + G_{i,j,k})$, where $G_{i,j,k} = G(x_{1i}, x_{2j}, \varphi_k)$. With the use of the centered difference formulas in (5), the difference equations

$$\frac{\tilde{\upsilon}_{i+1,j,k} - \tilde{\upsilon}_{i-1,j,k}}{2\Delta x_1} \cos(\varphi_k) + \frac{\tilde{\upsilon}_{i,j+1,k} - \tilde{\upsilon}_{i,j-1,k}}{2\Delta x_2} \sin(\varphi_k)$$
$$+ \frac{\tilde{\upsilon}_{i,j,k+1} - \tilde{\upsilon}_{i,j,k-1}}{2\Delta \varphi} \left(f_{2i,j} \cos(\varphi_k) - f_{1i,j} \sin(\varphi_k)\right) = \tilde{\mu}_{i,j,k} + F_{i,j,k} \tag{10}$$
$$(i = 1,2,...,I\ ,\ \ j = 1,2,...,J\ ,\ \ k = 1,2,...,K)$$

are obtained, where $\tilde{\mu}_{i,j,k}$ is the approximation to $\mu(x_{1i}, x_{2j}, \varphi_k) = \mu(a + i\Delta x_1, c + j\Delta x_2, k\Delta\varphi\ )$ and $F_{i,j,k} = F(x_{1i}, x_{2j}, \varphi_k)$. By using the approximate values $\tilde{\upsilon}_{i,j,k}$ and known values $f_{1i,j}$, $f_{2i,j}$ and $F_{i,j,k}$, equation (10) is used to approximate the function μ. So the approximation to the solution (u, μ) of Problem 1 is obtained and the algorithm, which is used to compute these approximate values, is the following:

Algorithm 1 *(Finite Difference Approximation)*

INPUT : Functions $f_1(x_1, x_2)$, $f_2(x_1, x_2)$ *and* $G(x_1, x_2, \varphi)$,
real numbers a, b, c, d, *integers* I, J, K.

OUTPUT : Approximation $\tilde{u}$ *for* u *and* $\tilde{\mu}$ *for* μ *at* $I \times J \times K$ *mesh points of* Ω.

Step 1 Set step sizes $\Delta x_1 = \frac{b-a}{I+1}$, $\Delta x_2 = \frac{d-c}{J+1}$, $\Delta\varphi = \frac{2\pi}{K}$,
Set inner and boundary mesh points $(x_{1i}, x_{2j}, \varphi_k) = (a + i\Delta x_1, c + j\Delta x_2, k\Delta\varphi)$;

Step 2 Construct the column matrix $\mathbf{b}$, *consisting of* $\mathscr{F}_{i,j,k}$,
Construct the block tridiagonal matrix $\tilde{A}$,
Solve the system of linear equations $\tilde{A}\tilde{\upsilon} = \mathbf{b}$,
Set $\tilde{u}_{i,j,k} = \exp(\tilde{\upsilon}_{i,j,k} + G_{i,j,k})$ *Output* $(\tilde{u}_{i,j,k})$

Step 3 Compute $\tilde{\mu}$ *from (10) using* $\tilde{\upsilon}$ *Output* $(\tilde{\mu}_{i,j,k})$

End.

3.2 Symbolic Approximation

Computation of approximate solution V_N of the Problem 5 is made by writing it in the form $V_N = \sum_{i,j,k=0}^{N} \left(\alpha_{i,j,k} v_{i,j,k} + \beta_{i,j,k} w_{i,j,k}\right) \eta$, where the function η defined in D is selected such that it vanishes on the boundary and outside of D. $\left\{v_{i,j,k}\right\}_{i,j,k=0}^{N}$ and $\left\{w_{i,j,k}\right\}_{i,j,k=0}^{N}$ are complete systems in $L_2(\Omega)$ where $v_{i,j,k} = x_1^i x_2^j \sin(k\varphi)$ and

