

Fictitious Time Integration Method of Fundamental Solutions with Chebyshev Polynomials for Solving Poisson-type Nonlinear PDEs

Chia-Cheng Tsai¹, Chein-Shan Liu² and Wei-Chung Yeih³

Abstract: The fictitious time integration method (FTIM) previously developed by Liu and Atluri (2008a) is combined with the method of fundamental solutions and the Chebyshev polynomials to solve Poisson-type nonlinear PDEs. The method of fundamental solutions with Chebyshev polynomials (MFS-CP) is an exponentially-convergent meshless numerical method which is able to solving non-homogeneous partial differential equations if the fundamental solution and the analytical particular solutions of the considered operator are known. In this study, the MFS-CP is extended to solve Poisson-type nonlinear PDEs by using the FTIM. In the solution procedure, the FTIM is introduced to convert a Poisson-type nonlinear PDE into a sequence of linear nonhomogeneous modified Helmholtz equations which are then formally solved by the MFS-CP. Several numerical experiments were carried out to validate the proposed methods.

Keywords: Fictitious time integration method, Method of fundamental solutions, Chebyshev polynomial, Nonlinear partial differential equation.

1 Introduction

Based on the Lie-group shooting method (Liu 2006, 2008a), the fictitious time integration method (FTIM) was invented by Liu and Atluri (2008a) to solve coupled nonlinear algebraic equations. In the solution procedure, a fictitious time is introduced to convert the original nonlinear algebraic equations into evolutionary ordinary differential equations. The fixed point of the evolutionary system, which is also the root of the original algebraic equations, is then obtained by applying numerical integrations on the resultant ordinary differential equations, which do not

¹ Department of Marine Environmental Engineering, National Kaohsiung Marine University, Kaohsiung City 811, Taiwan (tsaichiacheng@mail.nkmu.edu.tw)

² Department of Civil Engineering, National Taiwan University, Taipei, Taiwan

³ Department of Harbor & River Engineering and Computation and Simulation Center, National Taiwan Ocean University, Keelung, 202, Taiwan

require the derivative of nonlinear algebraic equations. After the invention of the FTIM, it was successfully applied to solve the discretized inverse Sturm-Liouville problems, m -point boundary value problems, problems involving Fredholm integral equations, backward advection-dispersion problems and Cauchy problems by Liu (2008f), Liu (2009a), Liu and Atluri (2009), Chang and Liu (2009) and Chi, Yeh and Liu (2009), respectively. In all of these applications, the problems were first discretized into nonlinear algebraic equations and then formally solved by the FTIM. Furthermore, the FTIM was found to be robust against the noises when it is applied to solve problems involving the Fredholm integral equations. Similarly, Liu (2008c) utilized the FTIM to solve some nonlinear obstacle problems. With the aid of NCP-functions a set of nonlinear algebraic equations are obtained and then the FTIM is used to solve these nonlinear equations.

In addition to nonlinear algebraic equations, Liu and Atluri (2008b) introduced the application of the FTIM to solve a nonlinear optimization problem (NOP) under multiple equality and inequality constraints. The Kuhn-Tucker optimality conditions are used to transform the NOP into a mixed complementary problem.

The FTIM has also been adopted to tackle two-dimensional nonlinear elliptic boundary value problems by Liu (2008d) and Liu (2009b). Six examples including Laplace, Poisson, reaction diffusion, Helmholtz, the minimal surface, and explosion equations were tested by Liu (2008d). In order to improve the efficiency, Ku, Yeh, Liu and Chi (2009) introduced a new time-like function to solve nonlinear PDEs. In their solution procedure, the spatial finite difference method (FDM) was applied to discretize the governing equation into a system of nonlinear algebraic equations which was then formally solved by the FTIM. In this study, we on the other hand directly apply the FTIM to convert a Poisson-type nonlinear PDE into a sequence of linear nonhomogeneous modified Helmholtz equations which are then solved by a meshless numerical method.

In the last two decades, there has been an increasing interest in the idea of meshless or mesh-free numerical methods for solving partial differential equations (PDEs). Generally speaking, such methods can be divided into two categories. The first one is domain-type methods in which both the differential equations and boundary conditions are approximated, such as the Kansa's method (or multiquadrics (MQ) method) [Kansa (1990a, 1990b); Young, Jane, Lin, Chiu and Chen (2004); Young, Chen and Wong (2005)] as well as the meshless local Petrov-Galerkin method (MLPG) [Wordelman, Aluru and Ravaioli (2000); Lin and Atluri (2000); Kim and Atluri (2000); Atluri (2004); Han and Atluri (2004)]. The second one is boundary-type methods where only boundary conditions are collocated, such as the method of fundamental solutions (MFS) [Kupradze and Aleksidze (1964); Mathon and Johnston (1977); Bogomolny (1985), Smyrlis and Karageorghis (2001); Tsai,

Young and Cheng (2002); Smyrlis and Karageorghis (2004); Chen, Fan, Young, Murugesan and Tsai (2005); Hon and Wei (2005); Young and Ruan (2005); Tsai, Lin, Young, and Atluri (2006); Young, Tsai, Lin and Chen (2006); Young, Chen, Chen and Kao (2007); Liu (2008b); Hu, Young, and Fan (2008); Marin (2008, 2009a, 2009b)] and the Trefftz method [Liu (2007a, 2007b, 2008e); Cozzano and Rodríguez (2007); Lee and Chen (2009); Li, Lu, Huang and Cheng (2009)]. In this paper, we are going to use the MFS to solve modified Helmholtz equations which are resulted from the Poisson-type nonlinear PDEs by using the FTIM.

