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Dispersion Relations of Axisymmetric Wave Propagation in Finite Pre-Stretched Compound Circular Cylinders Made from Highly Elastic Incompressible Materials

Surkay D. Akbarov[1,2,3]**, Mugan S. Guliev**[4] **and Ramazan Tekercioglu**[5]

Abstract: Dispersion relations of axisymmetric longitudinal wave propagation in a finite pre-strained compound (bi-material) cylinder made from high elastic incompressible materials are investigated within the scope of a piecewise homogeneous body model utilizing three-dimensional linearized theory wave propagation in the initially stressed body. The materials of the inner and outer cylinders are assumed to be neo-Hookean. The numerical results regarding the influence of the initial strains in the inner and outer cylinders on the wave dispersion are presented and discussed. These results are obtained for the case where the material of the inner solid cylinder is stiffer than that of the outer hollow cylinder. At the same time, the following results have been obtained for the following two cases (distinguished from each other by the thickness of the external hollow cylinder): Case 1: The thickness is infinite; Case 2: The thickness is finite.

Keywords: Compound cylinder; finite initial strain; non-linear dynamical effect; wave dispersion; incompressible highly elastic material.

1 Introduction

Initial strains (or stresses) in the construction elements are one of the reference particularities of those which must be taken into account under consideration of their statical and especially dynamical behaviour. It is known that these initial strains

[1] Corresponding author.
[2] Yildiz Technical University, Faculty of Mechanical Engineering, Department of Mechanical Engineering, Yildiz Campus, 34349, Besiktas, Istanbul, Turkey.E-mail address: akbarov@yildiz.edu.tr (S. D. Akbarov)
[3] Institute of Mathematics and Mechanics of National Academy of Sciences of Azerbaijan, 37041, Baku, Azerbaijan.
[4] Genje State University Fac. General Tech. Discip. Dep. Physic. Genje, Azerbaijan.
[5] Yildiz Technical University, Faculty of Chemistry and Metallurgy, Department of Mathematical Engineering, Davutpasa campus No 127, 34210 Esenler, Istanbul, Turkey.

(or stresses) occur in structural elements during their manufacture and assembly, in the Earth's crust under the action of geostatic and geodynamic forces, in composite materials, etc.

Moreover, the construction elements are loaded by external forces under working procedure and in the case where additional forces act on those in this procedure, and if it is necessary to learn the mechanical problems caused by these additional forces, then the stresses caused by the working load can be taken as initial stresses.

Consequently, the area of the problem regarding the initially stressed body is significantly wide and it is evident that to study these problems is of the utmost importance in the practical as well as in the theoretical sense. At the same time, there are also other types of reference particularities such as the existence of a crack in the body which also significantly influences its dynamics: see, for example, Guz, Menshikov, Zozulya and Guz (2007), Guz and Zozulya (2007). Nevertheless, we return to the discussion of the dynamic problems regarding the elastic body with initial stresses.

First, we note that at present the problems regarding the elastodynamics for initially stressed bodies are studied by utilizing a linearized theory constructed using the linearization principle from the general nonlinear theory of elasticity or its simplified modifications. Within the scope of elastoacoustics linearized equations make it possible to investigate all kinds of dynamical problems for initially stressed bodies. However, at this point it is necessary to distinguish the so called approximate and exact approaches. The first of these (i.e. the approximate approaches) is based on the Bernoulli, Kirchoff-Love and Timoshenko hypotheses and other methods of reducing three-dimensional (two-dimensional) problems to two-dimensional (one-dimensional) ones. Consequently, the approximate approaches simplify the mathematics involved in finding a solution. However, the acceptable fields of these approaches are bounded with a few propagating waves (modes) in rods, plates and shells. At the same time, within the scope of these approaches the near-surface dynamical processes for the initially stressed bodies cannot be described. It follows from these statements that it is preferable to use the exact approach; i.e., the Three-dimensional Linearized Theory of Elastic Waves in Initially Stressed Bodies (TLTEWISB) for investigations of the dynamical problems of elastic bodies with initial stresses. The general field equations and relations of the TLTEWISB have been elaborated in many investigations such as Biot (1965), Gren, Rivlin and Shield (1952), Eringen and Suhubi (1975a, 1975b), Guz (1986a, 1986b, 2004), Truestell (1961), etc.

It should be noted that the main part of the investigations made by employing TLTEWISB, (except Akbarov (2006a, 2006b, 2006c, 2006d, 2007a), Yahnioglu (2007) and some others listed therein) refer to the influence of the initial stresses on the speed and the dispersion of various types of waves. The details of the mentioned investigations can be found in papers by Hayes and Rivlin (1961), Chadwick and Jarvis (1979a, 1979b), Dowaik and Ogden (1991), Ogden and Sotiropoulos (1998), Fu and Mielke (2002), Daniel (2008), Akbarov and Guz (2004), Akbarov and Ozisik (2003, 2004), Rogerson and Sandiford (2000), Zhuk and Guz (2007), Guz, Rushchitsky and Guz (2007, 2008) and the papers listed therein. Reviews of these investigations were given in papers by Akbarov (2007b), Guz (2002, 2005), Guz and Makhort (2000). The systematic analysis of these investigations was given in the monographs by Guz (1986a, 1986b, 2004).

Analysis of the aforementioned references shows that a considerable part of the investigations refer to layered composite materials. Also there are a considerable number of investigations on wave propagation in pre-stressed cylinders in Belward (1976), Demiray and Suhubi (1970), Green (1961, 1963), Guz, Kushnir and Makhort (1975), Kushnir (1979) and others. However, in these investigations the subject of research was a homogeneous circular cylinder. Consequently, up to now, investigations on the wave propagation in pre-stressed compound (bi-material) cylinders are almost completely absent. One notable exception was an investigation on the axisymmetric longitudinal wave propagation in the compound cylinder which was made in the paper by Akbarov and Guz (2004) in which it was assumed that the materials of the cylinders are moderately rigid and the initial strains in them are small. Here by "small" it is meant that the strains can be neglected with respect to unit in the corresponding equations and relations of the TLTEWISB. According to the foregoing assumptions, in the paper by Akbarov and Guz (2004) it was concluded that the effect of the initial stresses (i.e. the initial uniaxial homogeneous stresses the values of which are less than the corresponding yield stresses) on the wave propagation velocity in compound cylinders is insignificant.

Taking this statement into account in the paper by Akbarov and Guliev (2009), the axisymmetric longitudinal wave propagation in a compound cylinder with finite initial strains has been studied. However, in the mentioned paper by Akbarov and Guliev (2009) it is assumed that the materials of the cylinders are compressible highly elastic ones.

In the present paper, the investigations started in the paper of Akbarov and Guliev (2009) is continued for high elastic incompressible materials, i.e. it is assumed that the cylinders' materials are incompressible neo-Hookean highly elastic materials. At the same time, in the present paper the resulting dispersion relations are obtained for the following two cases (distinguished from each other by the thickness of the external hollow cylinder): Case 1. The thickness of the external hollow cylinder is infinite; Case 2. The thickness of the external hollow cylinder is finite. However, in the paper by Akbarov and Guliev (2009) only Case 2 is considered.

2 Formulation of the problem and governing field equations

We consider the compound (composite) circular cylinder shown in Fig. 1 and assume that in the natural state the radius

of the inner solid cylinder is R, the thickness of the external hollow cylinder is h. In the natural state we determine the position of the points of the cylinders by the Lagrangian coordinates in the Cartesian system of coordinates $Oy_1y_2y_3$ as well as in the cylindrical system of coordinates $Or\theta y_3$. Assume that the cylinders have infinite length in the direction of the Oy_3 axis and the initial stress state in each component of the considered body is axisymmetric with respect to this axis and homogeneous. Such a stress field may be present with stretching of the considered body along the Oy_3 axis. The stretching may be conducted for the inner solid cylinder and the external hollow cylinder separately before they are compounded; this can also be done after compounding them. Note that in the later case the initial stress field in the constituents of the body will also be the homogeneous one because the materials of these constituents are assumed to be incompressible ones.

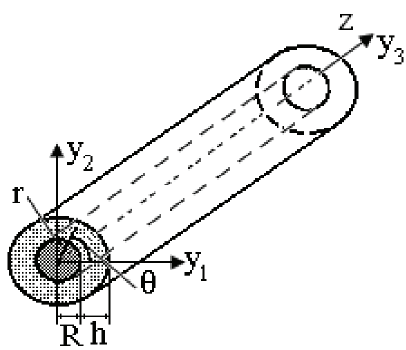


Figure 1: The geometry of the compound cylinder

With the initial state of the cylinders we associate the Lagrangian cylindrical system of coordinates $O'r'\theta'y'_3$ and the Cartesian system of coordinates $O'y'_1y'_2y'_3$. Assume that the mechanical relations of the materials of the components are the neo-Hookean materials and the values related to the inner solid cylinder and external hollow cylinder will be denoted by upper indices (2) and (1), respectively. Furthermore, we denote the values related to the initial state by an additional upper index, 0. Thus, the initial strain state in the fiber and matrix can be determined as follows.

$$u_m^{(k),0} = (\lambda_m^{(k)} - 1) y_m, \quad \lambda_1^{(k)} = \lambda_2^{(k)} \neq \lambda_3^{(k)}, \quad \lambda_m^{(k)} = const, \quad \lambda_1^{(k)} \lambda_2^{(k)} \lambda_3^{(k)} = 1,$$
$$m = 1,2,3; \quad k = 1,2, \quad (1)$$

where $u_m^{(k),0}$ is a displacement and $\lambda_m^{(k)}$ is the elongation along the Oy_m axis. We introduce the following notation

$$\lambda_3^{(k)} = \lambda^{(k)} \quad \lambda_1^{(k)} = \lambda_2^{(k)} = \left(\lambda^{(k)}\right)^{-1/2}, \quad (2)$$

It follows from the equation (1) that

$$y'_i = \lambda_i^{(k)} y_i, \quad r' = \left(\lambda^{(k)}\right)^{-1/2} r, \quad R' = \left(\lambda^{(2)}\right)^{-1/2} R. \quad (3)$$

The values related to the system of the coordinates associated with the initial state below, i.e. with $O'y'_1y'_2y'_3$, will be denoted by upper prime.

