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A State Parameter Based Generalized Plasticity Model for Unsaturated Soils

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Abstract: This paper presents an extension of the Generalized Plasticity model proposed by Pastor – Zienkiewicz in 1986. The extension is based on (i) incorporating a state dependant parameter to model the mechanical behaviour of sand under a wide range of relative densities and confining pressures (ii) the definition of the effective stress of Schrefler (1984) modified to obtain unique CSL for different suction and (iii) the work conjugated variable proposed by Houlsby (1997). Several examples are presented for saturated and unsaturated soils.

Keywords: Constitutive modelling, state parameter, unsaturated soils.

1 Introduction

Constitutive models are a fundamental part of simulations codes, together with mathematical and numerical model.

Much effort has been done during the past decades to improve our understanding of how geomaterials in general and soils in particular behave. Experimental techniques, such as tomography or the 3D testing devices (Desrues, 1984) have provided valuable information which has helped constitutive researchers to improve their models.

Indeed, new virtual testing machines for granular materials based on the discrete element method have also contributed to this effort (Calvetti, Combe and Lanier, 1997; Sibille, Prunier, Nicot and Darve, 2008) and it is not unreasonable to think that in a near future the constitutive equation at Gauss point level could be based on the discrete element drivers.

In continuous mechanics, modelling of geo-materials behaviour as soils, rock and concrete has attracted the attention of many researchers in the recent years (Gawin

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and Schrefler, 1995; Pijaudier-Cabot and Jason, 2002; Sanavia and Schrefler, 2002; Pastor and Mabssout, 2002; Ferguson and Palananthakumar, 2005; Jabbari and Gatmiri, 2007, Ozaki, Hashiguchi, Okayasu and Chen, 2007; Selvadurai and Ghiabi, 2008, Sageresan and Drathi, 2008; Kringos, Scarpas, and Selvadurai, 2008; Jönsthövel, van Gijzen, Vuik, Kasbergen, and Scarpas, 2009). However, modelling of many geotechnical problems as fast landslide induced by rainfall, wetting and drying cycles and its influence on shallow foundations require appropriate constitutive models.

In recent years, several constitutive models were developed to reproduce the behaviour of unsaturated soils. It is worth mentioning the contributions of Alonso, Gens and Josa(1990), Bolzon, Schrefler and Zienkiewicz (1996) and its extension by Santagiuliana and Schrefler (2006), Loret and Khalili (2000), Vaunat, Romero and Jommi (2000), Gallipoli, Gens, Sharma and Vaunat (2003), Tamagnini and Pastor (2004), Borja (2004), Fernandez Merodo, Tamagnini, Pastor and Mira (2005), Laloui and Nuth (2005), Russell and Khalili (2006), among others.

Most of the above mentioned constitutive models have shown a strong dependency on initial conditions (density, confining pressure, degree of saturation, suction) which implies that multiple model constants are required to reproduce the behaviour of the same material.

This paper is devoted to present a new state parameter Generalized Plasticity model for unsaturated soils. In this model, a unique set of parameter allows for describing the behaviour of the soil under a wide range of situations, from unsaturated to saturated conditions, under large variations of confining pressures e.g. in large earth dams.

The paper is structured as follows:

First of all, we will recall the main features of the basic Generalized Plasticity model proposed by Pastor, Zienkiewicz, Chan (1990), with a simple extension to bonded granular materials,

Next, we will introduce the state parameter based model,

Finally, we will complete the model with the extension for unsaturated soils.

Several examples will be presented in order to assess both the limitations and advantages of the proposed model.

2 Generalized Plasticity Theory

Generalized Plasticity Theory was introduced by Zienkiewicz and Mroz (1984) and later extended by Pastor et al. (1985,1986,1990) as a framework within which simple models accounting for material behaviour under monotonic and cyclic loading could be developed. The first generalized plasticity model was proposed by Pastor, Zienkiewicz and Chan in 1990 extending a Bounding Surface model proposed by Zienkiewicz et al (1985) and Pastor et al. (1985).

Recently, the basic model has been extended to bonded materials (Fernandez Merodo et al. 2004), granular soils incorporating state parameter (Manzanal,2008), and unsaturated soils (Tamagnini and Pastor, 2004 ; Manzanal, Pastor, Fernandez Merodo, 2009).

First, we will present the bases of Generalized Plasticity Theory for the sake of completeness. Generalized Plasticity Theory introduces the dependence of the constitutive tensor relating increments of stress and strain on the direction of the increment of stress via a unit tensor **n** which discriminates the states of "loading" and "unloading"

$$\begin{aligned} d\boldsymbol{\varepsilon} &= \mathbf{C}_L : d\boldsymbol{\sigma} \text{ for } \mathbf{n} : d\boldsymbol{\sigma}^e > 0 \\ d\boldsymbol{\varepsilon} &= \mathbf{C}_U : d\boldsymbol{\sigma} \text{ for } \mathbf{n} : d\boldsymbol{\sigma}^e < 0 \end{aligned} \tag{1}$$

where $d\boldsymbol{\sigma}^e$ is the elastic stress increment, which would be produced if the behaviour were elastic, $d\boldsymbol{\sigma}^e = \mathbf{D}^e : d\boldsymbol{\varepsilon}$, and $\mathbf{D}^e$ is the elastic constitutive tensor.

After imposing the condition of continuity between loading and unloading states, we arrive to

$$\begin{aligned} \mathbf{C}_L &= \mathbf{C}^e + \frac{1}{H_L} \cdot \mathbf{n}_{gL} \otimes \mathbf{n} \\ \mathbf{C}_U &= \mathbf{C}^e + \frac{1}{H_U} \cdot \mathbf{n}_{gU} \otimes \mathbf{n} \end{aligned} \tag{2}$$

In above, subindexes L and U refer to "loading" and "unloading". The scalars $H_{L/U}$ are referred to as loading and unloading plastic modulii, and the unit tensors $\mathbf{n}_{gL/U}$ give the direction of the plastic flow during loading and unloading.