The method of particular solution (MPS), invented by Golberg and Chen (1999), is needed in order to apply the MFS for solving nonhomogeneous PDEs since the MFS generally requires the governing equation to be homogeneous. Actually, the MPS is equivalent to the dual reciprocity method (DRM) invented by Nardini and Brebbia (1982). In the applications of the MPS, it is desired to have the analytical particular solutions associated to the basis functions and the considered partial differential operator. When the Chebyshev polynomials are adopted as the basis function, analytical particular solutions of the modified Helmholtz operator can be found in the literature [Golberg, Muleshkov, Chen, and Cheng (2003); Reutskiy and Chen (2006); Karageorghis, and Kyza (2007); Ding and Chen (2007); Tsai (2008); Tsai (2009)]. Overall, the method of fundamental solutions with Chebyshev polynomials (MFS-CP) is a well-developed exponentially-convergent meshless numerical method which is able to solve nonhomogeneous modified Helmholtz equations.

In this study, we combine the FTIM and MFS-CP as the FTIMFS-CP to solve the Poisson-type nonlinear PDEs. Numerical experiments in two- and three-dimensional domains are carried out to validate the proposed numerical scheme. Numerical results demonstrate the accuracy of the method. In addition, the meshless feature makes the proposed method easy to be applied in complex computational domains. This paper is organized as follows. In Section 2 the Poisson-type nonlinear PDE is introduced and the FTIM is applied to convert it into a sequence of linear nonhomogeneous modified Helmholtz equations. In Section 3, the MFS-CP formulation is introduced to solve the nonhomogeneous modified Helmholtz equations. Numerical experiments are presented in Section 4 and some conclusions are drawn in the last section.

2 Poisson-type nonlinear PDE and FTIM formulation

In this study, we consider the following Poisson-type nonlinear boundary value problem:

$$-\nabla^2 u(x, y, z) = G(x, y, z, u, u_x, u_y, u_z) \quad x, y, z \in \Omega \quad (1)$$

$$\mathbf{B}u(x, y, z) = H(x, y, z) \quad x, y, z \in \Gamma \tag{2}$$

where ∇^2 is the Laplacian operator, \mathbf{B} is a linear boundary differential operator, Γ is the boundary of the computational domain Ω , and G and H are given functions.

Two-dimensional nonlinear elliptic PDEs were first solved by Liu (2008d) using the FTIM. In order to improve the efficiency, Ku, Yeih, Liu and Chi (2009) introduced a new time-like function to solve two- and three-dimensional nonlinear PDEs. In their studies, spatial finite difference discretizations were introduced to convert Eq. (1) into nonlinear algebraic equations which were then solved formally by the FTIM.

In our study, we on the other hand directly apply the FTIM to Eq. (1) which essentially include two stages. First, a fictitious time is introduced to convert the Poisson-type nonlinear PDE into a quasilinear evolutionary system and numerical integrations are then applied to obtain a sequence of linear nonhomogeneous modified Helmholtz equations.

2.1 Evolutionary system

In order to apply the FTIM, we formally introduce $w(x, y, z, t)$ as follows:

$$w(x, y, z, t) = u(x, y, z) q(t) \tag{3}$$

or equivalently

$$u(x, y, z) = \frac{w(x, y, z, t)}{q(t)} \tag{4}$$

where t is a fictitious time and $q(t)$ is a time-like function. Taking the derivative of Eq. (3) with respect to t , we have

$$\frac{\partial w}{\partial t} = u \frac{dq}{dt} \tag{5}$$

Using Eqs. (4) and (1), we have

$$-\nabla^2 \frac{w}{q} = G\left(x, y, z, \frac{w}{q}, \frac{w_x}{q}, \frac{w_y}{q}, \frac{w_z}{q}\right) \tag{6}$$

Addition of Eqs. (5) and (6) results in

$$\frac{\partial w}{\partial t} - \nabla^2 \left(\frac{w}{q}\right) = u \frac{dq}{dt} + G\left(x, y, z, \frac{w}{q}, \frac{w_x}{q}, \frac{w_y}{q}, \frac{w_z}{q}\right) \tag{7}$$