Within this framework, let us investigate the axisymmetric wave propagation along the $O'y'_3$ axis in the considered body. We make this investigation by the use of coordinates r' and y'_3 in the framework of the TLTEWISB under construction of which one considers two states of a deformable solid. The first is regarded as the initial or unperturbed state and the second is a perturbed state with respect to the unperturbed. By "the state of a deformable solid" both motion and equilibrium (as a particular case of motion) are meant. It is assumed that all values in a perturbed state can be represented as a sum of the values in the initial state and perturbations. The latter is also assumed to be small in comparison with the corresponding values in the initial state. It is also assumed that both initial (unperturbed) and perturbed states are described by the equations of non-linear solid mechanics. Owing to the fact that the perturbations are small, the relationships for the perturbed state in the vicinity of appropriate values for the unperturbed state are linearized, and then the relations for the perturbed state are subtracted from them. The results are the equations of the TLTEWISB.

The general problems of the TLTEWISB have been elaborated in many investigations such as Guz (1986a, 1986b, 2004) and others. In the present paper we will follow the style and notation used in the monograph Guz (2004).

Thus, according to Guz (2004), we write the basic relations of the TLTEWISB for the incompressible body under an axisymmetrical state. These relations are satisfied within each constituent of the considered body because we use the piecewise homogeneous body model.

The equations of motion are

$$\frac{\partial}{\partial r'}Q'^{(k)}_{r'r'}+\frac{\partial}{\partial y'_3}Q'^{(k)}_{r'3}+\frac{1}{r'}\left(Q'^{(k)}_{r'r'}-Q'^{(k)}_{\theta'\theta'}\right)=\rho'^{(k)}\frac{\partial^2}{\partial t^2}u'^{(k)}_{r'},$$

$$\frac{\partial}{\partial r'}Q'^{(k)}_{3r'}+\frac{\partial}{\partial y'_3}Q'^{(k)}_{33}+\frac{1}{r'}Q'^{(k)}_{3r'}=\rho'^{(k)}\frac{\partial^2}{\partial t^2}u'^{(k)}_{3}. \tag{4}$$

The mechanical relations are

$$Q'^{(k)}_{r'r'}=\chi'^{(k)}_{1111}\frac{\partial u'^{(k)}_{r'}}{\partial r'}+\chi'^{(k)}_{1122}\frac{u'^{(k)}_{r'}}{r'}+\chi'^{(k)}_{1133}\frac{\partial u'^{(k)}_{3}}{\partial y'_3}+p'^{(k)},$$

$$Q'^{(k)}_{\theta'\theta'}=\chi'^{(k)}_{2211}\frac{\partial u'^{(k)}_{r'}}{\partial r'}+\chi'^{(k)}_{2222}\frac{u'^{(k)}_{r'}}{r'}+\chi'^{(k)}_{2233}\frac{\partial u'^{(k)}_{3}}{\partial y'_3}+p'^{(k)},$$

$$Q'^{(k)}_{33}=\chi'^{(k)}_{3311}\frac{\partial u'^{(k)}_{r'}}{\partial r'}+\chi'^{(k)}_{3322}\frac{u'^{(k)}_{r'}}{r'}+\chi'^{(k)}_{3333}\frac{\partial u'^{(k)}_{3}}{\partial y'_3}+p'^{(k)},$$

$$Q'^{(k)}_{r'3}=\chi'^{(k)}_{1313}\frac{\partial u'^{(k)}_{r'}}{\partial y'_3}+\chi'^{(k)}_{1331}\frac{\partial u'^{(k)}_{3}}{\partial r'},\quad Q'^{(k)}_{3r'}=\chi'^{(k)}_{3113}\frac{\partial u'^{(k)}_{r'}}{\partial y'_3}+\chi'^{(k)}_{3131}\frac{\partial u'^{(k)}_{3}}{\partial r'}. \tag{5}$$

In (4) and (5), from $Q'^{(k)}_{r'r'},\ldots,Q'^{(k)}_{3r'}$ the perturbation of the components of the Kirchoff stress tensor are denoted. The notation $u'^{(k)}_{r'}$, $u'^{(k)}_{3}$ shows that the perturbations of the components of the displacement vector, $p'^{(k)}=p'^{(k)}(r',y'_3,t)$ is an unknown function (a Lagrangian multiplier). The constants $\chi'^{(k)}_{1111},\ldots,\chi'^{(k)}_{3333}$ in (4), (5) are determined through the mechanical constants of the fiber and matrix materials and through the initial stress state. $\rho'^{(k)}$ is a density of the k-th material.

As noted above, in the present investigation we assume that the fiber's and matrix's materials are incompressible neo-Hookean one and the elasticity relations for that is given by the following potential:

$$\Phi=C_{10}(I_1-3),\quad I_1=3+2A_1,\quad A_1=\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{33}, \tag{6}$$

where C_{10} is an elastic constant; A_1 is the first algebraic invariant of Green's strain tensor, ε_{rr}, $\varepsilon_{\theta\theta}$ and ε_{33} are the components of this tensor. For the considered axisymmetric case, the components of Green's strain tensor are determined through the components of the displacement vector by the following expressions:

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r}+\frac{1}{2}\left(\frac{\partial u_r}{\partial r}\right)^2+\frac{1}{2}\left(\frac{\partial u_3}{\partial r}\right)^2,\quad \varepsilon_{\theta\theta}=\frac{u_r}{r}+\frac{1}{2}\left(\frac{u_r}{r}\right)^2,$$

$$\varepsilon_{r3}=\frac{1}{2}\left(\frac{\partial u_3}{\partial r}+\frac{\partial u_r}{\partial y_3}+\frac{\partial u_r}{\partial r}\frac{\partial u_r}{\partial y_3}+\frac{\partial u_3}{\partial r}\frac{\partial u_3}{\partial y_3}\right),$$

$$\varepsilon_{zz}=\frac{\partial u_3}{\partial y_3}+\frac{1}{2}\left(\frac{\partial u_3}{\partial y_3}\right)^2+\frac{1}{2}\left(\frac{\partial u_r}{\partial y_3}\right)^2. \tag{7}$$

In this case the components S_{ij} of the Lagrange stress tensor are determined as follows:

$$S_{rr}=\frac{\partial\Phi}{\partial\varepsilon_{rr}}+pg^*_{rr},\quad S_{\theta\theta}=\frac{\partial\Phi}{\partial\varepsilon_{\theta\theta}}+pg^*_{\theta\theta},$$

$$S_{33}=\frac{\partial\Phi}{\partial\varepsilon_{zz}}+pg^*_{zz},\quad S_{r3}=\frac{\partial\Phi}{\partial\varepsilon_{r3}},\quad S_{r3}=S_{3r},$$

$$g^*_{rr}=1+2\frac{\partial u_r}{\partial r}+\left(\frac{\partial u_r}{\partial r}\right)^2+\left(\frac{\partial u_3}{\partial r}\right)^2,\quad g^*_{zz}=1+2\frac{\partial u_3}{\partial y_3}+\left(\frac{\partial u_3}{\partial y_3}\right)^2+\left(\frac{\partial u_r}{\partial y_3}\right)^2,$$

$$g^*_{\theta\theta}=1+2\frac{u_r}{r}+\left(\frac{u_r}{r}\right)^2. \tag{8}$$

Note that the expressions (6)-(8) are written in the arbitrary system of cylindrical coordinate system without any restriction related to the association of this system to the natural or initial state of the considered compound cylinders.

For the considered case the relations between the perturbation of the Kirchoff stress tensor and the perturbation of the components of the Lagrange stress tensor can be written as follows:

$$Q'^{(k)}_{r'r'}=\left(\lambda^{(\underline{k})}\right)^{-1}S^{(\underline{k})}_{r'r'},\quad Q'^{(k)}_{\theta'\theta'}=\left(\lambda^{(\underline{k})}\right)^{-1}S^{(\underline{k})}_{\theta'\theta'},$$

$$Q'^{(k)}_{33}=\left(\lambda^{(\underline{k})}\right)^{2}S^{(\underline{k})}_{33}+\lambda^{(\underline{k})}S^{(\underline{k}),0}_{33}\frac{\partial u^{(\underline{k})}_3}{\partial y_3},$$

$$Q'^{(k)}_{r'3}=\left(\lambda^{(\underline{k})}\right)^{1/2}S^{(\underline{k})}_{r'3},\quad Q'^{(k)}_{3r'}=\left(\lambda^{(\underline{k})}\right)^{1/2}S^{(\underline{k})}_{3r'}+\lambda^{(\underline{k})}S^{(\underline{k})}_{33}\frac{\partial u^{(\underline{k})}_r}{\partial y_3}. \tag{9}$$

According to Guz (1986a, 1986b, 2004), by linearization of equation (8) and taking (9), (1) and (2) into account, we obtain the following expressions for the constants $\chi'^{(k)}_{1111},\ldots,\chi'^{(k)}_{3333}$ in (5) for the potential (6):

$$\chi'^{(k)}_{1111}=\chi'^{(k)}_{2222}=4C^{(k)}_{10}(\lambda^{(k)})^{-1},$$

$$\chi'^{(k)}_{1122}=\chi'^{(k)}_{1133}=\chi'^{(k)}_{2233}=\chi'^{(k)}_{3311}=\chi'^{(k)}_{2211}=\chi'^{(k)}_{3322}=0,$$

$$\chi'^{(k)}_{1331} = 2C^{(k)}_{10}(\lambda^{(k)})^{-1}, \quad \chi'^{(k)}_{1221} = 2C^{(k)}_{10}(\lambda^{(k)})^{-1},$$
$$\chi'^{(k)}_{3333} = 2C^{(\underline{k})}_{10}\left(1+\left(\lambda^{(\underline{k})}\right)^{-3}\right)(\lambda^{(k)})^{2}, \quad \chi'^{(k)}_{1313} = \chi'^{(k)}_{3131} = 2C^{(\underline{k})}_{10}\left(\lambda^{(\underline{k})}\right)^{-1},$$
$$\chi'^{(k)}_{3113} = 2C^{(k)}_{10}(\lambda^{(k)})^{2}. \tag{10}$$

It should be noted that to the above equations the incompressibility condition of the fiber and matrix materials must be added. This condition for the considered case can be written as follows:

$$\frac{\partial u'^{(k)}_{r'}}{\partial r'} + \frac{u'^{(k)}_{r'}}{r'} + \frac{\partial u'^{(k)}_{3}}{\partial y'_3} = 0. \tag{11}$$

Thus, the wave propagation in the considered body will be investigated by the use of the equations (4), (5), (10) and (11). In this case we will assume that the following complete contact conditions are satisfied.

$$Q'^{(1)}_{r'r'}\Big|_{r'=R'} = Q'^{(2)}_{r'r'}\Big|_{r'=R'}, \quad Q'^{(1)}_{r'3}\Big|_{r'=R'} = Q'^{(2)}_{r'3}\Big|_{r'=R'},$$
$$u'^{(1)}_{r'}\Big|_{r'=R'} = u'^{(2)}_{r'}\Big|_{r'=R'}, \quad u'^{(1)}_{3}\Big|_{r'=R'} = u'^{(2)}_{3}\Big|_{r'=R'},$$
$$Q'^{(2)}_{r'r'}\Big|_{r'=R'+h'} = 0, \quad Q'^{(2)}_{r'3}\Big|_{r'=R'+h'} = 0. \tag{12}$$

With this we have exhausted the formulation of the problem and the consideration of the governing field equations. It should be noted that in the case where $\lambda^{(k)} = 1$, $(k = 1,2)$ equations (4), (5), (10), (11) and (12) for the k-th constituent transform to the corresponding ones of the classical linear theory of elastodynamics for incompressible bodies.