The limit case, $\mathbf{n} : d\boldsymbol{\sigma}^e = 0$ is called "neutral loading", and with the assumption done in (2), it can be seen that response is continuous as:

$$\begin{aligned} d\boldsymbol{\varepsilon}_L &= \mathbf{C}_L : d\boldsymbol{\sigma} = \mathbf{C}^e : d\boldsymbol{\sigma} \\ d\boldsymbol{\varepsilon}_U &= \mathbf{C}_U : d\boldsymbol{\sigma} = \mathbf{C}^e : d\boldsymbol{\sigma} \end{aligned}$$

In small strain theory the strain increment can be descomposed into two parts, elastic and plastic as:

$$\begin{aligned} d\boldsymbol{\varepsilon} &= d\boldsymbol{\varepsilon}^e + d\boldsymbol{\varepsilon}^p \\ d\boldsymbol{\varepsilon}^e &= \mathbf{C}^e : d\boldsymbol{\sigma} \\ d\boldsymbol{\varepsilon}^p &= \frac{1}{H_{L/U}} \cdot \left(\mathbf{n}_{gL/U} \otimes \mathbf{n}\right) : d\boldsymbol{\sigma} \end{aligned} \tag{3}$$

The main advantage of Generalized Plasticity Theory (GPT) is that all ingredients can be postulated without introducing any yield or plastic potential surface. Moreover, it can be seen that both Classical Plasticity and Bounding Surface Plasticity models are special cases of the GPT.

We will describe next a simple model proposed by Pastor, Zienkiewicz, Leung and Chan (1985,1990) which is able to reproduce the following basic features of the sand behaviour under monotonic and cyclic loading:

(i) Volumetric deformations depend both on density and the stress ratio $\eta = q/p'$, where q is the deviatoric stress and p' is the principal effective stress. There is a characteristic value $\eta = M_g$ at which the behaviour changes from contractive to dilative. Failure at constant volume takes place also at this line, referred to as "Characteristic State Line" by Luong (1980), and it can be interpreted as a Critical State Line for granular soils. The basic idea behind is that the soil, before failure, crosses a state at which there is no volume change, and comes back to it at residual conditions.

(ii) Very loose and loose sands exhibit compaction under shearing, which results on an increase of pore pressures when the loading process is not fully drained. In the limit, liquefaction can happen.

(iii) Dense sands exhibit dilation once the Characteristic State Line has been crossed. Dilation causes softening, and the strength decreases after a peak has been reached. Here, localization of strain in shear bands obscures the experimental results as the specimen is not homogeneous. On the other hand, undrained paths show an important decrease of the pore water pressure which may cause cavitation (Mokni and Desrues, 1998).

(iv) Under cyclic loading we observe the same compaction and dilation patterns. Plastic deformation occurs and the soil compacts progressively or the pore pressure increases. Liquefaction under cyclic loading is just the result of the increase of the pore pressure and the mechanism which is observed in monotonic loading.

(v) Medium dense sands under undrained cyclic loading develop an special type of behaviour which is referred to as 'cyclic mobility'. The difference with liquefaction consists on dilation which causes the pore pressure to decrease, hardening in turn the soil.

Taking into account all experimental facts described above, it is possible to develop a model within the Generalized Plasticity Theory as follows:

First of all, the direction of the plastic flow in the (p,q) plane is postulated as:

$$\begin{aligned} &\mathbf{n}_g^T = (n_{gv}, n_{gs}) \\ &n_{gv} = d_g/(1+d_g^2)^{1/2} n_{gs} = 1/(1+d_g^2)^{1/2} \end{aligned} \tag{4}$$

where the dilatancy d_g is defined as the ratio between the increments of plastic volumetric and shear strain is given by:

$$d_g = (1+\alpha)\cdot(M_g-\eta)$$

The loading-unloading discriminating relation **n** is obtained in a similar way:

$$\begin{aligned}&\mathbf{n}^T = (n_v, n_s)\\&n_v = d_f/(1+d_f^2)^{1/2} n_s = 1/(1+d_f^2)^{1/2}\\&d_f = (1+\alpha)\cdot(M_f-\eta)\end{aligned} \tag{5}$$

In above, α, M_g and M_f are model parameters.

The third ingredient is the plastic modulus, which has to be defined both for loading and unloading. During loading, we will assume:

$$H_L = H_0 \cdot p' \cdot H_f \cdot (H_v + H_s) \cdot H_{DM} \tag{6}$$

where H_0 is a constitutive parameter. In above, H_f is given by

$$H_f = \left(1-\frac{\eta}{\eta_f}\right)^4$$

and

$$\eta_f = \left(1+\frac{1}{\alpha}\right)\cdot M_f$$

This factor varies between 1 at q= 0 to 0 at the straight line tangent to the Yield surface at the origin.

The terms H_v, H_s and H_{DM} refer, respectively, to volumetric and deviatoric strain hardening and the discrete memory. They are given by:

$$\begin{aligned}&H_v = \left(1-\frac{\eta}{M_g}\right)\\&H_s = \beta_0\cdot\beta_1\cdot\exp\left(-\beta_0\cdot\xi_{dev}\right)\\&H_{DM} = \left(\frac{\zeta_{\max}}{\zeta}\right)^{\gamma}\end{aligned} \tag{7}$$

where β_0, β_1 and γ are model parameters, and ξ_{dev} the accumulated deviatoric plastic strain.

Let us now consider each term. The volumetric term is zero at the CSL, and therefore, failure would take place there if H_s were zero. It can be observed in triaxial tests that both in drained and undrained processes, the stress paths are able to cross this line. The role of H_s is to prevent failure at this stage, but to allow it at residual conditions. This is achieved by making H_s to depend on the accumulated deviatoric plastic strain ξ_{dev} defined from $d\xi_{dev} = (d\mathbf{e}^p : d\mathbf{e}^p)^{1/2}$ $d\xi_{dev} = (d\mathbf{e}^p : d\mathbf{e}^p)^{1/2}$ where $d\mathbf{e}^p$ is the increment of the plastic deviatoric strain tensor.

Finally, the variable ζ is a measure of the stress intensity, which has been taken as

$$\zeta = p' \cdot \left\{ 1 - \left(\frac{1+\alpha}{\alpha} \right) \cdot \frac{\eta}{M_g} \right\}^{1/\alpha} \tag{8}$$

and ζ_{max} (see eq. (7)) is the maximum value of ζ reached during past history of the material.