Then, using Eqs. (4) and (7) results in

$$\frac{\partial w}{\partial t} - \nabla^2 \left(\frac{w}{q} \right) = \frac{w}{q} \frac{dq}{dt} + G \left(x, y, z, \frac{w}{q}, \frac{w_x}{q}, \frac{w_y}{q}, \frac{w_z}{q} \right) \tag{8}$$

or equivalently as

$$\frac{\partial}{\partial t} \left(\frac{w}{q} \right) - \frac{1}{q} \nabla^2 \left(\frac{w}{q} \right) = \frac{1}{q} G \left(x, y, z, \frac{w}{q}, \frac{w_x}{q}, \frac{w_y}{q}, \frac{w_z}{q} \right) \tag{9}$$

by using

$$\frac{\partial}{\partial t} \left(\frac{w}{q} \right) = \frac{1}{q} \frac{\partial w}{\partial t} - \frac{w}{q^2} \frac{dq}{dt} \tag{10}$$

Using Eq. (4), Eq. (9) can also be rewritten as

$$\frac{\partial u}{\partial t} - \frac{1}{q} \nabla^2 u = \frac{1}{q} G(x, y, z, u, u_x, u_y, u_z) \tag{11}$$

In this study, we typically select the following time-like function:

$$q(t) = (1 + t)^M \tag{12}$$

where M is a parameter to arrange the convergence of the FTIM as was done in Ku, Yeh, Liu and Chi (2009). Eq. (11) is just the evolutionary equation to be solved. In the solution procedure, the boundary condition in Eq. (2) should also be imposed and an initial condition should be set up.

2.2 Euler scheme

We may employ a forward Euler scheme to integrate Eq. (11), which results in [Press, Teukolsky, Vetterling and Flannery (2007)]:

$$\nabla^2 u^{I+1} - \frac{q^{I+1}}{\Delta t} u^{I+1} = -\frac{q^I}{\Delta t} u^I - G(x, y, z, u^I, u_x^I, u_y^I, u_z^I) \quad I = 0, 1, 2, \dots \tag{13}$$

which is a sequence of linear nonhomogeneous modified Helmholtz equations to be solved. In addition to the governing equation in Eq. (13) we also need the following boundary condition

$$\mathbf{B}u^{I+1}(x, y, z) = H(x, y, z) \quad I = 0, 1, 2, \dots \tag{14}$$

The numerical procedure starts from an initial value of $u^0(x, y, z)$ which can be arbitrarily chosen. Then, the FTIM integrates Eq. (13) from $t=0$ to a selected final

time. In the numerical integration process, the convergence criterion of $u^{I+1}(x, y, z)$ is written as

$$\|u^{I+1}(x, y, z) - u^I(x, y, z)\|_\infty < \varepsilon \tag{15}$$

where $\| \cdot \|_\infty$ is the maximum norm and ε is a given convergence criterion. If at a certain time the above criterion is satisfied, then the solution of $u(x, y, z)$ is obtained.

3 MFS-CP formulation

In this section, we introduce the application of the method of fundamental solution with Chebyshev polynomials (MFS-CP) [Golberg, Muleshkov, Chen, and Cheng (2003)] to solve the nonhomogeneous modified Helmholtz equation addressed in Eq. (13). We begin our formulation formally by the principle of superposition as follows:

$$u^{I+1}(x, y, z) = u_h^{I+1}(x, y, z) + u_p^{I+1}(x, y, z) \tag{16}$$

where the particular solution $u_p^{I+1}(x, y, z)$ should satisfy

$$\nabla^2 u_p^{I+1} - \frac{q^{I+1}}{\Delta t} u_p^{I+1} = -\frac{q^I}{\Delta t} u^I - G(x, y, z, u^I, u_x^I, u_y^I, u_z^I) \quad I = 0, 1, 2, \dots \tag{17}$$

without any boundary condition. And the homogeneous solution $u_h^{I+1}(x, y, z)$ is governed by

$$\nabla^2 u_h^{I+1} - \frac{q^{I+1}}{\Delta t} u_h^{I+1} = 0 \quad I = 0, 1, 2, \dots \tag{18}$$

and boundary condition

$$\mathbf{B}u_h^{I+1}(x, y, z) = H(x, y, z) - \mathbf{B}u_p^{I+1}(x, y, z) \quad I = 0, 1, 2, \dots \tag{19}$$

Now, we are in a position to solve the particular solution and the homogeneous solution respectively by the MPS and the MFS.

3.1 Chebyshev Polynomials

To solve the particular solution governed by Eq. (17), we begin with the trivariate Chebyshev polynomial interpolation for

$$F^I(x, y, z) = -\frac{q^I}{\Delta t} u^I(x, y, z) - G(x, y, z, u^I(x, y, z), u_x^I(x, y, z), u_y^I(x, y, z), u_z^I(x, y, z)) \tag{20}$$