3 Solution procedure

To solve the dynamical problems of deformable solid body mechanics, various types of numerical and semi-analytical methods have been developed. The detailed analyses of these methods are given, for example, in papers by Yoda and Kodama (2006), Lu and Zhu (2007), Chen, Fu and Zhang (2007), Gato and Shie (2008), Liu, Chen, Li and Cen (2008), Lin, Lee, Tsai, Chen, Wang and Lee (2008), Wang and Wang (2008), Yao (2009), Dziatkiewicz and Fedelinski (2007), Gato and Shie (2008) and in many others. The systematic consideration of these methods was made in monographs by Atluri (2004, 2005). It should be noted that these methods are realized by employing modern computer modeling. At the same time, there are also other methods, so called analytical+numerical methods (see Mitra and Gopalakrishnan (2008), Willner (2009), Wei and Su (2008), Akbarov and Guliev (2009, 2010), Fu and Mielke (2002)) according to which, up to a certain stage of the solution procedure, analytical expressions are obtained for the sought values, but after this stage procedures based on visual numerical results arrived at with modern PC modeling are also employed. As in the previous paper by Akbarov and Guliev (2009), in the present paper also the latest version of computer modeling is employed. Thus we return to the solution procedure of the formulated problem above.

Substituting equation (5) in (4), we obtain the following equation of motion for the displacement terms:

$$\chi'^{(k)}_{1111}\frac{\partial^2 u'^{(k)}_{r'}}{\partial r'^2}+\chi'^{(k)}_{1122}\frac{\partial}{\partial r'}\left(\frac{u'^{(k)}_{r'}}{r'}\right)+\left(\chi'^{(k)}_{1133}+\chi'^{(k)}_{1331}\right)\frac{\partial^2 u'^{(k)}_{3}}{\partial r'\partial y'_3}+$$
$$\chi'^{(k)}_{1313}\frac{\partial^2 u'^{(k)}_{r'}}{\partial y'^2_3}+\frac{1}{r'}\left(\chi'^{(k)}_{1111}-\chi'^{(k)}_{2211}\right)\frac{\partial u'^{(k)}_{r'}}{\partial r'}+$$
$$\left(\chi'^{(k)}_{1122}-\chi'^{(k)}_{2222}\right)\frac{u'^{(k)}_{r'}}{r'^2}+\left(\chi'^{(k)}_{1133}-\chi'^{(k)}_{2233}\right)\frac{1}{r'}\frac{\partial u'^{(k)}_{3}}{\partial y'_3}=\rho'^{(k)}\frac{\partial^2 u'^{(k)}_{r'}}{\partial t^2}-\frac{\partial p'^{(k)}}{\partial r'};$$

$$\chi'^{(k)}_{3322}\frac{\partial^2 u'^{(k)}_{r'}}{\partial r'\partial y'_3}+\chi'^{(k)}_{3131}\frac{\partial^2 u'^{(k)}_{3}}{\partial r'^2}+\frac{1}{r'}\chi'^{(k)}_{3113}\frac{\partial u'^{(k)}_{r'}}{\partial y'_3}+\frac{1}{r'}\chi'^{(k)}_{3131}\frac{\partial u'^{(k)}_{3}}{\partial r'}+$$
$$\chi'^{(k)}_{3311}\frac{\partial^2 u'^{(k)}_{r'}}{\partial y'_3\partial r'}+\chi'^{(k)}_{3322}\frac{1}{r'}\frac{\partial u'^{k)}_{r'}}{\partial y'_3}+\chi'^{(k)}_{3333}\frac{\partial^2 u'^{(k)}_{3}}{\partial y'^2_3}=\rho'^{(k)}\frac{\partial^2 u'^{(k)}_{3}}{\partial t^2}-\frac{\partial p'^{(k)}}{\partial y'_3}. \quad (13)$$

Equations (11) and (13) compose the complete system with respect to the unknown functions $u'^{(k)}_{r'}$, $u'^{(k)}_{3}$ and $p'^{(k)}$. According to Guz (1986a), we use the following representation for the displacement and unknown function $p'^{(k)}$:

$$u'^{(k)}_{r'}=-\frac{\partial^2}{\partial r'\partial y'_3}\mathrm{X}'^{(k)},\quad u'^{(k)}_{3}=\Delta'_1\mathrm{X}'^{(k)},$$

$$p'^{(k)}=\left[\left(\chi'^{(k)}_{1111}-\chi'^{(k)}_{1133}-\chi'^{(k)}_{1313}\right)\Delta'_1+\chi'^{(k)}_{3113}\frac{\partial^2}{\partial y'^2_3}-\rho'^{(k)}\frac{\partial^2}{\partial t^2}\right]\frac{\partial}{\partial y'_3}\mathrm{X}'^{(k)}, \quad (14)$$

where

$$\Delta'_1=\frac{\partial^2}{\partial r'^2}+\frac{1}{r'}\frac{\partial}{\partial r'}, \quad (15)$$

The function $X'^{(k)}$ satisfies the following equation:

$$\left[\left(\Delta'_1+\left(\xi'^{(k)}_2\right)^2\frac{\partial^2}{\partial y'^2_3}\right)\left(\Delta'_1+\left(\xi'^{(k)}_3\right)^2\frac{\partial^2}{\partial y'^2_3}\right)-\frac{\rho'^{(k)}}{\chi'^{(k)}_{1331}}\left(\Delta'_1+\frac{\partial^2}{\partial y'^2_3}\right)\frac{\partial^2}{\partial t^2}\right]X'^{(k)} = 0, \quad (16)$$

where for the considered case

$$\left(\xi'^{(k)}_2\right)^2=\left(\lambda^{(k)}\right)^3, \quad \left(\xi'^{(k)}_3\right)^2=1, \quad \chi'^{(k)}_{1331}=2C^{(k)}_{10}\left(\lambda^{(k)}\right)^{-1}. \quad (17)$$

We represent the function $X'^{(k)}=X'^{(k)}\left(r',y'_3,t\right)$ as

$$X'^{(k)}=X'^{(k)}_1\left(r'\right)\cos(ky'_3-\omega t). \quad (18)$$

Substituting (18) in (16) and doing some mathematical manipulations we obtain the following equation for $X'^{(k)}_1\left(r'\right)$:

$$\left(\Delta'_1+\left(\zeta'^{(k)}_2\right)^2\right)\left(\Delta'_1+\left(\zeta'^{(k)}_3\right)^2\right)X'^{(k)}_1(r')=0. \quad (19)$$

The constants $\zeta'^{(k)}_{2,3}$ are determined from the following equation for the considered case:

$$\left(\zeta'^{(k)}_2\right)^2\left(\zeta'^{(k)}_3\right)^2=\left(\lambda^{(k)}\right)^3-\frac{\lambda^{(k)}c^2}{\left(c^{(k)}_2\right)^2},$$

$$\left(\zeta'^{(k)}_2\right)^2+\left(\zeta'^{(k)}_3\right)^2=\frac{\lambda^{(k)}c^2}{\left(c^{(k)}_2\right)^2}-1-\left(\lambda^{(k)}\right)^3,$$

$$\left(c^{(k)}_2\right)^2=\frac{2C^{(k)}_{10}}{\rho^{(k)}}, \quad c=\frac{\omega}{k}. \quad (20)$$

Introducing the notation $s^{(m)}=c\big/c^{(m)}_2$, we obtain from (20) that

$$\left(\zeta'^{(m)}_2\right)^2=\lambda^{(m)}\left(\left(s^{(m)}\right)^2-\left(\lambda^{(m)}\right)^2\right), \quad \left(\zeta'^{(m)}_3\right)^2=-1. \quad (21)$$

According to (19), (21) we can write:

$$X'^{(m)}_1\left(r'\right)=X'^{(m)}_{12}\left(r'\right)+X'^{(m)}_{13}\left(r'\right), \quad \left(\Delta'_1-1\right)X'^{(m)}_{12}\left(r'\right)=0,$$

$$\left[\Delta_1' + \lambda^{(m)}\left(\left(s^{(m)}\right)^2 - \left(\lambda^{(m)}\right)^2\right)\right] \mathrm{X}'^{(m)}_{13} = 0. \tag{22}$$

From (23) we determine the following expressions for the functions $\mathrm{X}'^{(1)}_1(r')$ and $\mathrm{X}'^{(2)}_1(r')$.

$$\mathrm{X}'^{(2)}_1(r') = A^{(2)} I_0(kr') + B^{(2)} E_0(kr'),$$

$$\mathrm{X}'^{(1)}_1(r') = A^{(1)} I_0(kr') + B^{(1)} G_0(kr') + C^{(1)} K_0(kr') +$$

$$D^{(1)} F_0(kr'), \tag{23}$$

where

$$G_0\left(kr'\right) = \begin{cases} J_0\left(\sqrt{\lambda^{(1)}\left(\left(s^{(1)}\right)^2 - \left(\lambda^{(1)}\right)^2\right)} kr'\right) & \text{if } s^{(1)} > \lambda^{(1)} \\ I_0\left(\sqrt{\lambda^{(1)}\left(\left(s^{(1)}\right)^2 - \left(\lambda^{(1)}\right)^2\right)} kr'\right) & \text{if } s^{(1)} < \lambda^{(1)} \end{cases},$$

$$F_0\left(kr'\right) = \begin{cases} Y_0\left(\sqrt{\lambda^{(1)}\left(\left(s^{(1)}\right)^2 - \left(\lambda^{(1)}\right)^2\right)} kr'\right) & \text{if } s^{(1)} > \lambda^{(1)} \\ K_0\left(\sqrt{\lambda^{(1)}\left(\left(s^{(1)}\right)^2 - \left(\lambda^{(1)}\right)^2\right)} kr'\right) & \text{if } s^{(1)} < \lambda^{(1)} \end{cases},$$

$$E_0\left(kr'\right) = \begin{cases} J_0\left(\sqrt{\lambda^{(2)}\left(\left(s^{(2)}\right)^2 - \left(\lambda^{(2)}\right)^2\right)} kr'\right) & \text{if } s^{(2)} > \lambda^{(2)} \\ I_0\left(\sqrt{\lambda^{(2)}\left(\left(s^{(2)}\right)^2 - \left(\lambda^{(2)}\right)^2\right)} kr'\right) & \text{if } s^{(2)} < \lambda^{(2)} \end{cases} \tag{24}$$

In (23) and (24) the functions $J_0(x)$ and $Y_0(x)$ are Bessel functions of the first and second kind of order zero; $I_0(x)$ and $K_0(x)$ are a Bessel function of a purely imaginary argument in order zero and a Macdonald function in order zero, in turn; in (23), (24) the constant k is a wave number of the propagating wave. Note that the expressions for the functions $E_0\left(kr'\right)$ and $F_0\left(kr'\right)$ for other cases which are not considered in (24), (i.e. for the cases where $s^{(m)} = \lambda^{(m)}$), can easily be determined. Therefore the corresponding expressions for $E_0\left(kr'\right)$ and $F_0\left(kr'\right)$ are not given here.