This basic model for sands was able to reproduce most salient aspects of sand behaviour under both drained and undrained conditions, the main limitation being the necessity of using different model parameters for different relative densities.

3 Bonded soils and weak rocks

An extension of the Generalized Plasticity model has been recently proposed by the authors to reproduce the mechanical behaviour of bonded soils, weak rocks and other materials of a similar kind.

The model is based on the ideas of Gens and Nova (1993) and Lagioia and Nova (1995), who assumed that yielding of the materials causes a progressive debonding until a final state is reached where the materials is fully unstructured. The mechanical propeties of the model are thus made dependant on bonding. It is interesting to note that this type of model could be used for the inverse process, i.e., the building of bonds in fresh concrete.

As the amount of bonding increases the yield surface must increase. Two parameters are used to define the new enlarged yield locus: p_{c0} that controls the yielding of the bonded soil in isotropic compression and p_t which is related to the cohesion and tensile strength of the material. Both p_{c0} and p_t increase with the magnitude of bonding.

We can assume that the degradation of the material (decrease in bonding) is related to some kind of damage measure, that will in turn depend on plastic strains. Lagioia and Nova (1995) proposed simple laws to describe the debonding effect on a calcarenite material. The evolution of p_t is governed by:

$$p_t = p_{t0} \cdot \exp\left(-\rho_t \cdot \xi_{vol}\right) \tag{9}$$

Where p_{t0} and ρ_t are two constitutive parameters and ξ_{vol} is the accumulated plastic volumetric strain defined as $\xi_{vol} = \int \left| d\varepsilon_v^p \right|$. It appears reasonable to assume that changes of the yield locus will be controlled by two different phenomena: conventional plastic hardening (or softening) for an unbonded material and bond degradation. In that case, the plastic modulus of the sand model proposed by Pastor et al. (1990) can be improved introducing a term H_b in the definition of the plastic modulus:

$$H_L = (H_0 \cdot p^* - H_b) \cdot H_f^* \cdot (H_v^* + H_s) \cdot H_{DM}^* \tag{10}$$

where

$$\begin{aligned}
&p^* = p' + p_t \\
&H_f^* = \left(1 - \frac{\eta^*}{\eta_f}\right)^4 \\
&H_v^* = \left(1 - \frac{\eta^*}{M_g}\right) \\
&H_b = b_1 \cdot \xi_{vol} \cdot \exp\left(-b_2 \cdot \xi_{vol}\right) \\
&H_{DM}^* = \left(\frac{\zeta_{\max}^*}{\zeta^*}\right)^\gamma \\
&\eta * = q / \left(p' + p_t\right) \\
&\zeta^* = \left(p' + p_t\right) \cdot \left\{1 - \left(\frac{\alpha}{1+\alpha}\right) \cdot \frac{\eta^*}{M_g}\right\}^{-1/\alpha}
\end{aligned} \tag{11}$$

where $\xi_{\max}^* = p_t + p_m + p_s$ according to the original expression proposed by Lagioia and Nova (1995); p_t evolution follows (9), $p_s = p_{s0} \cdot \exp\left[\left(\varepsilon_v^p + \xi \xi_{dev}\right)/B_p\right]$ and $p_m = p_{m0} \cdot \exp\left[-\rho_m \left(\xi_{vol}\right)^3\right]$. p_{s0}, ξ, B_p, p_{m0} and ρ_m are constitutive parameters.

It can be seen that the value of the new term H_b decreases when the volumetric plastic strain increases (i.e. when debonding occurs) and in the limit case, when destructuration is complete, H_b becomes zero. In this case, the new plastic modulus defined above coincides with the original plastic modulus.

It is possible to reproduce with this improvement the laboratory tests of Lagioia and Nova (1995) on the Gravina calcarenite. Fig. 1. (Fernández Merodo et al 2004) compares experimental data and model predictions for an isotropic compression test.

This type of behaviour -destructuration with an important compaction- is a mechanism which in our opinion plays a paramount role on the generation of pore pressures and catastrophic failure of soils. This is the case of the landslide of Las

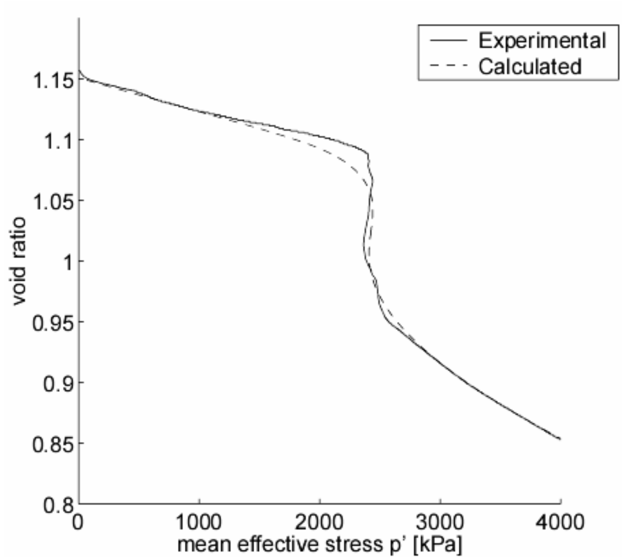

Figure 1: Isotropic compression test: experimental data from Lagioia and Nova and model predictions.

Colinas (El Salvador), triggered by the first 2001 earthquake. The soil presented cementation and was unsaturated. When sheared, this material can collapse, and if the loading is fast enough, pore pressures can cause the material to liquefy. In order to show qualitatively the phenomenon, we have performed a simulation on an ideal material, a fine grained soil with cementation. The parameters have been determinated by the set of experimental data of Lagioia and Nova (1995) which are given in Table I. Fig. 2 and 3 show the results of a consolidated undrained triaxial test. We have depicted in Fig. 2 the stress path and in Fig. 3 the deviatoric stress vs axial strain for both the bonded and the unbonded materials.

Table 1: Parameters of the constitutive model for Las Colinas landslide.