The Chebyshev interpolant using the Gauss-Lobatto nodes for cubic domain $[x_a, x_b] \times [y_a, y_b] \times [z_a, z_b]$ takes the form:

$$F^I(x, y, z) \cong \sum_i^l \sum_j^m \sum_k^n a_{ijk}^I T_i\left(\frac{2x - x_b - x_a}{x_b - x_a}\right) T_j\left(\frac{2y - y_b - y_a}{y_b - y_a}\right) T_k\left(\frac{2z - z_b - z_a}{z_b - z_a}\right) \quad (21)$$

where

$$a_{ijk}^I = \frac{8}{lmn\bar{c}_{l,i}\bar{c}_{m,j}\bar{c}_{n,k}} \sum_{\alpha}^l \sum_{\beta}^m \sum_{\gamma}^n \frac{F^I(x_{\alpha}, y_{\beta}, z_{\gamma})}{\bar{c}_{l,\alpha}\bar{c}_{m,\beta}\bar{c}_{n,\gamma}} \cos \frac{i\pi\alpha}{l} \cos \frac{j\pi\beta}{m} \cos \frac{k\pi\gamma}{n} \quad (22)$$

$$\bar{c}_{i,0} = \bar{c}_{i,j} = 2, \quad \bar{c}_{i,i} = 1, \quad 1 \leq i \leq l-1 \quad (23)$$

Note that l , m and n are the numbers of Gauss-Lobatto nodes in the x , y and z directions, respectively. And the cube should be big enough to enclose the computational domain. For two dimensions, it is depicted in Fig. 1. Furthermore, it is well known that the use of Gauss-Lobatto nodes ensure the spectral convergence for Chebyshev interpolation. Details of Chebyshev interpolation can be found in a recent excellent review book of Mason and Handscomb (2003).

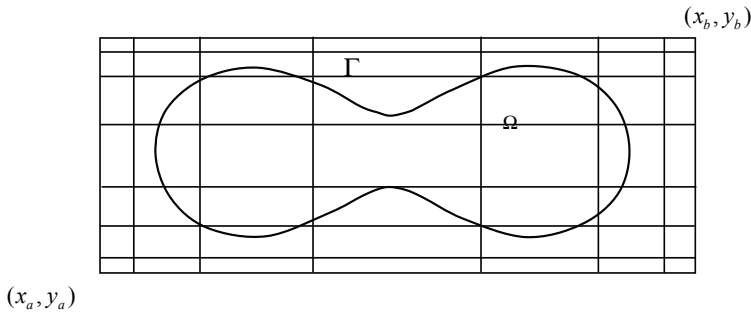


Figure 1: Geometry configuration of two-dimensional MPS.

Eq. (21) can also be rewritten as

$$F^I(x, y, z) \cong \sum_i^l \sum_j^m \sum_k^n b_{ijk}^I x^i y^j z^k \quad (24)$$

where b_{ijk}^I are constants. Then the particular solution can be approximated by

$$u_p^{i+1}(x, y, z) \cong \sum_i^l \sum_j^m \sum_k^n \sum_{\alpha=0}^{\lfloor \frac{i}{2} \rfloor} \sum_{\beta=0}^{\lfloor \frac{j}{2} \rfloor} \sum_{k=0}^{\lfloor \frac{k}{2} \rfloor} \frac{-b_{ijk}^I (\alpha + \beta + \gamma)! i! j! k! x^{i-2\alpha} y^{j-2\beta} z^{k-2\gamma}}{(q/\Delta t)^{\alpha+j+\gamma+1} \alpha! \beta! \gamma! (i-2\alpha)! (j-2\beta)! (k-2\gamma)!} \quad (25)$$

Eqs. (24) and (25) can be coded by multiple loops without book keeping and matrix inverse as described in Tsai (2008, 2009).

To impose the boundary condition of homogeneous solution in Eq. (19), we also need

$$\mathbf{B}u_p^{I+1}(x, y, z) \cong \sum_i^l \sum_j^m \sum_k^n \sum_{\alpha=0}^{\lfloor \frac{i}{2} \rfloor} \sum_{\beta=0}^{\lfloor \frac{j}{2} \rfloor} \sum_{k=0}^{\lfloor \frac{k}{2} \rfloor} \frac{-b_{ijk}^I (\alpha + \beta + \gamma)! i! j! k! \mathbf{B} (x^{i-2\alpha} y^{j-2\beta} z^{k-2\gamma})}{(q/\Delta t)^{\alpha+j+\gamma+1} \alpha! \beta! \gamma! (i-2\alpha)! (j-2\beta)! (k-2\gamma)!} \quad (26)$$

Similarly, the coding of Eq. (26) is straightforward if we have a subroutine for dealing with the derivatives of monomials.

3.2 Method of fundamental solution

Eqs. (18) and (19) together with Eq. (26) form a well-posed boundary value problem and thus can be solved by the MFS. Since the modified Helmholtz fundamental solution satisfies the modified Helmholtz equation, we may assume the homogeneous solution is a linear combination of the fundamental solution of the modified Helmholtz operator, i.e.,

$$u_h^{I+1}(x, y, z) \cong \sum_{i=1}^N A_i g(|\mathbf{x} - \mathbf{s}_i|) \quad (27)$$

where

$$g(\mathbf{x}; \mathbf{s}) = \begin{cases} \frac{1}{2\pi} K_0 \left(r \sqrt{\frac{q}{\Delta t}} \right) & \text{for 2D} \\ \frac{e^{-r \sqrt{\frac{q}{\Delta t}}}}{4\pi r} & \text{for 3D} \end{cases} \quad (28)$$