Thus, using (23), (24), (18), (14) and (5) we obtain the following dispersion equation from the contact conditions (12).

$$\det\left\|\alpha_{ij}\right\| = 0, \quad i;j = 1,2,3,4,5,6, \tag{25}$$

where

$$\alpha_{11}=\frac{C_{10}^{(2)}}{\lambda^{(2)}}\left[-\lambda^{(2)}\left(\left(\lambda^{(2)}\right)^2-\left(s^{(2)}\right)^2\right)I_0\left(kR'\right)-I_2\left(kR'\right)\right],\quad \alpha_{21}=\frac{C_{10}^{(2)}}{\lambda^{(2)}}I_1\left(kR'\right),$$

$$\alpha_{31}=I_1\left(kR'\right),\quad \alpha_{41}=I_0\left(kR'\right),$$

$$\alpha_{51}=0,\quad \alpha_{61}=0,$$

$$\alpha_{12}=\frac{C_{10}^{(2)}}{\lambda^{(2)}}\begin{cases}-\left(q^{(2)}\right)^2\left[I_0\left(q^{(2)}kR'\right)+I_2\left(q^{(2)}kR'\right)\right] & \text{if } \lambda^{(2)}<s^{(2)}\\ \left(q_1^{(2)}\right)^2\left[J_0\left(q_1^{(2)}kR'\right)-J_2\left(q_1^{(2)}kR'\right)\right] & \text{if } \lambda^{(2)}>s^{(2)}\end{cases},$$

$$\alpha_{22}=\frac{C_{10}^{(2)}}{2\lambda^{(2)}}\begin{cases}\left(q^{(2)}+\left(q^{(2)}\right)^3\right)I_1\left(q^{(2)}kR'\right) & \text{if } \lambda^{(2)}<s^{(2)}\\ \left(q_1^{(2)}-\left(q_1^{(2)}\right)^3\right)J_1\left(q_1^{(2)}kR'\right) & \text{if } \lambda^{(2)}>s^{(2)}\end{cases},$$

$$\alpha_{32}=\begin{cases}q^{(2)}I_1\left(q^{(2)}kR'\right) & \text{if } \lambda^{(2)}<s^{(2)}\\ q_1^{(2)}J_1\left(q_1^{(2)}kR'\right) & \text{if } \lambda^{(2)}>s^{(2)}\end{cases},$$

$$\alpha_{42}=\begin{cases}\left(q^{(2)}\right)^2I_0\left(q^{(2)}kR'\right) & \text{if } \lambda^{(2)}<s^{(2)}\\ -\left(q_1^{(2)}\right)^2J_0\left(q_1^{(2)}kR'\right) & \text{if } \lambda^{(2)}>s^{(2)}\end{cases},$$

$$\alpha_{52}=0,\quad \alpha_{62}=0,$$

$$\alpha_{13}=-\frac{C_{10}^{(1)}}{\lambda^{(1)}}\left[-\lambda^{(1)}\left(\left(\lambda^{(1)}\right)^2-\left(s^{(1)}\right)^2\right)I_0\left(kR'\right)-I_2\left(kR'\right)\right],$$

$$\alpha_{23}=-\frac{C_{10}^{(1)}}{\lambda^{(1)}}I_1\left(kR'\right),\quad \alpha_{33}=-I_1\left(kR'\right),\quad \alpha_{43}=-I_0\left(kR'\right),$$

$$\alpha_{53}=-\frac{C_{10}^{(1)}}{\lambda^{(1)}}\left[-\lambda^{(1)}\left(\left(\lambda^{(1)}\right)^2-\left(s^{(1)}\right)^2\right)I_0\left(kR'(1+h'/R')\right)-I_2\left(kR'(1+h'/R')\right)\right],$$

$$\alpha_{63}=I_1\left(kR'(1+h'/R')\right),$$

$$\alpha_{14}=-\frac{C_{10}^{(1)}}{\lambda^{(1)}}\left[-\lambda^{(1)}\left(\left(\lambda^{(1)}\right)^2-\left(s^{(1)}\right)^2\right)K_0\left(kR'\right)-K_2\left(kR'\right)\right],$$

$$\alpha_{24}=\frac{C_{10}^{(1)}}{\lambda^{(1)}}K_1\left(kR'\right),\quad \alpha_{34}=K_1\left(kR'\right),\quad \alpha_{44}=-K_0\left(kR'\right),$$

$$\alpha_{54} = -\frac{C_{10}^{(1)}}{\lambda^{(1)}}\left[-\lambda^{(1)}\left(\left(\lambda^{(1)}\right)^2 - \left(s^{(1)}\right)^2\right)K_0\left(kR'(1+h'/R')\right) - K_2\left(kR'(1+h'/R')\right)\right],$$

$$\alpha_{64} = -K_1\left(kR'(1+h'/R')\right),$$

$$\alpha_{15} = \frac{C_{10}^{(1)}}{\lambda^{(1)}}\begin{cases}\left(q^{(1)}\right)^2\left[I_0\left(q^{(1)}kR'\right) + I_2\left(q^{(1)}kR'\right)\right] & \text{if } \lambda^{(1)} < s^{(1)} \\ -\left(q_1^{(1)}\right)^2\left[J_0\left(q_1^{(1)}kR'\right) - J_2\left(q_1^{(1)}kR'\right)\right] & \text{if } \lambda^{(1)} > s^{(1)}\end{cases},$$

$$\alpha_{25} = -\frac{C_{10}^{(1)}}{2\lambda^{(1)}}\begin{cases}((q^{(1)})^3 + q^{(1)})I_1(q^{(1)}kR') & \text{if } \lambda^{(1)} < s^{(1)} \\ ((q^{(1)})^3 - q^{(1)})J_1(q^{(1)}kR') & \text{if } \lambda^{(1)} > s^{(1)}\end{cases},$$

$$\alpha_{35} = \begin{cases}-q^{(1)}I_1(q^{(1)}kR') & \text{if } \lambda^{(1)} < s^{(1)} \\ q^{(1)}J_1(q^{(1)}kR') & \text{if } \lambda^{(1)} > s^{(1)}\end{cases},$$

$$\alpha_{45} = \begin{cases}-(q^{(1)})^2I_0(q^{(1)}kR') & \text{if } \lambda^{(1)} < s^{(1)} \\ (q^{(1)})^2J_0(q^{(1)}kR') & \text{if } \lambda^{(1)} > s^{(1)}\end{cases},$$

$$\alpha_{55} = \frac{C_{10}^{(1)}}{\lambda^{(1)}}\begin{cases}\left(q^{(1)}\right)^2\left[I_0\left(q^{(1)}kR'(1+h'(R')\right) + I_2\left(q^{(1)}kR'(1+h'(R')\right)\right] \\ \text{if } \lambda^{(1)} < s^{(1)} \\ -\left(q_1^{(1)}\right)^2\left[J_0\left(q_1^{(1)}kR'(1+h'(R')\right) - J_2\left(q_1^{(1)}kR'(1+h'(R')\right)\right] \\ \text{if } \lambda^{(1)} > s^{(1)}\end{cases}$$

$$\alpha_{65} = -\frac{C_{10}^{(1)}}{2\lambda^{(1)}}\begin{cases}((q^{(1)})^3 + q^{(1)})I_1(q^{(1)}kR'(1+h'/R')) & \text{if } \lambda^{(1)} < s^{(1)} \\ ((q^{(1)})^3 - q^{(1)})J_1(q^{(1)}kR'(1+h'/R')) & \text{if } \lambda^{(1)} > s^{(1)}\end{cases},$$

$$\alpha_{16} = \frac{C_{10}^{(1)}}{\lambda^{(1)}}\begin{cases}\left(q^{(1)}\right)^2\left[K_0\left(q^{(1)}kR'\right) + K_2\left(q^{(1)}kR'\right)\right] & \text{if } \lambda^{(1)} < s^{(1)} \\ -\left(q_1^{(1)}\right)^2\left[Y_0\left(q_1^{(1)}kR'\right) - Y_2\left(q_1^{(1)}kR'\right)\right] & \text{if } \lambda^{(1)} > s^{(1)}\end{cases},$$

$$\alpha_{26} = \frac{C_{10}^{(1)}}{2\lambda^{(1)}}\begin{cases}((q^{(1)})^3 + q^{(1)})K_1(q^{(1)}kR') & \text{if } \lambda^{(1)} < s^{(1)} \\ -((q^{(1)})^3 - q^{(1)})Y_1(q^{(1)}kR') & \text{if } \lambda^{(1)} > s^{(1)}\end{cases},$$

$$\alpha_{36} = \begin{cases}q^{(1)}K_1(q^{(1)}kR') & \text{if } \lambda^{(1)} < s^{(1)} \\ q^{(1)}Y_1(q^{(1)}kR') & \text{if } \lambda^{(1)} > s^{(1)}\end{cases},$$

$$\alpha_{46} = \begin{cases}-(q^{(1)})^2K_0(q^{(1)}kR') & \text{if } \lambda^{(1)} < s^{(1)} \\ (q^{(1)})^2Y_0(q^{(1)}kR') & \text{if } \lambda^{(1)} > s^{(1)}\end{cases},$$

$$\alpha_{56}=\frac{C_{10}^{(1)}}{\lambda^{(1)}}\begin{cases}-\left(q^{(1)}\right)^2\left[K_0\left(q^{(1)}kR'(1+h'(R')\right)+K_2\left(q^{(1)}kR'(1+h'(R')\right)\right]\\ \text{if } \lambda^{(1)}<s^{(1)}\\ -\left(q_1^{(1)}\right)^2\left[Y_0\left(q_1^{(1)}kR'(1+h'(R')\right)-Y_2\left(q_1^{(1)}kR'(1+h'(R')\right)\right]\\ \text{if } \lambda^{(1)}>s^{(1)}\end{cases}$$

$$\alpha_{66}=-\frac{C_{10}^{(1)}}{2\lambda^{(1)}}\begin{cases}-((q^{(1)})^3+q^{(1)})K_1(q^{(1)}kR'(1+h'/R')) & \text{if } \lambda^{(1)}<s^{(1)}\\ ((q^{(1)})^3-q^{(1)})Y_1(q^{(1)}kR'(1+h'/R')) & \text{if } \lambda^{(1)}>s^{(1)}\end{cases},$$

$$q^{(m)}=\sqrt{\lambda^{(\underline{m})}\left(\left(\lambda^{(\underline{m})}\right)^2-\left(s^{(\underline{m})}\right)^2\right)}\quad q_1^{(m)}=\sqrt{\lambda^{(\underline{m})}\left(\left(s^{(\underline{m})}\right)^2-\left(\lambda^{(\underline{m})}\right)^2\right)}. \tag{26}$$

Thus, the dispersion equation for the considered wave propagation problem has been derived in forms (25) and (26).