M_g	M_f	α	β_0	β_I	γ	p_{s0}	p_{m0}	p_{t0}	β_p	ρ_m	ρ_t	ξ	b_1	b_2
1.47	0.3	0.45	1.	0.2	9.0	120 kPa	240 kPa	24 kPa	0.06	8333	1000	–0.1	5.0e3	36

4 State parameter based modelling

It is a well known fact that sands have different volumetric and stress – strain responses according to density and mean effective stress level. Contractive behaviour

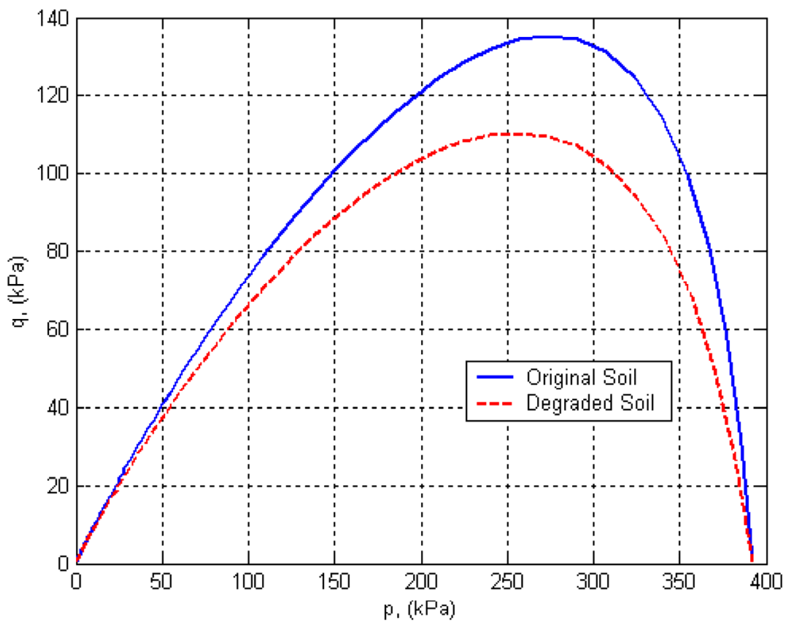

Figure 2: Undrained behaviour of a bonded loose granular soil: stress path.

and strain hardening is observed in loose sands while dense sands show dilative behaviour and strain softening during shearing under drained loading. Moreover, for a given density, sands may show strong dilative behaviour at low confining pressures and fully contractive response at high confining pressures. This means that neither density nor confining pressure alone can fully characterize sand behaviour, but a combination of both. The idea of a unified parameter including this double dependency of sand behaviour was studied since the early works of Roscoe and Poorooshasb (1963), Wroth and Basset (1965), and Seed and Lee (1967). The first constitutive model incorporating a state parameter is the Harmonic Response model of Uriel (1975). In recent years several attempts have been made to deal with the influence of density and confining pressure in soil modelling. It is worth mentioning the work of Jefferies & Shuttle (2005), Yang and Ling (2005), Larsson Faleskog and Massih (2004), Taiebat and Dafalias (2008), among others. The state parameter, as nowadays is known, has various definitions depending of the different combinations of the current state and critical state (Been and Jefferies, 1985; Ishihara, 1993; Wang, Dafalias, Li and Makdisi, 2002). The most widely accepted state parameter today is that proposed by Been and Jefferies (1985). It is defined

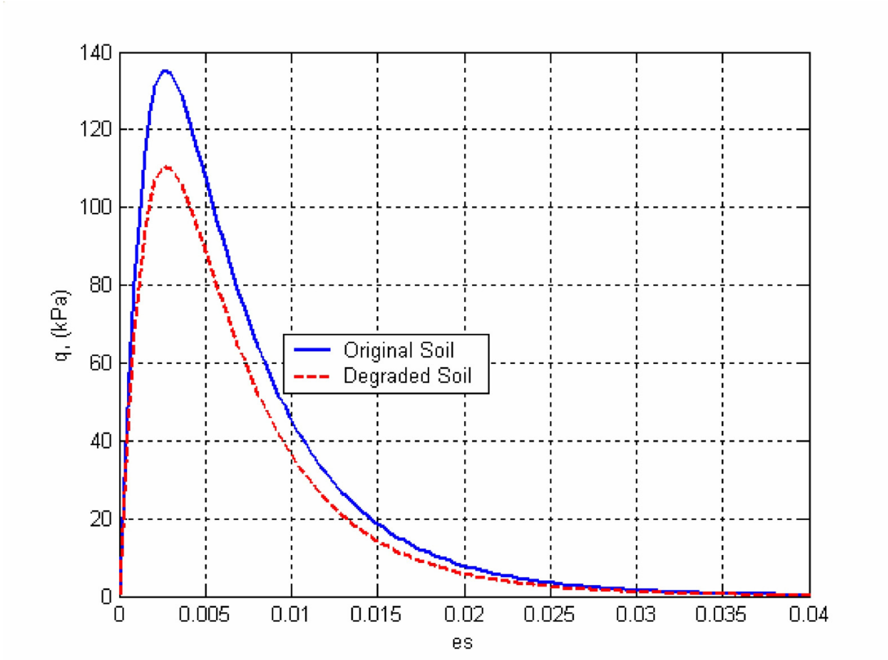

Figure 3: Undrained behaviour of a bonded loose granular soil: Deviatoric stress versus axial strain.

as the difference of the current voids ratio and the voids ratio at critical state under the same confining pressure.

$$\psi = e - e_c \tag{12}$$

As the state parameter definition is based on the critical state line, we have chosen that proposed by Li (1997)

$$e_c = e_\Gamma - \lambda \left(\frac{p'_c}{p'_a}\right)^{\zeta_c} \tag{13}$$

where e_Γ is void ratio at a confining pressure of 1 kPa, λ is the slope of critical state line in a $e-(p'/p'_a)^{\zeta_c}$plane$'$ e_c and p'_c are the void ratio and the confining pressure at critical state, respectively and p'_a is the atmospheric pressure. The parameters e_Γ and λ can be determined by fitting the experimental data at critical state in a $e-(p'/p'_a)^{\zeta_c}$ plane and ζ_c varies between 0.60 to 0.80 as Li (1997) stated.

One limitation of the basic model for sands described in the preceding sections is that specimens of a given sand with different densities require different set of parameters to reproduce the observed behaviour. The method we will follow to extend the basic generalized plasticity model for sand is based on how the basic ingredients of the model depend on confining pressure and void ratio through the state parameters defined above. The state parameter will enter the definitions of the three main ingredient of a Generalized Plasticity model: the directions $\mathbf{n}_g$ and $\mathbf{n}$ and the plastic modulus H_L.