is the fundamental solution of the modified Helmholtz operator [Golberg and Chen (1999)], $\mathbf{x} = (x, y, z)$ represents the location of the field points, \mathbf{s}_i gives the location of the source points, $r = |\mathbf{x} - \mathbf{s}_i|$ is the distance, and N is the number of source points and field points. The source points are typically distributed away from the

boundary field points to avoid the singularity as shown in Fig. 2. By collocating Eq. (27) on N boundary field points \mathbf{x}_i , a linear square matrix system can be found and the N unknowns A_i can be solved by numerical methods. This MFS procedure is quite formal, more details can be found in the literature [Kupradze and Aleksidze (1964); Mathon and Johnston (1977); Bogomolny (1985), Smyrlis and Karageorghis (2001); Tsai, Young and Cheng (2002); Smyrlis and Karageorghis (2004); Chen, Fan, Young, Murugesan and Tsai (2005); Hon and Wei (2005); Young and Ruan (2005); Tsai, Lin, Young, and Atluri (2006); Young, Tsai, Lin and Chen (2006)].

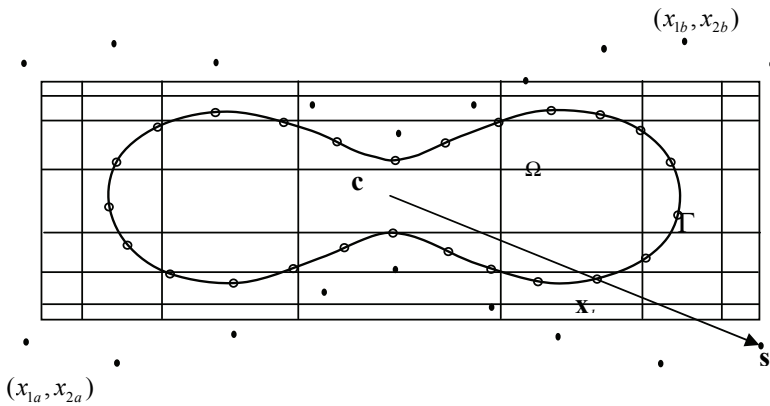


Figure 2: Geometry configuration of MFS-MPS.

After we have both the homogeneous and particular solution, we can proceed to next time step until the criterion in Eq. (15) be satisfied. These finish the development of FTIMFS-CP formulation.

4 Numerical experiments

Now we are ready to perform some numerical examples. The four numerical experiments in Ku, Yeh, Liu and Chi (2009) are first carried out to validate the proposed FTIMFS-CP. Then the method is extended to a peanut-shaped domain to demonstrate the flexibility of the present meshless method. In these numerical experiments, the root mean square error (RMSE) is defined by

$$\sqrt{\frac{\sum_{j=1}^L (u^{NUM}(\mathbf{x}_j) - u^{ANA}(\mathbf{x}_j))^2}{L}} \quad (29)$$

where $u^{NUM}(\mathbf{x}_j)$ is the numerical solution at the j -th point of the L considered positions and $u^{ANA}(\mathbf{x}_j)$ is the corresponding analytical solution of the problem. Here, L shall be chosen large enough to ensure the utilization of the RMSEs.

4.1 Example 1

The first example to be considered is

$$-\nabla^2 u = -3u^2 \quad (30)$$

Correspondingly, the analytical solution is given by

$$u = \frac{4}{(3+x+y)^2} \quad (31)$$

In the implementation, the boundary constraints are set up by using the exact solution in Eq. (31) and directly assign the Dirichlet values on the boundaries of the computational domain $\Omega = \{(x, y) \mid |x| \leq \frac{1}{2} \wedge |y| \leq \frac{1}{2}\}$.

Figs. 3 and 4 describe the RMSEs obtained for different time-like functions. In these numerical experiments, we observed that the FTIM fails to converge for too large and small Δt . And bigger feasible Δt generally gives better convergences. In addition, the two figures also demonstrate that the smaller M of the time-like function gives the faster convergence. In all of our numerical experiments, arbitrary small M can be safely chosen. However, further decreasing of M does not improve the convergence further. Also, Fig. 5 gives the RMSEs for different discretizations and finer discretization gives better accuracy as expected. It can be noted that smaller M and Δt should be utilized for the finest discretization and this is caused by the ill-conditioning of the MFS and/or Chebyshev polynomials. Finally, Figs. 6 and 7 describe the numerical result obtained by the FTIMFS-CP and its comparison with the analytical solution. In this numerical experiment, the RMSE reaches 2.44×10^{-10} for 40 boundary nodes and 100 interior nodes after 2000 fictitious time steps. This result demonstrates the high accuracy and stability of the proposed numerical scheme for solving a nonlinear PDE.