4 Numerical results and discussions

The quantities regarding the fiber (matrix) will be denoted below by the upper symbol $(f)((m))$ instead of the upper index (2) ((1)). Now we investigate the influence of the initial

strains determined by the elongations $\lambda^{(f)}$ and $\lambda^{(m)}$ on the dispersion curves

$$c=c\left(kR,h/R,\rho^{(m)},\rho^{(f)},C_{10}^{(f)}/C_{10}^{(m)},\lambda^{(m)},\lambda^{(f)}\right). \tag{27}$$

At this point, the notation $e=C_{10}^{(f)}/C_{10}^{(m)}$ is introduced. It should be noted that in the framework of the paper we cannot consider the analyses and discussions of numerical results regarding the possible problem parameters. Therefore, we must bound the change range of these parameters. In connection with this, in the present paper we assume that $e=5.0$.

The packet programs by employing of which the numerical results are obtained have been tested on known problems such as axisymmetric wave propagation in a solid cylinder as well as in a hollow cylinder, separately. Some fragments of theses tests will be indicated below.

To establish the effect of the waves reflected from the outer surface of the external hollow cylinder on the dispersion of the wave propagation in the considered compound (composite) cylinder, first we consider the case where $h/R=\infty$; i.e. the case which corresponds the wave propagation in the solid cylinder contained in the infinite body. Note that the dispersion equation corresponding to this case is attained from equations (25), (26) by the evident corrections.

4.1 Numerical results regarding the case where $h/R = \infty$

According to the physical and mechanical considerations the results obtained in this case may be more reasonable under $h >> R$, $e >> 1$ for a certain finite interval of time. First we consider the case where $\lambda^{(f)} = \lambda^{(m)} = 1.0$, i.e. where the initial strains in the constituents of the considered body are absent. The dispersion curve for this case is given in Fig. 2 for the first three modes for the various values of $\rho^{(m)}/\rho^{(f)}$(where $\rho^{(m)}$ ($\rho^{(f)}$) is a density of the infinite body (solid cylinder)). Moreover, in Fig. 2, dashed lines show the corresponding dispersion curves for the solid cylinder. It follows from these graphs that the contact of the solid cylinder with surrounding infinite material causes the values $c/c_2^{(f)}$ to decrease because $c_2^{(f)} > c_2^{(m)}$. Moreover, it follows from these graphs that the mode1 for the solid cylinder has a finite limit as $kR \to 0$and this limit is equal to the corresponding bar velocity. At the same time, the wave propagation velocity for this mode approaches to the Rayleigh wave velocity as $kR \to \infty$. However, such types of modes; i.e. the mode for which the wave propagation velocity has a finite limit as $kR \to 0$, disappears in the case where the cylinder is surrounded by an infinite body.

Detail analyses of the numerical results show that around the points indicated by the small circles (Fig. 2) in the dispersion curves, stop band (narrow) zones arise.

Let us to introduce notation $s = c/c_2^{(f)}$, where the values of s for the aforementioned points denote $s_1 < s_2 < ... < s_n < ... < s_N$. To analyze the meaning of the parts of the dispersion curves we consider the dispersion diagrams given in Fig. 3 for the case $\rho^{(f)}/\rho^{(m)} = 0.7$. It follows from these diagrams that for the considered modes in which the points s_1 and s_2 exist only, the parts of the dispersion curves determined by the relations $s > s_1$, $s_2 < s < s_1$ and $s < s_2$ correspond to the backward wave, anomalous dispersion and normal dispersion, respectively.

According to the expression $c_2^{(f)}/c_2^{(m)} = \sqrt{C_{10}^{(f)}\rho^{(m)}/\left(C_{10}^{(m)}\rho^{(f)}\right)}$, for fixed $C_{10}^{(f)}/C_{10}^{(m)}$ the decreasing of the values of $c_2^{(f)}/c_2^{(m)}$corresponds to the decreasing of the $\rho^{(m)}/\rho^{(f)}$ and this statement causes an increase in the values of $c/c_2^{(f)}$ with reducing of the $\rho^{(m)}/\rho^{(f)}$. Note that this conclusion agrees well with the physical considerations and with the corresponding results obtained in paper Parnes (1981) which regards the study of the waves propagated in a system consisting of a linear elastic rod embedded in a compressible linear elastic medium. However, in paper Parnes (1981) the equation of motion for the rod is written within the framework of the Bernoulli hypotheses. From the viewpoint of the authors, therefore, effects of the stop band zones discussed above were not observed in investigation Parnes (1981). Because the backward wave, band zones and other similar type particularities of the wave propagation in the rods or in the compound rods is caused namely with the radial

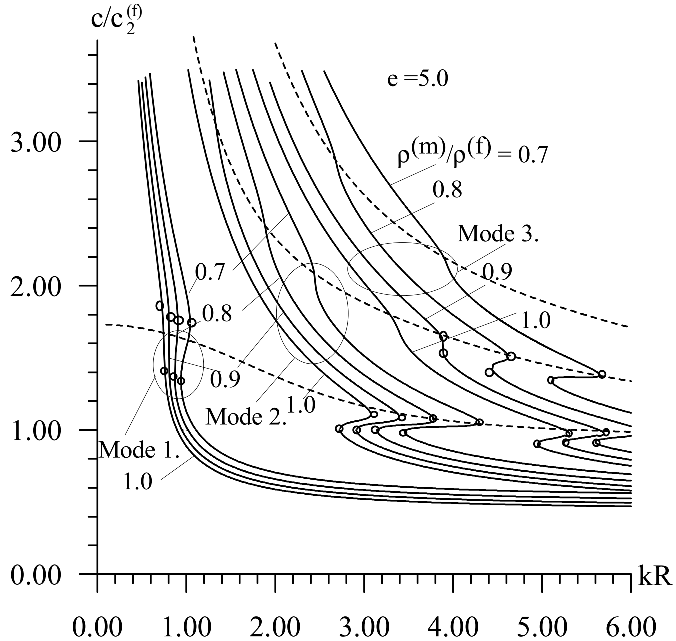


Figure 2: Dispersion curves attained in the case where $h/R = \infty$ for various $\rho^{(m)}/\rho^{(f)}$ and for the first three modes.

displacement which within the scope of the Bernoulli hypotheses is not taken into account.

Now we consider the influence of the initial strains of the constituents on the analyzed wave dispersion. For this purpose we consider only the case where $\rho^{(m)}/\rho^{(f)} = 0.7$. Note that in the qualitative sense the same results are obtained for other values of the relation $\rho^{(m)}/\rho^{(f)}$.

Fig. 4 shows the influence of the pre-stretching of the cylinder on the dispersion curves in the case where the initial strain in the surrounding infinite body is absent, i.e. $\lambda^{(m)} = 1.0$. It follows from the graphs that in modes 2 and 3, the number of points s_n increases with initial strains. Nevertheless, the part of the dispersion curves for which $s < s_N$ corresponds to normal dispersion. However, those parts which are similar to the part for which $s \in (s_N, s_{n-1})$ $(s \in (s_{N-1}, s_{N-2}))$correspond to an anomalous dispersion (backward wave).

We denote the values of kR corresponding to the values $s_1 < s_2 < ... < s_n < ... < s_N$

through the $(kR)_1$, $(kR)_2$,...,$(kR)_n$,..., $(kR)_N$. Moreover, we introduce the notation

$$(kR)_* = \min\{(kR)_1\,,\quad (kR)_2,\ldots,\quad (kR)_n,\ldots,\quad (kR)_N\} \tag{28}$$

The analyses of the results show that the values of $(kR)_*$ depends significantly of the initial strains of the cylinder; i.e. the values of $(kR)_*$ decrease with $\lambda^{(f)}$. At the same time, the values of s_N increase with $\lambda^{(f)}$. Consequently, for the fixed mode, $\min s_N$ is the $s_N$attained for the case where $\lambda^{(f)} = 1.0$, but $\min(kR)_*$ is the $(kR)_*$ attained for the $\max(\lambda^{(f)})$ (for the considered case $\max(\lambda^{(f)}) = 1.90$). According to the aforementioned notation and according to the results illustrated in Fig. 4, we can conclude that under $kR < (kR)_N^*$ (where $(kR)_N^*$ corresponds to the $\min s_N$) and under $kR > \min(kR)_*$ (for the parts which correspond to normal dispersion) the wave propagation velocity increase with the initial stretching of the solid cylinder.

Note that similar results occur also for the case where initial stretching exists only in the surrounding infinite body. The dispersion curves for this case are given in Fig. 5, according to which the wave propagation velocity decreases (increases) with $\lambda^{(m)}$ under $kR < (kR)_N$ ($kR > (kR)_*$ for normal dispersion parts). The same results also occur for the case where the constituents of the considered body are pre-stretched simultaneously. The dispersion curves attained for this case are illustrated in Fig. 6. However in this case, the length of the stop band zones increase with $\lambda(= \lambda^{(f)} = \lambda^{(m)})$. Moreover, in Fig. 3, for example, the dispersion diagrams are given for the case where $\lambda = 1.20$. These diagrams illustrate more clearly the influence of the simultaneous pre-stretching of the constituents of the compound cylinder on the values of the s_n and $(kR)_n$.