$w_{i,j,k}=x_1^i x_2^j\cos(k\varphi)$. Unknown coefficients $\alpha_{i,j,k}$ and $\beta_{i,j,k}$, $i,j,k=0,...,N$, are determined from the following system of linear algebraic equations;

$$\sum_{i,j,k=0}^{N}\left(A\left(\alpha_{i,j,k}v_{i,j,k}+\beta_{i,j,k}w_{i,j,k}\right)\eta,v_{i',j',k'}\eta\right)_{L_2(\Omega)} = \left(\mathscr{F},v_{i',j',k'}\eta\right)_{L_2(\Omega)}, \quad (11)$$

$$\sum_{i,j,k=0}^{N}\left(A\left(\alpha_{i,j,k}v_{i,j,k}+\beta_{i,j,k}w_{i,j,k}\right)\eta,w_{i',j',k'}\eta\right)_{L_2(\Omega)} = \left(\mathscr{F},w_{i',j',k'}\eta\right)_{L_2(\Omega)}, \quad (12)$$

where $i',j',k'=0,...,N$. The approximation to u is obtained by setting $U_N=\exp(V_N+G)$ and by using V_N in (5) we obtain μ_N and the algorithm to obtain symbolic approximation (U_N,μ_N) to the solution of Problem 1, is given below.

Algorithm 2 *(Symbolic Approximation)*

INPUT : Functions $f_1(x_1,x_2)$, $f_2(x_1,x_2)$ *and* $G(x_1,x_2,\varphi)$, *integer* N.

OUTPUT : U_N *and* μ_N *symbolic approximations to* u *and* μ.

Step 1 Construct the left side of $(11a)$ *and* $(11b)$

Procedure SysA(i',j',k') *for* $i,j,k=0,...,N$

$$Left := Left+\left(A\left(\alpha_{i,j,k}v_{i,j,k}+\beta_{i,j,k}w_{i,j,k}\right)\eta,\ v_{i',j',k'}\eta\right)_{L_2(\Omega)};$$

Procedure SysB(i',j',k') *for* $i,j,k=0,...,N$

$$Left := Left+\left(A\left(\alpha_{i,j,k}v_{i,j,k}+\beta_{i,j,k}w_{i,j,k}\right)\eta,\ w_{i',j',k'}\eta\right)_{L_2(\Omega)};$$

Step 2 Construct the system of equations (11)

Procedure SYS *Set* $:=\{\}$, $F:=-LG$, $\mathscr{F}:=\hat{L}F$, *for* $i',j',k'=0,...,N$

$$Set := Set\cup\left\{SysA(i',j',k')=\left(\mathscr{F},\ v_{i',j',k'}\eta\right)_{L_2(\Omega)},\ SysB(i',j',k')=\left(\mathscr{F},\ w_{i',j',k'}\eta\right)_{L_2(\Omega)}\right\};$$

Step 3 Solve the system of equations (11)

Solve $\left(SYS,\{\alpha_{i,j,k}\},\{\beta_{i,j,k}\}\right)$ *for* $i,j,k=0,...,N$

$V_N=V_N+\left(\alpha_{i,j,k}v_{i,j,k}+\beta_{i,j,k}w_{i,j,k}\right)\eta$;

$U_N=\exp(V_N+G)$, $\mu_N=L(V_N)-F$ *Output* (U_N,μ_N)

End.

3.3 Implementation of the Approximation Algorithms

We have made some numerical experiments related to the proposed approximation algorithms for the solution of Problem 1 under different boundary conditions and the functions f_1 and f_2, and implementation of Algorithm 1 and 2 is presented on the following example.

Example 1 *Consider the problem on the domain* $\Omega = (0,1/2)\times(0,1/2)\times(0,2\pi)$, *where* $f_1(x_1,x_2) = f_2(x_1,x_2) = 2(x_1+x_2)$ *and*

$$u_0(x_1,x_2,\varphi) = \begin{cases} \exp\left(\exp(x_2)^2(\cos\varphi+\sin\varphi)\right), \\ \qquad x_1 = 0, x_2 \in [0,1/2], \varphi \in (0,2\pi) \\ \exp\left(\exp(x_2+1/2)^2(\cos\varphi+\sin\varphi)\right), \\ \qquad x_1 = 1/2, x_2 \in [0,1/2], \varphi \in (0,2\pi) \\ \exp\left(\exp(x_1)^2(\cos\varphi+\sin\varphi)\right), \\ \qquad x_2 = 0, x_1 \in [0,1/2], \varphi \in (0,2\pi) \\ \exp\left(\exp(x_1+1/2)^2(\cos\varphi+\sin\varphi)\right), \\ \qquad x_2 = 1/2, x_1 \in [0,1/2], \varphi \in (0,2\pi) \end{cases}$$

are given. Under these conditions, it can be easily verified that the pair of functions (u,μ), *where*