4.2 Example 2

To further validate the FTIMFS-CP, we then consider another Poisson-type nonlinear PDE in the same computational domain as follows:

$$-\nabla^2 u = -4u^3 \quad (32)$$

The analytical solution is given by

$$u = \frac{1}{4+x+y} \quad (33)$$

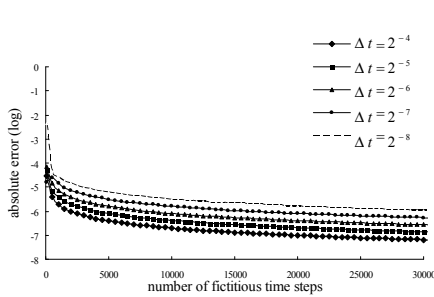


Figure 3: RMSEs of $M = 10^{-2}$ for different Δt (example 1, $N = 32$, $l = m = 8$).

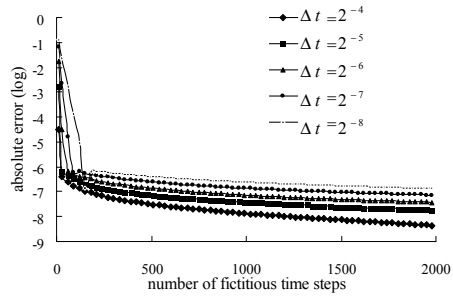


Figure 4: RMSEs of $M = 10^{-4}$ for different Δt (example 1, $N = 32$, $l = m = 8$).

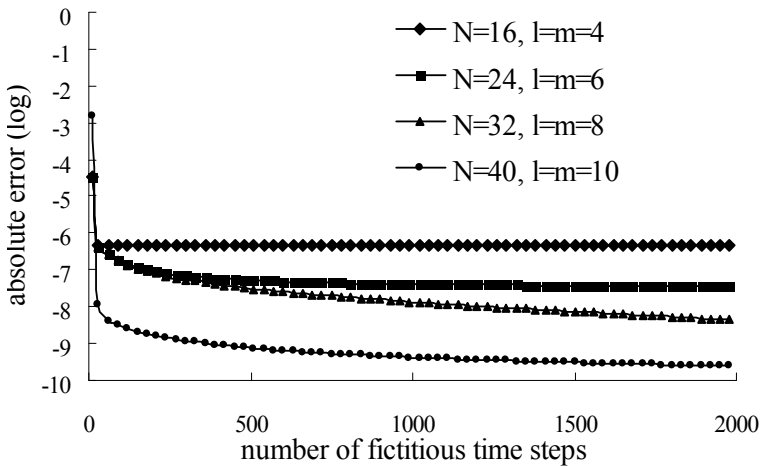


Figure 5: RMSEs of example 1 for different discretizations ($M = 10^{-6}$, $\Delta t = 2^{-5}$ for $N = 40$ and $M = 10^{-4}$, $\Delta t = 2^{-4}$ for others).

and Dirichlet boundary conditions are set up by using the exact solution of Eq. (33). Figs. 8, 9, and 10 describe the convergent statuses for $M = 10^{-2}$, $M = 10^{-4}$ and different meshless discretizations respectively. These results are very similar to those in the previous example. In Figs. 11 and 12, we also give the surfaces of the numerical result and its error which reaches a remarkable RMSE value 1.41×10^{-9} .

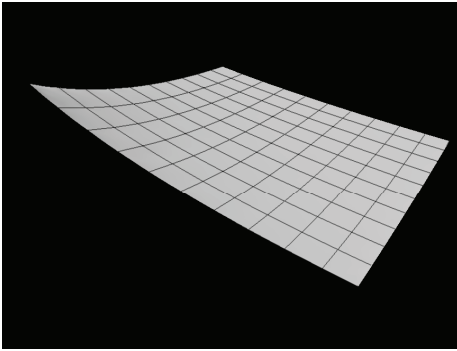


Figure 6: Results obtained by FTIM (example 1, $M = 10^{-6}$, $\Delta t = 2^{-5}$, $N = 40$, $l = m = 10$).

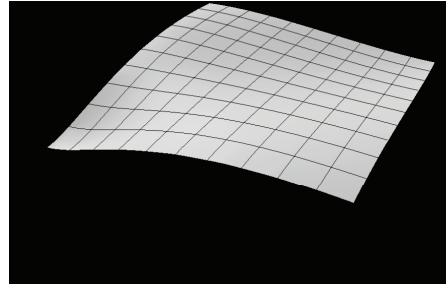


Figure 7: Results obtained by FTIM (example 1, $M = 10^{-4}$, $\Delta t = 2^{-5}$, $N = 40$, $l = m = 10$).

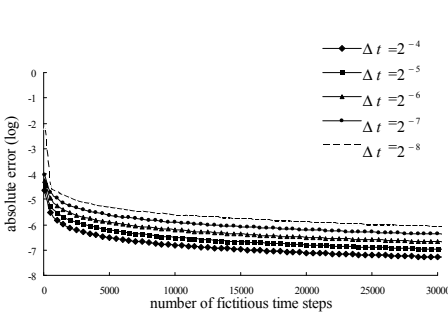


Figure 8: RMSEs of $M = 10^{-2}$ for different Δt (example 2, $N = 32$, $l = m = 8$).