With the preceeding, we restrict ourselves to consideration of the numerical results attained for the system which comprises the solid cylinder and the surrounding infinite body. The character of the obtained results can be explained by the increase in the material rigidity with the initial stretching and by the kind of nonlinearity of the materials.

4.2 Numerical results regarding to the compound cylinder

As in the previous subsection, first we consider the case where $\lambda^{(f)} = \lambda^{(m)} = 1.0$; i.e. the case where the initial stretching in the constituents of the compound cylinder is absent. The dispersion curves for this case are given in Fig. 7 for the first three modes under various h/R. These curves show that, as a result of taking into account the waves, reflection from the free surface of the external hollow cylinder, the wave propagation velocity in the first mode (Fig. 7, a) has a finite limit as $kR \to 0$. Note that this limit is a bar velocity (denote it byc_b) of the considered compound cylinder which is determined by the expression

$$\frac{c_b}{c_2^{(f)}} = \sqrt{3}\left(\frac{C_{10}^{(m)}}{C_{10}^{(f)}}\eta^{(m)} + \eta^{(f)}\right)^{\frac{1}{2}}\left(\frac{\rho^{(m)}}{\rho^{(f)}}\eta^{(m)} + \eta^{(f)}\right)^{-\frac{1}{2}} \tag{29}$$

where

$$\eta^{(m)} = \left(2\frac{h}{R} + \left(\frac{h}{R}\right)^2\right)\left(\left(1+\frac{h}{R}\right)^2\right)^{-1}, \quad \eta^{(f)} = \left(1+\frac{h}{R}\right)^{-2}. \tag{30}$$

Moreover, in the first mode the wave propagation velocity approach also to the Rayleigh wave velocity for the external hollow cylinder material (denote it by $c_R^{(m)}$); i.e.

$$\frac{c_R^{(m)}}{c_2^{(f)}} = \sqrt{\frac{C_{10}^{(m)}}{C_{10}^{(f)}}\frac{\rho^{(f)}}{\rho^{(m)}}(1-x_*^2)} \tag{31}$$

where $x_* \approx 0.2916$.

It follows from the graphs given in Fig. 7 that as a result of accounting for the wave reflection from the free surface of the external hollow cylinder, the parts of the dispersion curves which correspond to the anomalous dispersion observed for the case $h/R = \infty$ disappear. Also Fig. 7 shows that the wave propagation velocity in the considered modes decrease with h/R. This decreasing can be explained with the increasing of the volumetric concentration (i.e. of the $\eta^{(m)}$) of the external hollow cylinder material in the compound cylinder with h/R and with the assumption $c_2^{(m)}/c_2^{(f)} < 1$. In other words, the increasing of the surround cylinder material the wave propagation velocity for which is less than that in the inner cylinder material, the wave propagation velocity in the compound cylinder decreases. At the same time, the graphs given in Fig. 7 show that for fixed kR the values of $c/c_2^{(f)}$approach to the certain limit value of $c/c_2^{(f)}$ as h/R increases. However this asymptotic value does not coincide with corresponding value of the $c/c_2^{(f)}$ obtained for the case where $h/R = \infty$. This statement is characteristic for dynamical problems and is explained with neglecting the wave reflection from the external surface surrounding the body in the direction which is perpendicular to the wave propagation velocity.

Consider the numerical results illustrating the influence of the initial stretching of the components of the compound cylinder on the wave propagation velocity. These results for the first mode are shown in Fig. 8 under $h/R = 0.1$ (Fig. 8, a and Fig. 8,

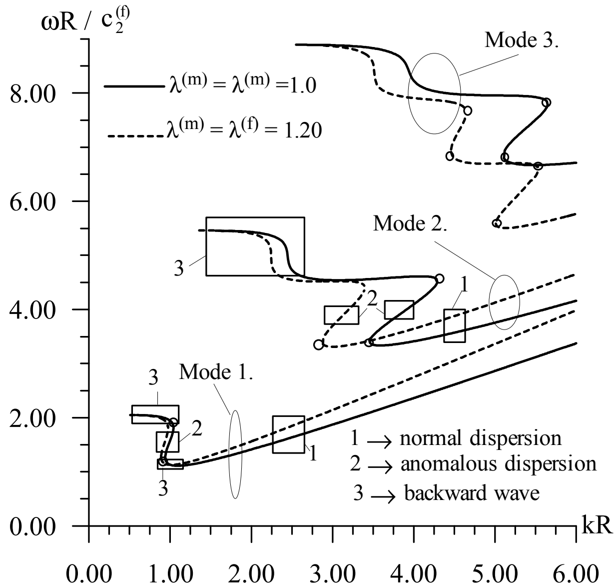


Figure 3: Dispersion diagrams constructed for the case where $\rho^{(m)}/\rho^{(f)} = 0.7$ under $h/R = \infty$ for the first three modes.

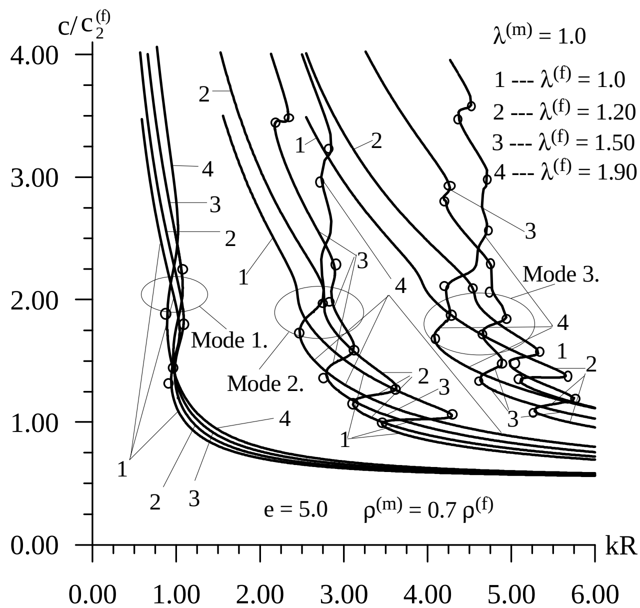


Figure 4: The influence of the pre-stretching of the solid cylinder on the wave dispersion curves in the case where $h/R = \infty$.

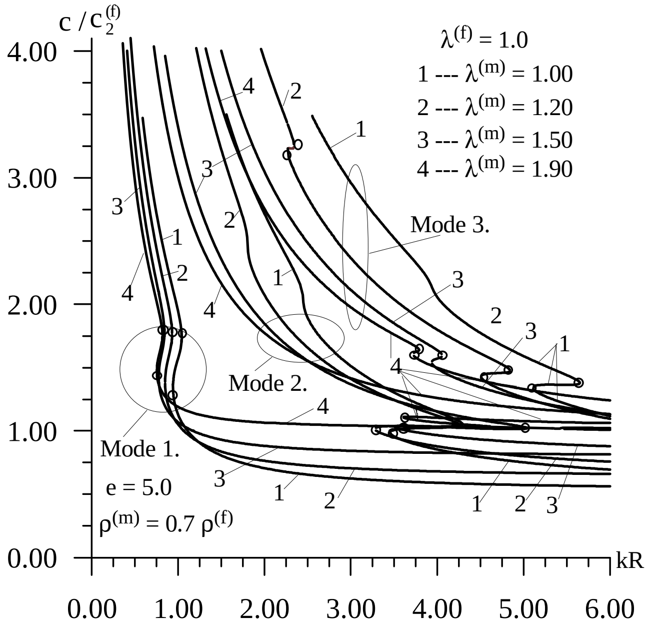


Figure 5: The influence of the pre-stretching of the surrounding infinite body on the wave dispersion curves in the case where $h/R = \infty$.

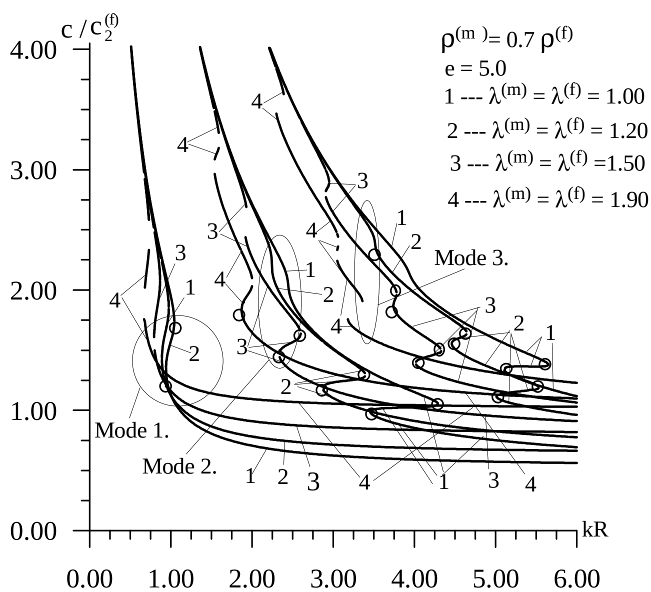


Figure 6: The influence of the simultaneous pre-stretching of the solid cylinder and surrounding infinite body on the dispersion curves in the case where $h/R = \infty$.