Concerning the plastic flow direction $\mathbf{n}_g$, we have introduced a new dilatancy law following Li and Dafalias (2000)

$$d = \frac{d_0}{M_g} \cdot (\eta_{PTS} - \eta) \text{ where } \eta_{PTS} = M_g \cdot \exp(m\psi) \tag{14}$$

where d_0 and m are model constants. ψ is the state parameter defined by equation (12) ; η is the stress ratio and M_g is the Critical State Line in the plot $q--p'$. Finally, η_{PTS} is the stress ratio at the phase transformation point which depends on the state parameter ψ. Equation (14) shows the existence of a family of stress - dilatancy curves for different densities and confining pressures. The model constants d_0 and m can be obtained from the experimental data in drained or undrained triaxial tests as explained by Li & Dafalias (2000).

The second ingredient which was found in the basic model to depend on void ratio and confining pressure was the loading-unloading discriminating direction $\mathbf{n}$. Here we have kept the same basic structure (see eq. (5)) and we will assume d_f to be of the form:

$$d_f = \frac{d_0}{M_f} \cdot (M_f \cdot Exp(m\psi) - \eta) \tag{15}$$

The proposed expression includes a material parameter M_f which in the basic PZ model is constant. Zienkiewicz, Chan, Pastor, Schrefler and Shiomi (1999) proposed that the ratio between M_f and M_g was similar to the sand relative density. Here we propose the following relation which allows determination of M_f once M_g is known:

$$\frac{M_f}{M_g} = h_1 - h_2 \cdot \left(\frac{e}{e_c}\right)^{\beta} \tag{16}$$

where h_1 and h_2 are model constants and β is equal to 1.80. The ratio e/e_c varies between $e_{\min}/e_{\max}$ and $e_{\max}/e_{\min}$ as stated by Verdugo and Ishihara (1996). When e/e_c reaches its lower limit, M_f/M_g ratio is close to one. Parameters h_1 and h_2 can be calibrated based on the $q-p$ curve form undrained triaxial test on loose states for different values of M_f/M_g and the ratio e/e_c equal to $e_{\max}/e_{\min}$.

Finally, the third ingredient which was found to depend on the void ratio and the confining pressure was the loading plastic modulus. Here we have kept the same basic structure of the plastic modulus proposed by Pastor, Zienkiewicz and Chan (1990), which is expressed as:

$$H_L = H_0 \cdot \sqrt{p' \cdot p'_a} \cdot H_{DM} \cdot H_f \cdot (H_v + H_s) \tag{17}$$

In above equation H_f, H_s and H_{DM} are defined by equation (7) and M_f is given by equation (16).

H_0 has been assumed to depend on the state parameter. Here we have chosen the law:

$$H_0 = H'_0 \cdot \exp\left[-\beta'_0 \cdot (e/e_c)^{\beta}\right] \tag{18}$$

where H'_0 and β'_0 are additional model parameters. It can be seen that we have introduced a dependency of H_0 on void ratio in order to improve the model accuracy in tests run at constant stress ratio, including as a special case the isotropic compression test for which it is zero. The model constants H'_0 and β'_0 can be determined adjusting the volumetric response of the model with the experimental counterpart of the isotropic triaxial test.

Finally, taking into account that the peak stress ratio η_p depends on the initial conditions of the soil, we have modified H_v by making it dependent on ψ. The proposed expression is:

$$H_v = H_{v0} \cdot [\eta_p - \eta] \text{ with } \eta_p = M_g \cdot \exp(-\beta_v \cdot \psi) \tag{19}$$

where H_{v0} and β_v are model parameters. It can be easily verified that $\eta_p < M_g$ for loose states while $\eta_p > M_g$ for dense states. The expression of η_p (see eq. (19)) is similar to the one proposed by Li and Dafalias (2000). Parameter β_v can be determined at a peak stress in the drained test as shown by Li and Dafalias (2000). Assuming that for saturated soils the model constants β_0 and β_1 are zero, we can obtain H_{v0} by fitting the model predictions with the experimental results of drained triaxial tests.

As in other constitutive models for soils, the proposed model assumes a non-linear reversible response through the expression of the shear modulus proposed by Richard, Hall and Woods (1970),

$$G = G_{eso} \cdot \frac{(2.97 - e)^2}{(1+e)} \cdot \sqrt{p' \cdot p'_a} \tag{20}$$

and the elastic bulk modulus is assumed to be:

$$K = K_{evo} \cdot \frac{(2.97 - e)^2}{(1 + e)} \cdot \sqrt{p' \cdot p'_a} \tag{21}$$

where G_{eso} and K_{evo} are model constants. e is the void ratio, p'_c is the confining pressure and p'_a is the atmospheric pressure in kPa.

In order to assess the model predictive capability, we have reproduced well known experimental results obtained on Toyoura sand (Verdugo and Ishihara, 1996). We have obtained a single set of constitutive parameters which have been used for all densities, confining pressures and types of tests –drained and undrained (Table 2). Details about parameter calibration can be found in Manzanal (2008).

Table 2: Constitutive model parameters for Toyoura sand.

G_{eso}	K_{evo}	M_g	e_Γ	λ	ζ_c	d_0	m	h_1/h_2
125	167	1.25	0.934	0.019	0.70	0.88	3.50	1.31 /0.85
H'_0	β'_0	β	H_{v0}	β_v	β_1	β_0	γ	α
125	1.90	1.80	175	1.50	0	0	0	0.45

Fig. 4 shows the experimental results and the model predictions for undrained triaxial tests on Toyoura sand for dense (e = 0.735 – Dr = 63.7%), and loose (e = 0.907 - Dr = 18.5%) samples under a range of confining pressures between 100 kPa and 3000 kPa. The model predictions agree well with the experimental results.

Fig. 5 compares the model predictions and experimental data of drained triaxial tests on dense, medium dense and loose samples of Toyoura sand under two initial confining pressures of 100 and 500kPa. In general, good agreement between predicted and measured data for drained triaxial tests is found.