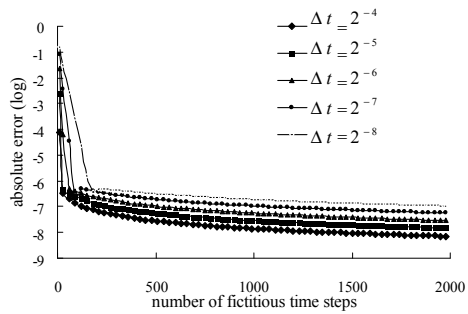


Figure 9: RMSEs of $M = 10^{-4}$ for different Δt (example 2, $N = 32$, $l = m = 8$).

4.3 Example 3

Then we also consider another Poisson-type nonlinear PDE arisen from the Dupuit-Forscheimer theory of groundwater flow as follows:

$$-\nabla^2 u = \frac{|\nabla u|^2}{u} \tag{34}$$

The considered domain is given by $\Omega = \{(x, y) \mid |x| \leq \frac{1}{2} \wedge |y| \leq \frac{1}{2}\}$. And boundary

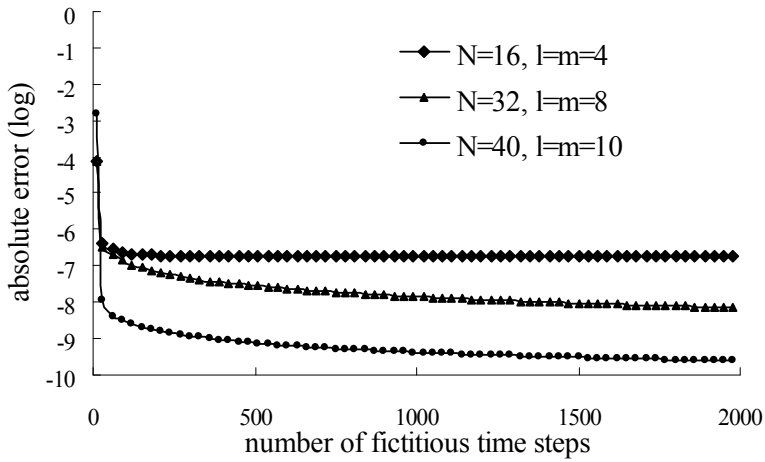


Figure 10: RMSEs of example 2 for different discretizations ($M = 10^{-6}$, $\Delta t = 2^{-5}$ for $N = 40$ and $M = 10^{-4}$, $\Delta t = 2^{-4}$ for others).

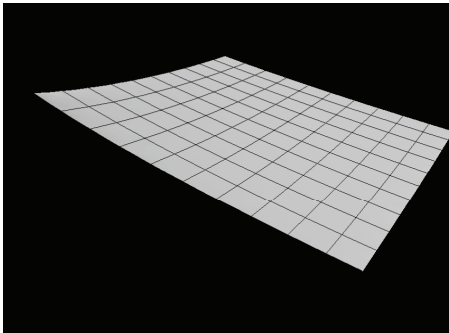


Figure 11: Results obtained by FTIM (example 2, $M = 10^{-6}$, $\Delta t = 2^{-5}$, $N = 40$, $l = m = 10$).

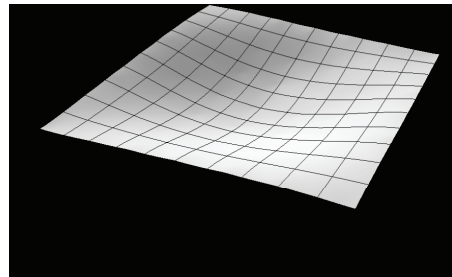


Figure 12: Results obtained by FTIM (example 2, $M = 10^{-4}$, $\Delta t = 2^{-5}$, $N = 40$, $l = m = 10$).

conditions are as following:

$$\begin{cases} u(-1, y) = 8 \\ u(1, y) = 2 \\ \frac{\partial u}{\partial y}(x, -1) = \frac{\partial u}{\partial y}(x, 1) = 0 \end{cases} \quad (35)$$

Correspondingly, we have the analytical solution as follows:

$$u = \sqrt{34 - 30x} \quad (36)$$

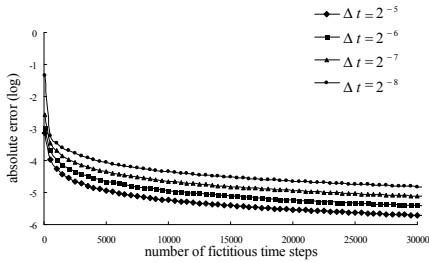


Figure 13: RMSEs of $M = 10^{-2}$ for different Δt (example 3, $N = 32$, $l = m = 8$).

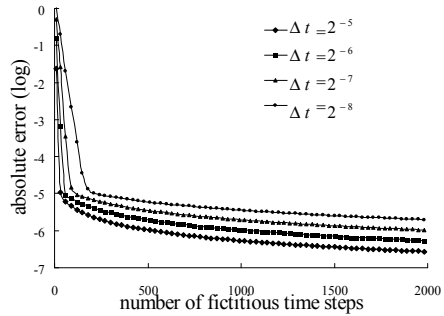


Figure 14: RMSEs of $M = 10^{-4}$ for different Δt (example 3, $N = 32$, $l = m = 8$).