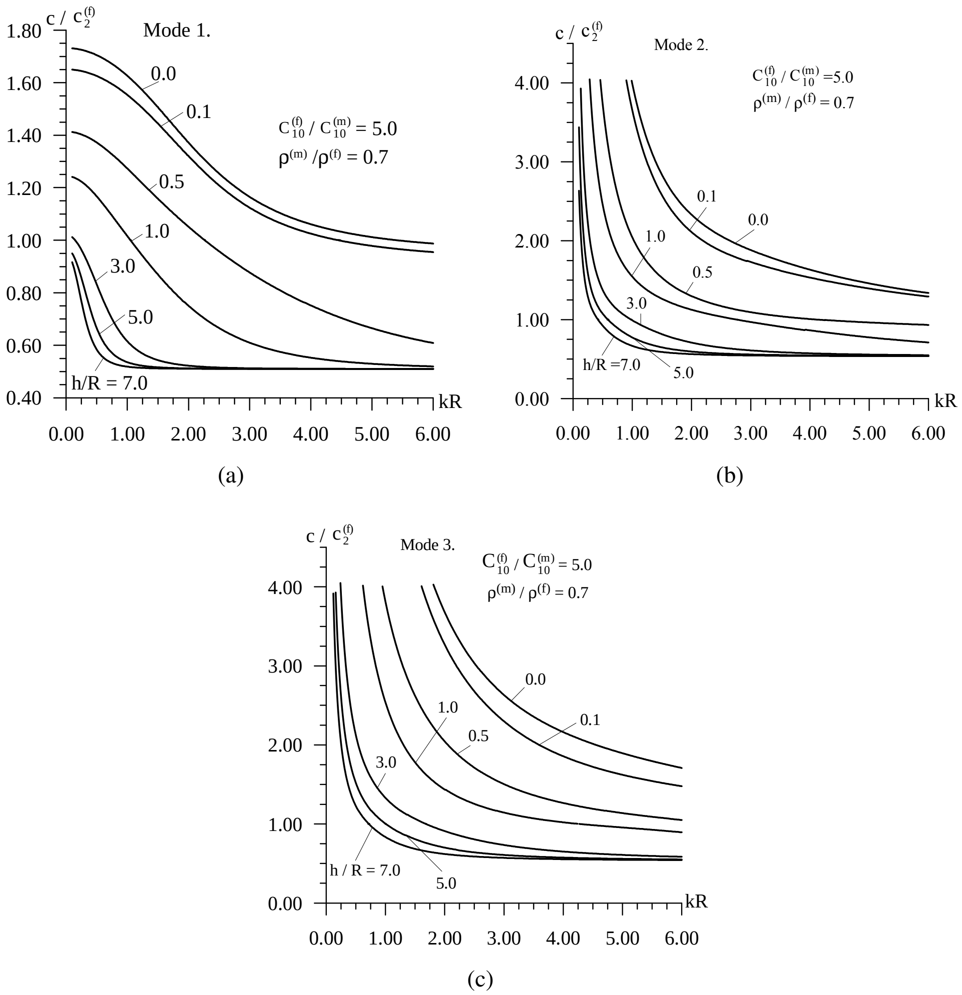
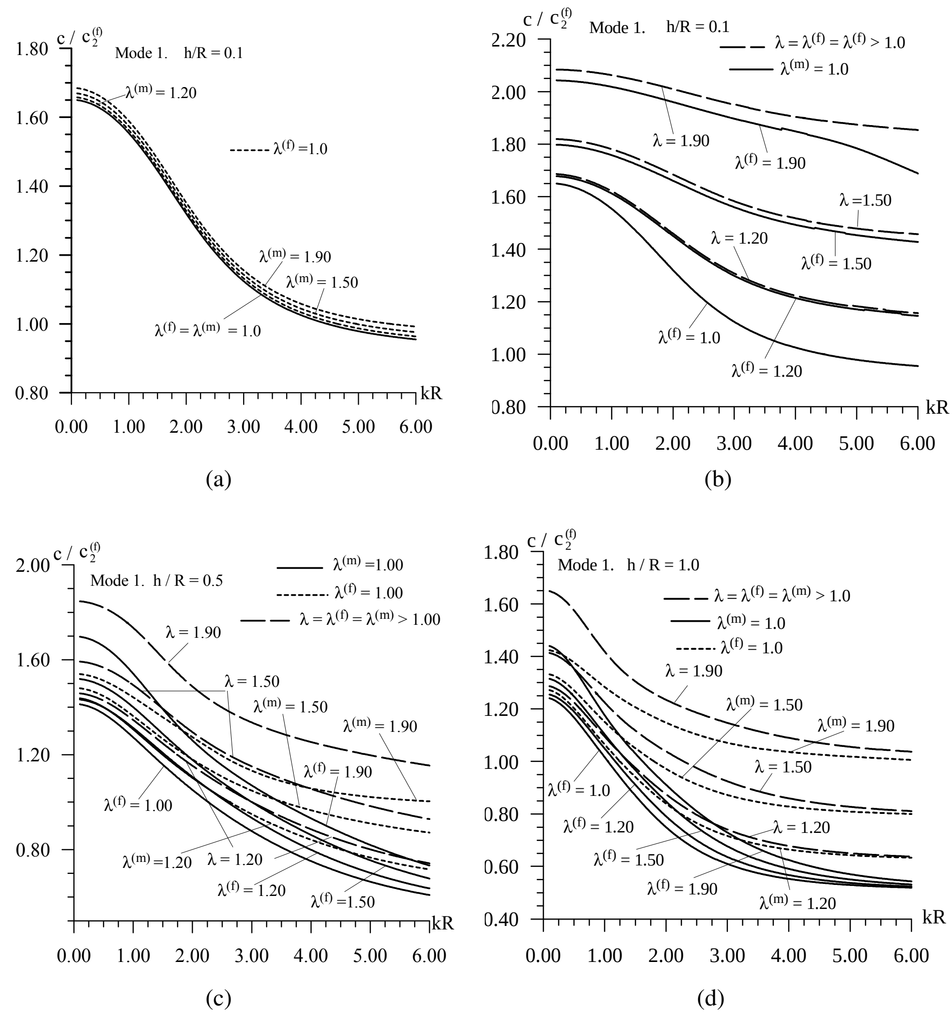
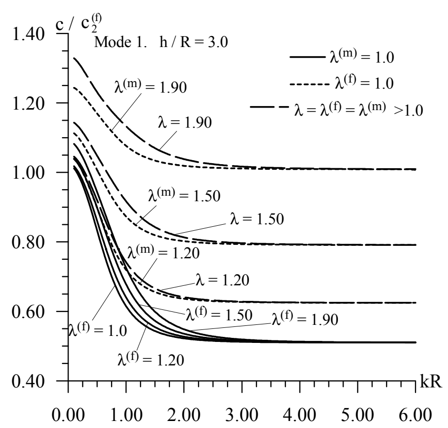


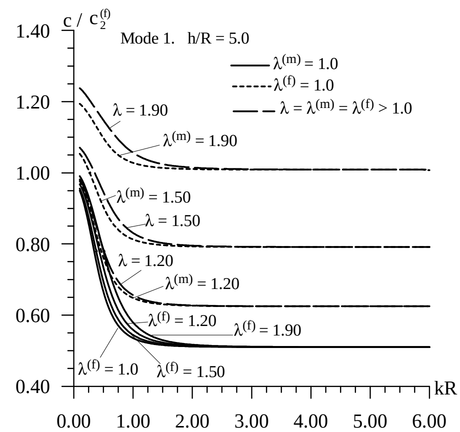
Figure 7: Dispersion curves for the compound cylinder constructed for various h/R: (a) mode 1; (b) mode 2; (c) mode 3.

b), 0.5 (Fig. 8,c), 1.0 (Fig. 8,d), 3.0 (Fig. 8, e), 5.0 (Fig. 8, f) and 7.0 (Fig. 8, g). It follows from these graphs that in all considered cases the initial stretching causes the wave propagation velocity for all considered values of kR to increase. In these cases the influence of the $\lambda^{(m)}(\lambda^{(f)})$on the values of $c/c_2^{(f)}$, predictably, increases (decreases) with h/R. At the same time, the values of $c/c_2^{(f)}$ have a limit as $kR \to 0$ which can be taken as the bar velocity of the pre-strained compound cylinder. This

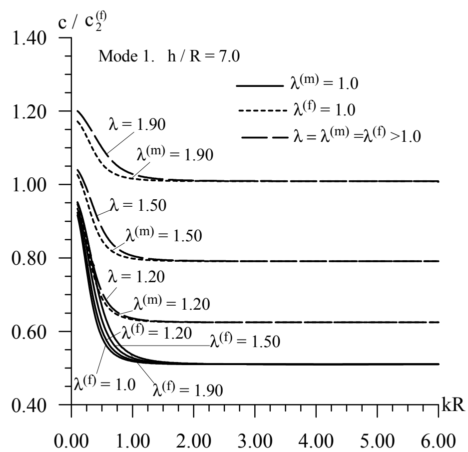

c / $c_2^{(f)}$
Mode 1. h/R = 0.1
$\lambda^{(m)} = 1.20$
$\lambda^{(f)} = 1.0$
$\lambda^{(m)} = 1.90$
$\lambda^{(m)} = 1.50$
$\lambda^{(f)} = \lambda^{(m)} = 1.0$
kR
(a)
Mode 1. h/R = 0.1
$\lambda = \lambda^{(f)} = \lambda^{(f)} > 1.0$
$\lambda^{(m)} = 1.0$
$\lambda = 1.90$
$\lambda^{(f)} = 1.90$
$\lambda = 1.50$
$\lambda^{(f)} = 1.50$
$\lambda = 1.20$
$\lambda^{(f)} = 1.0$
$\lambda^{(f)} = 1.20$
kR
(b)
Mode 1. h / R = 0.5
$\lambda^{(m)} = 1.00$
$\lambda^{(f)} = 1.00$
$\lambda = \lambda^{(f)} = \lambda^{(m)} > 1.00$
$\lambda = 1.90$
$\lambda = 1.50$
$\lambda^{(m)} = 1.50$
$\lambda^{(m)} = 1.90$
$\lambda^{(f)} = 1.90$
$\lambda^{(f)} = 1.00$
$\lambda^{(m)} = 1.20$
$\lambda = 1.20$
$\lambda^{(f)} = 1.20$
$\lambda^{(f)} = 1.50$
kR
(c)
Mode 1. h / R = 1.0
$\lambda = \lambda^{(f)} = \lambda^{(m)} > 1.0$
$\lambda^{(m)} = 1.0$
$\lambda^{(f)} = 1.0$
$\lambda = 1.90$
$\lambda^{(m)} = 1.50$
$\lambda^{(m)} = 1.90$
$\lambda = 1.50$
$\lambda^{(f)} = 1.0$
$\lambda^{(f)} = 1.20$
$\lambda = 1.20$
$\lambda^{(f)} = 1.50$
$\lambda^{(f)} = 1.90$
$\lambda^{(m)} = 1.20$
kR
(d)



(e)



(f)



(g)

Figure 8: The influence of the pre-stretching of the components of the compound cylinder on the wave dispersion curves in the mode 1 : (a) $h/R = 0.1$, $\lambda^{(f)} = 1.0$; (b) $h/R = 0.1$ for the cases where $\lambda^{(m)} = 1.0$ and $\lambda^{(f)} = \lambda^{(m)} > 1.0$; (c) $h/R = 0.5$; (d) $h/R = 1.0$; (e) $h/R = 3.0$; (f) $h/R = 5.0$; (g) $h/R = 7.0$.

bar velocity is determined by the following expression:

$$\frac{c_b}{c_2^{(f)}} = \left(\frac{C_{10}^{(m)}}{C_{10}^{(f)}} \eta^{(m)} \left(\left(\lambda^{(m)} \right)^2 + \frac{2}{\lambda^{(m)}} \right) + \right.$$

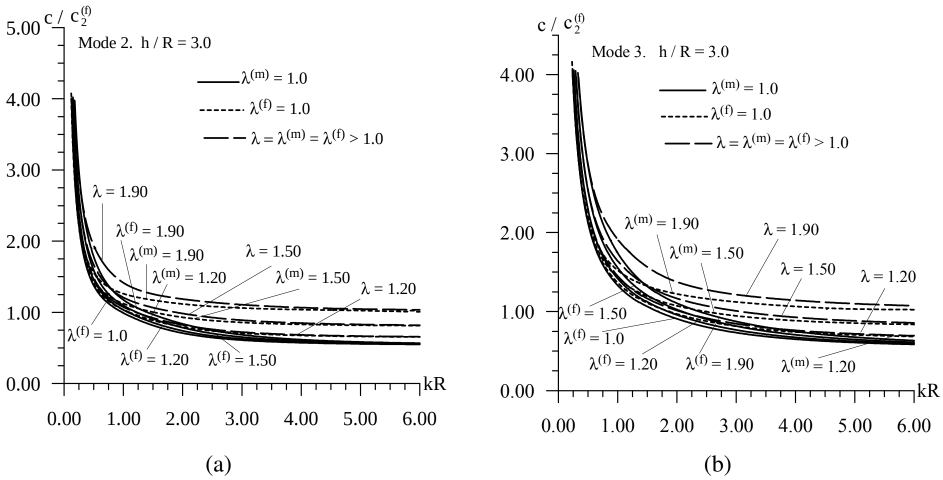


Figure 9: The influence of the pre-stretching of the components of the compound cylinder on the wave dispersion curves under $h/R = 3.0$: (a) mode 2; (b) mode 3.