5 Unsaturated soil modelling

In this section we present an extension of the basic Generalized Plasticity constitutive model (Pastor, Zienkienwicz, Chan, 1990) to reproduce the main features of the behaviour of unsaturated soils from state parameter point of view. The proposed model has been inspired by previous work of Tamagnini and Pastor (2004) and Tamagnini (2004). Tamagnini and Pastor model was able to reproduce some salient aspects of unsaturated soils, such as the volumetric collapse when the soil is saturated, but presented some limitations which have been addressed by Manzanal (2008) and which will be described next.

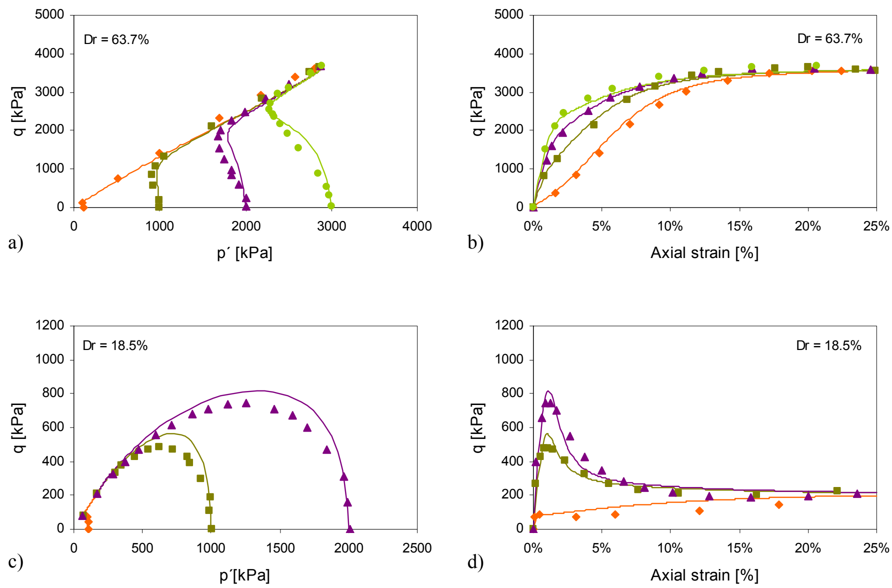

Figure 4: Experimental results (symbols) and model predictions (continuous lines) for undrained triaxial tests of Toyoura sand at different initial conditions. a) Stress path and b) deviatoric stress vs axial strain for e = 0.735; c) stress path and d) deviatoric stress vs axial strain for e = 0.907.

The model is formulated using two set of stress – strain work conjugated variables (Houlsby, 1997) coupling the hydraulic and the mechanical behaviour of unsaturated soils within a Generalized Plasticity framework. Stress variables are the effective stress tensor and the matrix suction s, and strain variables are the soil skeleton strain and the degree of saturation. The effective stress is given by

$$\sigma'_{ij} = \sigma_{ij} - p_a \cdot \delta_{ij} + S_{re} \cdot (p_a - p_w) \cdot \delta_{ij} \tag{22}$$

where σ_{ij} is the total stress tensor, p_a is the pore air pressure, p_w is the pore water pressure, $p_a - p_w$ is the matrix suction s, δ_{ij} is the Kronecker delta and S_{re} is the relative degree of saturation which is given by

$$S_{re} = \frac{S_r - S_{r0}}{1 - S_{r0}} \tag{23}$$

where S_{r0} is the residual degree of saturation. We found an important dispersion on the experimental data even when we used the effective stress definition introduced

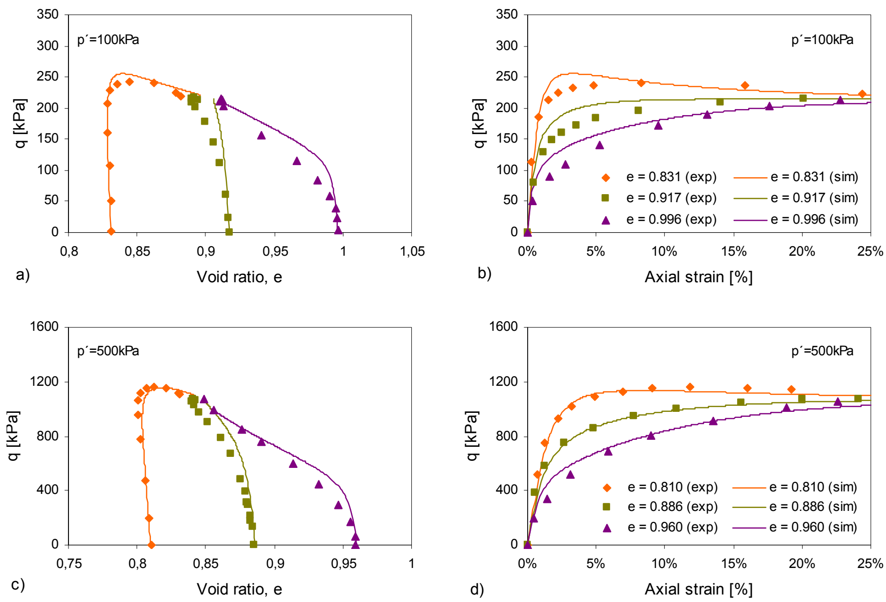

Figure 5: Experimental results (symbols) and model predictions (continuous lines) for drained triaxial test of Toyoura sand at different initial conditions. a) deviatoric stress vs void ratio and b) deviatoric stress vs axial strain for $p' = 100$kPa; c) stress path and d) deviatoric stress vs axial strain for $p' = 500$kPa.

by Schrefler (1984) with a modified scalar factor of Bishop effective stress defined by $\chi = S_r$. The improvement obtained by using S_{re} in the effective stress definition can be seen in figure 7 which shows the predictive and experimental shear strength with both approaches, $\chi = S_{re}$ and $\chi = S_r$, for the experimental data described in Toll (1990) and Sivakumar (1993).