$$\begin{aligned} u(x_1,x_2,\varphi) &= \exp\left(\exp(x_1+x_2)^2(\cos\varphi+\sin\varphi)\right), \\ \mu(x_1,x_2) &= 4(x_1+x_2)\exp(x_1+x_2)^2, \end{aligned}$$

is the exact solution of Problem 1. The obtained results from Algorithm 1 and 2 are compared with the exact solution in Figures 1-3. In the computations, we choose the function $G \in C^3_\pi(\bar{\Omega})$ *(see Section 2) as* $G(x_1,x_2,\varphi) = \exp(x_1+x_2)^2(\cos\varphi+\sin\varphi) - x_1x_2(1/2-x_1)(1/2-x_2)$ *and for the symbolic approximation we take*

$$\eta = \eta(x_1,x_2) = \begin{cases} x_1x_2(1/2-x_1)(1/2-x_2), & (x_1,x_2) \in D \\ 0, & (x_1,x_2) \notin D \end{cases}.$$

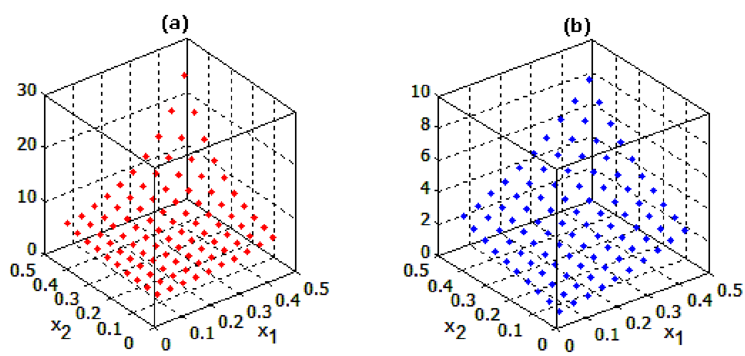

Figure 1: Finite difference approximation at $\varphi_1 = \pi/4$; (a) Approximate values of u; (b) Approximate values of μ.

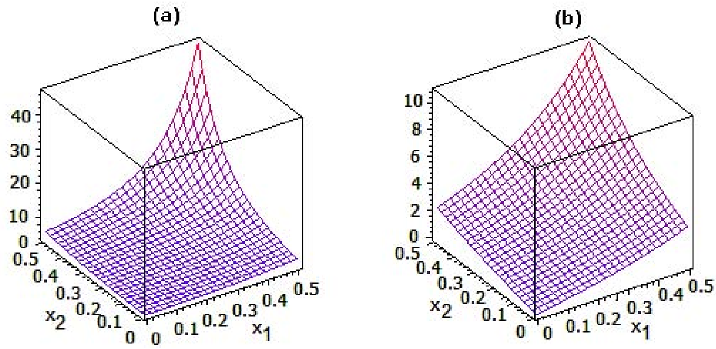

Figure 2: Symbolic approximation at $\varphi = \pi/4$; (a) Approximate u; (b) Approximate μ.

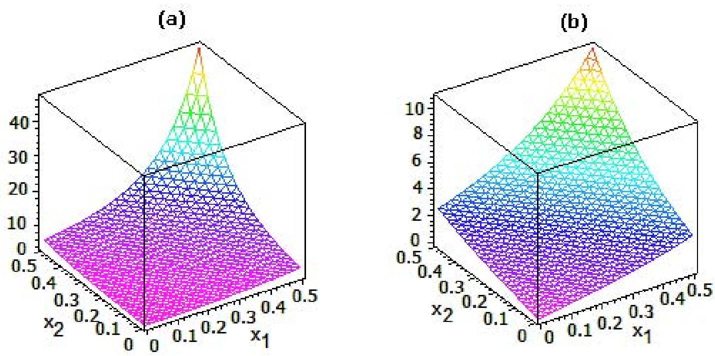

Figure 3: Exact solution at $\varphi = \pi/4$; (a) Exact u; (b) Exact μ.

MATLAB program is used for Algorithm 1 and MAPLE program is used for Algorithm 2, and the computations are performed on a PC with Intel Core 2 T7200 2.00 GHz CPU, 1 Gb memory, running under Windows Vista. In finite difference approximations we take $I = J = 10$ and $K = 8$ (see Figure 1) and the presented symbolic approximations in Figure 2 are computed for $N = 2$. Since u is a function of three variables, the comparison of the results are presented at a selected value φ ($\varphi = \pi/4$).

The results of computational experiments show that the proposed approximation methods in Section 3.1 and 3.2 are feasible to solve Problem 1 numerically.

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