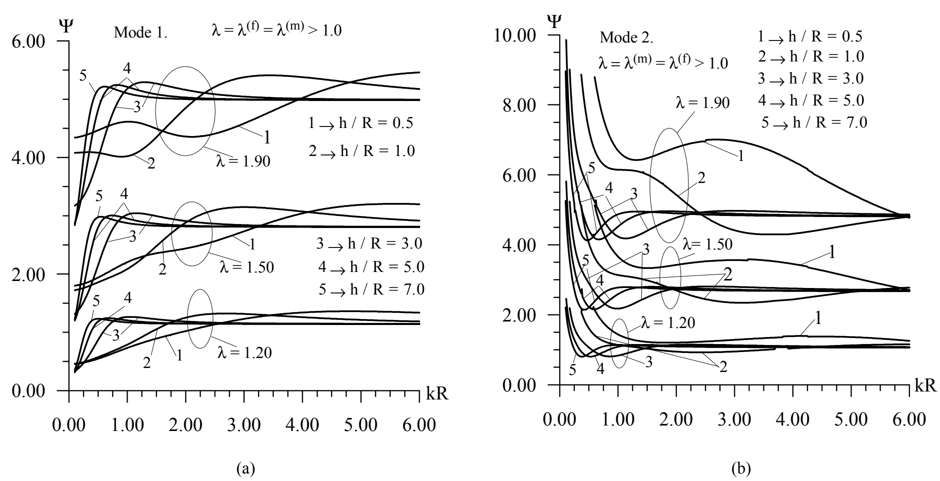


Figure 10: The graphs of the dependencies between Ψ (34) and kR attained for various values of h/R: (a) mode 1; (b) mode 2.

$$\eta^{(f)}\left(\left(\lambda^{(f)}\right)^2+\frac{2}{\lambda^{(f)}}\right)\right)^{\frac{1}{2}}\left(\frac{\rho^{(m)}}{\rho^{(f)}}\eta^{(m)}+\eta^{(f)}\right)^{-\frac{1}{2}} \tag{32}$$

Note that this expression is obtained by the following manner. First from the equation $\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}=0$ (which is a dispersion relation of the axisymmetric longitudinal wave propagation in the solid cylinder we determine the asymptotic root as $kR\to\infty$. This root is determined to be $c=\sqrt{C_{10}^{(f)}\left(\left(\lambda^{(f)}\right)^2+\left(\lambda^{(f)}\right)^{-1}\right)/\rho^{(f)}}$. Taking the expression $C_{10}^{(f)}\left(\left(\lambda^{(f)}\right)^2+\left(\lambda^{(f)}\right)^{-1}\right)$ as a "modulus of elasticity'' for the pre-stretched cylinder, we determine the effective (normalized) "modulus of elasticity" for the compound cylinder by the use of the well-known expression $\left[C_{10}^{(f)}\left(\left(\lambda^{(f)}\right)^2+\left(\lambda^{(f)}\right)^{-1}\right)\eta^{(f)}+C_{10}^{(m)}\left(\left(\lambda^{(m)}\right)^2+\left(\lambda^{(m)}\right)^{-1}\right)\eta^{(m)}\right]$. Dividing this expression into the averaged density $\left(\rho^{(f)}\eta^{(f)}+\rho^{(m)}\eta^{(m)}\right)$, we determine the expression (32) which coincide with the expression (29) under $\lambda^{(f)}=\lambda^{(m)}=1.0$.

Moreover, it follows from the results given in Fig. 8 that under $kR\to\infty$ the wave propagation velocity in the first mode approach to the Rayleigh wave propagation velocity of the pre-strained external hollow cylinder material. According to Guz (2004), this velocity for the considered case can be expressed by the expression

$$\frac{c_R^{(m)}}{c_2^{(f)}}=\lambda^{(m)}\sqrt{\frac{C_{10}^{(m)}}{C_{10}^{(f)}}\frac{\rho^{(f)}}{\rho^{(m)}}\left(1-x_*^2\left(\lambda^{(m)}\right)^{-4}\right)}, \qquad (33)$$

which coincides with (31) for $\lambda^{(m)}=1.0$.

Thus we can explain the character of the graphs given in Fig. 8. The numerical results which are not given here show that in the higher order modes of the considered wave propagation, the initial stretching of the components of the compound cylinder also causes the wave propagation velocity to increase. This concluding is also confirmed with the graphs illustrated in Fig. 9. Note that these graphs show the dependencies between $c/c_2^{(f)}$and $kR$for the modes 2 (Fig. 9,a) and 3 (Fig. 9,b) for the various $\lambda=\lambda^{(m)}=\lambda^{(f)}>1.0$ under $h/R=3.0$. For a more clear illustration of the previously discussed influence in Fig. 10, the graphs of the dependencies between

$$\psi=\left(\left.\frac{c}{c_2^{(f)}}\right|_{\lambda^{(m)}=\lambda^{(f)}>1.0}-\left.\frac{c}{c_2^{(f)}}\right|_{\lambda^{(m)}=\lambda^{(f)}=1.0}\right)\times 10 \qquad (34)$$

and kR are given for the first (a) and second (b) modes. It follows from these graphs that the influence of the initial stretching on the compound cylinder depends significantly on the values of the propagating wavelength (i.e. kR) and on the values of the thickness of the external hollow cylinder (i.e. h/R).

5 Conclusions

In the present paper, within the scope of the piecewise homogeneous body model with the use of the TLTEWISB, the axisymmetric longitudinal wave propagation in a finite pre-strained compound (composite) cylinder is investigated. The materials of the inner and outer cylinder are assumed to be incompressible neo-Hookean ones. The numerical results regarding the influence of the initial strains in the inner and outer cylinders on the wave dispersion, i.e. the dependencies between $c/c_2^{(f)}$ and kR, where $c_2^{(f)}$ is a distortion wave velocity in the inner solid cylinder material, have been presented and discussed. These results are obtained for the case where the material of the inner solid cylinder is more stiffer than that of the outer hollow cylinder; i.e. it is assumed that $C_{10}^{(f)}/C_{10}^{(m)} = 5$, where $C_{10}^{(f)}(C_{10}^{(m)})$ is a material constant of the inner (outer) cylinder which characterizes its stiffer. At the same time, these results are obtained for the following two cases (distinguished with each other by the thickness of the external hollow cylinder): Case 1. The thickness is infinite (i.e. $h/R = \infty$). Case 2. The thickness is finite (i.e. $h/R < \infty$).

According to the obtained numerical results, it can be drown the following conclusions for the Case 1.

- There exist the values $s_1 < s_2 < ... < s_n < ... < s_N$ of s $(= c/c_2^{(f)})$ around of which the narrow stop band zones arise (Figs. 2 and 3);

- The part of the dispersion curves for which $s < s_N$ corresponds to normal dispersion, however, those parts which are similar to the part for which $s \in (s_N, s_{N-1})$ $(s \in (s_{N-1}, s_{N-2}))$ correspond to an anomalous dispersion (backward wave);

- The values of $(kR)_*$ (determined by expression (28)) and s_N depends significantly on the initial strain of the solid cylinder; i.e. the values of $(kR)_*$ (s_N) decrease (increase) with $\lambda^{(f)}$;

- For $kR < (kR)_N^*$ (where $(kR)_N^*$corresponds to $\min s_N$; $\min s_N$ is the s_N attained for $\lambda^{(f)} = 1.0$) and $kR > \min(kR)_*$ (for the parts which correspond to the normal dispersion) the wave propagation velocity increases with the initial stretching of the solid cylinder;

- The wave propagation velocity decrease (increase) with $\lambda^{(m)}$ under $kR < (kR)_N$ $(kR > (kR)_*$for the normal dispersion parts).

- The concrete numerical results attained in the Case 2 can be summered as follows.

- The stop band zones in the dispersion curves observed in the Case 1 disappear for cases where $h/R < \infty$. Consequently, the arising of the aforementioned stop band zones for the considered problem can be explained with the neglecting of the wave reflection from the outer free surface of the external cylinder;
- The pre-stretching of the components of the compound cylinder causes to increase of the wave propagation velocity in all considered modes;
- In the first mode the wave velocity approach to the corresponding bar velocity determined by the expression (32) as $kR \to 0$;
- The wave propagation velocity in the first mode approach to the Rayleigh wave propagation velocity in outer cylinder material determined by the expression (33) as $kR \to \infty$;
- The character of the influence of the initial stretching on the wave propagation velocity significantly depends on the values of $kR$and h/R. This character is illustrated by the graphs given in Fig. 10.

Although these results were obtained for a concrete s elected value of the problem parameter $C_{10}^{(f)}/C_{10}^{(m}$, they also have a general validity in a qualitative sense for selected type of pairs of materials. At the same time, these results are also new ones for the classical linear theory of elastodynamics under absent of initial strains in the components of the compound cylinder. The results have many fields of application in the theoretical and practical sense. For example, the present results can be applied under nondestructive analyses of the residual stress-strain state in compound cylinders. Moreover, note that the obtained numerical results can be also applied to nano-composite materials by taking into consideration the restrictions described in papers Guz and Guz (2006), Guz, Roger and Guz (2005).

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