The first ingredient of this model is the definition of the state parameter defined on the previous Section, which is based on the critical state line. In the case of unsaturated soils the CSL depends on suction, it is of paramount importance to define the dependence of CSL on suction. Recently, Gallipoli, Gens, Sharma, and Vaunat (2003a) proposed a normalization of CSL for non saturated soils by using the bonding variable ξ as:

$$\xi = f(s) \cdot (1 - S_r) \tag{24}$$

where the function $f(s)$ is the ratio between the stabilizing pressure at a given suction s and at zero suction introduced by Haines (1925) and Fisher (1926) and it

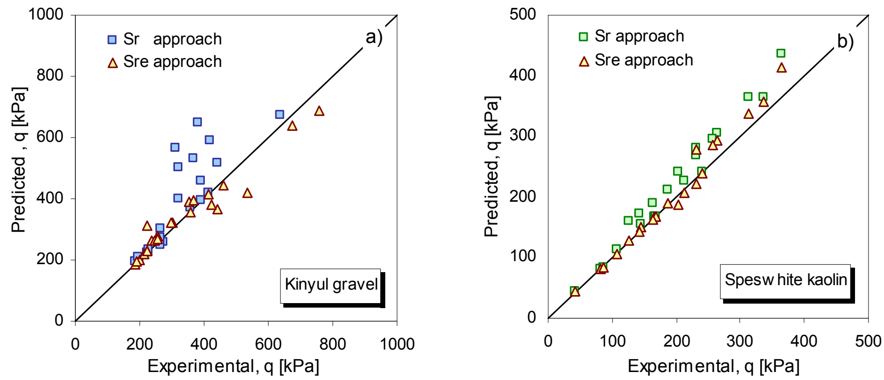

Figure 6: Comparison between predicted and experimental deviatoric stress for a) kinyul gravel (Experimental data from Toll, 1990) and b) speswhite kaolin (Experimental data from Sivakumar, 1993).

is given by

$$f(s) = \frac{3}{4}\left\{2 - \frac{1}{2s}\left[-\frac{3T_s}{R} + \sqrt{\left(\frac{3T_s}{R}\right)^2 + \frac{8T_s}{R}s}\right]\right\} \quad (25)$$

where T_s is the surface tension and R the radius of the spherical particles. There are two limit cases, when suction tends to zero and to infinity. In the former, $f(s) = 1$ and in the latter, when suction tends to infinity $f(s) = 3/2$.

Here, we will use the following alternative relation linking the values of p' at saturation and at a given suction for a fixed void ratio:

$$\frac{p'^{unsat}_{CS}}{p'^{sat}_{CS}} = 1 + g(\xi) \quad (26)$$

where

$$g(\xi) = a \cdot [exp(b \cdot \xi) - 1] \quad (27)$$

and ξ is bonding parameter defined by Gallipoli et al. (2003a). The function g(ξ) depends on the degree of saturation and on suction and takes a zero value at saturation. The parameters a and b are calibrated from experimental data as shown by Gallipoli et al (2003a). In fig. 7 we have depicted the CSL for saturated and unsaturated state on the plane $(p'_{CS}/p'_a)^{\zeta_c} - e$ and the normalization effect of the function $g(\xi)$ (see eq.(27)).

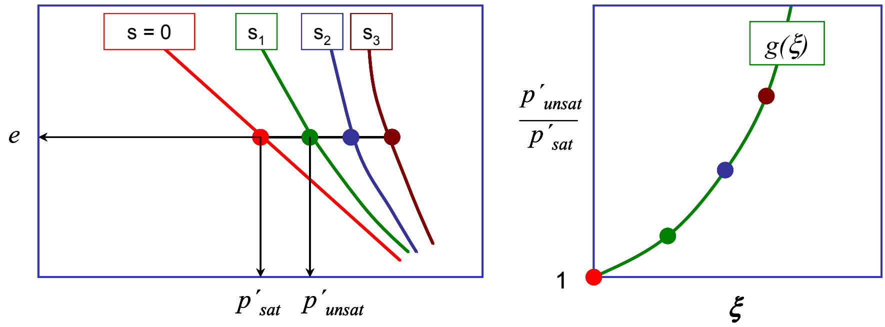

Figure 7: CSLs for saturated and unsaturated state.

By combining equation (26) and (27) with a suitable definition of a CSL for saturated states, we will obtain a generalization of the critical state line to unsaturated states. We provide in fig. 8 an example using the experimental data described in Ng and Chiu (2003) which illustrate the effectiveness of the proposed approach.

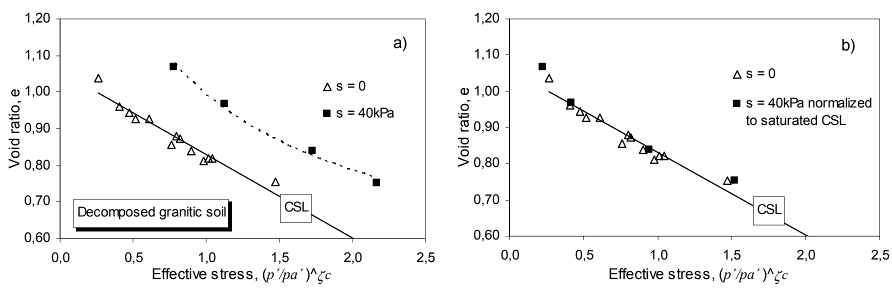

Figure 8: a) Critical state for decomposed granitic soil at different suctions b) Normalization of CSLs (Experimental data from Ng and Chiu, 2003).

The increment of strain is assumed to be:

$$d\varepsilon = \mathbf{C}^e : d\sigma' + \frac{1}{H_{L/U}} \cdot \mathbf{n}_{gL/U} \otimes \mathbf{n} : d\sigma' + \frac{1}{H_b} \cdot \mathbf{n}_{gL/U} \cdot ds \tag{28}$$

where the two first terms are the elastic and plastic strain which have already been described and the last term is the plastic strain develop during wetting – drying cycles.

The plastic modulus H_b is given by

$$H_b = w(\xi) \cdot H_0 \cdot \sqrt{p' \cdot p_{atm}} \cdot H_{DM} \cdot H_f \tag{29}$$

where

$$H_{DM} = \left(\frac{\zeta_{\max} \cdot J_s}{\zeta}\right)^{\gamma} \tag{30}$$

is a modified discrete memory function incorporating the effect of the suction and degree of saturation,

$$J_s = \exp\left(c.g\left(\xi\right)\right) \tag{31}$$

where c is a model parameter and $g(\xi)$ is defined by equation (27).

$w(\xi)$ incorporates the effect of the bonding parameter above defined.

$$w(\xi) = \begin{cases} -\left\{1 - \exp\left[g\left(\xi\right)\right]^2\right\}^2 & (wetting) \\ 1 & (drying) \end{cases} \tag{32}$$

and H_0, H_f, H_v and H_s are the same functions defined for saturated soils.

The model is completed with a suitable hydraulic equation which takes into account both the hydraulic hysteresis during a drying – wetting cycle and its dependency on past history. We have chosen a modified version of the water retention curve proposed by Fredlund & Xing (1994):

$$Sr = Sr_0 + (1 - Sr_0) \cdot \left\{\ln\left[\exp(1) + \left(\frac{s^*}{a_w \cdot p_0}\right)^n\right]\right\}^{-m} \tag{33}$$

where s^* is the normalized suction proposed by Gallipoli, Wheeler and Karstunen (2003) to account for void ratio dependency

$$s* = e^{\Omega} \cdot s \tag{34}$$

where Ω, a_w, n and m are model parameters, e is the void ratio and s the matrix suction. The main wetting and drying curves are obtained by assuming different values for a_w, n and m.

Using this state parameter based model, it is possible to reproduce the set of tests on Kurnell sand reported by Russell (2004) with a single set of parameters for saturated and non saturated conditions (See Table N^{o}3).

Fig. 9 compares the model predictions and experimental data of fully saturated drained triaxial tests under three initial confining pressure of 50, 157 and 301kPa.

Concerning unsaturated tests, the predicted behaviour and the experimental data of triaxial tests at constant water and drained triaxial tests at constant suction are shown in fig. 10 and 11 respectively. The overall behaviour of Kurnell sand is well reproduced by the model.

Table 3: Constitutive model parameters for Kurnell sand.

G_{eso}	K_{evo}	M_g	e_Γ	λ	ζ_c	d_0	m	h_1/h_2
135	292	1.475	0.932	0.0328	0.60	0.80	3.32	1 /0.55
H_0'	β_0'	β	H_{v0}	β_v	β_1	β_0	γ	α
135	1.10	1.80	20	0.95	4.20	1.8	0	0.45
a	b	c	S_{r0}	Ω	a_w/a_d	n_w/n_d	m_w/m_d	β_w
0.20	2.00	0	0.009	2.10	0.03/0.05	6.00/10	0.80/1.00	2

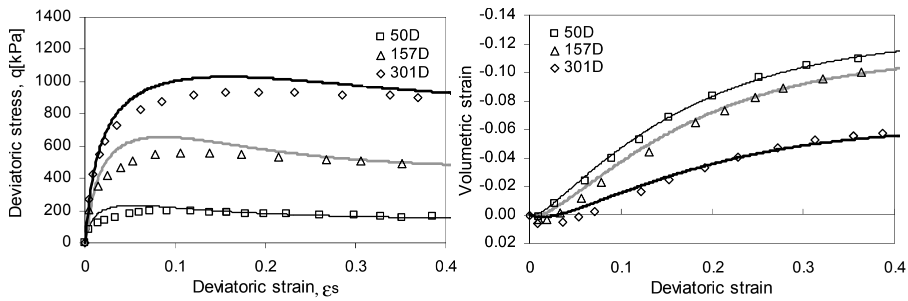

Figure 9: Comparisons between model simulations and fully saturated drained triaxial compression test results. (Experimental data from Russell, 2004). a) deviatoric stress vs deviatoric strain and b) volumetric strain vs deviatoric strain.

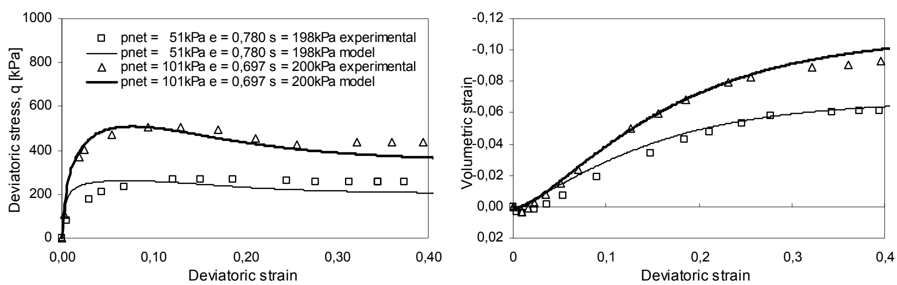

Figure 10: Comparisons between model simulations and drained triaxial compression test results at constant suction (Experimental data from Russell, 2004) a) deviatoric stress vs deviatoric strain and b) volumetric strain vs deviatoric strain.

6 Conclusion

We have presented a new Generalized Plasticity model based in the original Pastor-Zienkiewicz model.

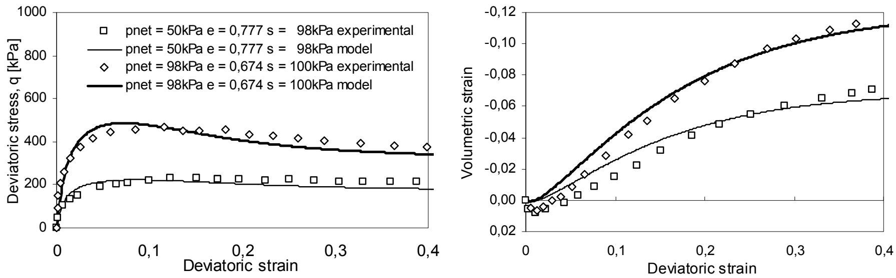

Figure 11: Comparisons between model simulations and triaxial compression test results at constant water content. (Experimental data by Russell, 2004). a) deviatoric stress vs deviatoric strain and b) volumetric strain vs deviatoric strain.

The model is based on:

- a state parameter which allows to describe with a single set of parameter the behaviour of the soil under a wide range of confining pressures and relative densities,
- The effective stress concept proposed by Schrefler (1984) for unsaturated soils, which has been modified to obtain a unique Critical State line for different suction,
- The set of work conjugated variables introduced by Houlsby (1997) coupling the hydraulic and the mechanical behaviour of unsaturated soils within a Generalized Plasticity framework.

The model has been applied to unsaturated soils under a wide range of conditions, and the quality of the obtained results is reasonably good.

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