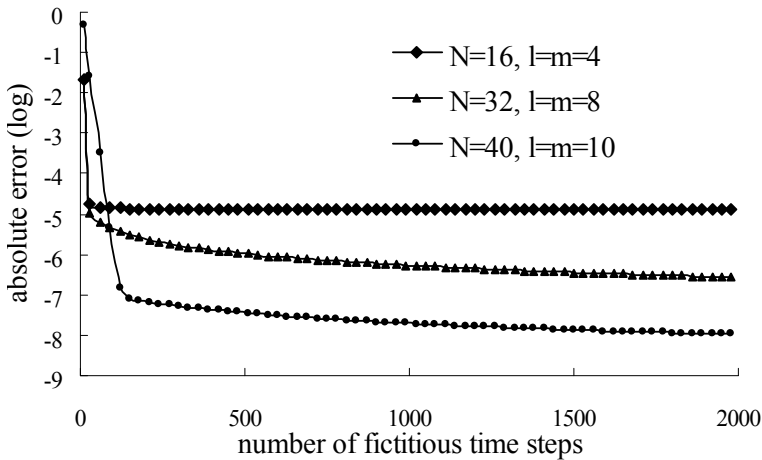


Figure 15: RMSEs of example 3 for different discretizations ($M = 10^{-6}$, $\Delta t = 2^{-5}$ for $N = 40$ and $M = 10^{-4}$, $\Delta t = 2^{-4}$ for others).

Figs. 13 and 14 depict the convergent situations of this example for the time-like functions of $M = 10^{-2}$ and $M = 10^{-4}$ respectively. Also, Fig. 15 describes the result for different discretizations. These results behave similarly compared with the previous two numerical experiments except that a worse accuracy and higher requirements of M and Δt . The spatial derivatives in the right-hand side of Eq. (34) should be a reasonable cause. In the last, Figs. 16 and 17 depict the obtained numerical result and its error. In this numerical experiment, we find a RMSE of 1.04×10^{-8} after 2000 fictitious time steps.

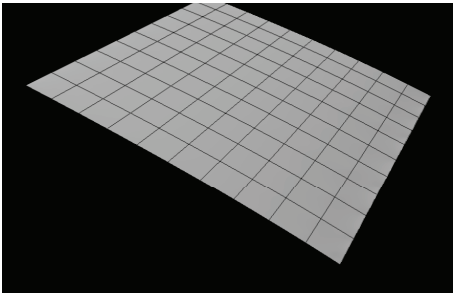


Figure 16: Results obtained by FTIM (example 3, $M = 10^{-6}$, $\Delta t = 2^{-5}$, $N = 40$, $l = m = 10$).

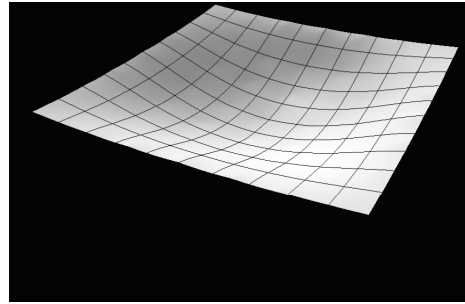


Figure 17: RMSEs of results obtained by FTIM (example 3, $M = 10^{-4}$, $\Delta t = 2^{-5}$, $N = 40$, $l = m = 10$).

4.4 Example 4

Our meshless numerical method can be extended to three-dimensional problems easily. Thus, we consider the following three-dimensional example

$$-\nabla^2 u = -6u^3 \tag{37}$$

The analytical solution is given by

$$u = \frac{1}{4 + x + y + z} \tag{38}$$

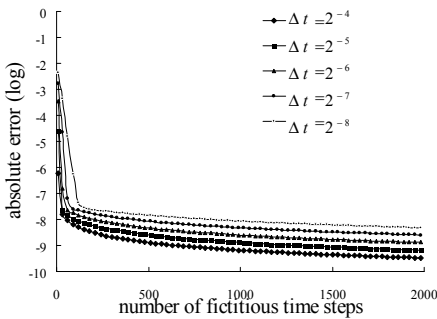


Figure 18: RMSEs of $M = 10^{-4}$ for different Δt (example 4, $N = 218$, $l = m = n = 8$).

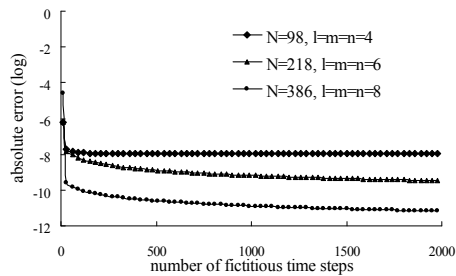


Figure 19: RMSEs of example 4 for different discretizations ($M = 10^{-6}$, $\Delta t = 2^{-5}$ for $N = 386$ and $M = 10^{-4}$, $\Delta t = 2^{-4}$ for others).

The boundary constraints can be obtained by using the exact solution of Eq. (38) and directly assigned the Dirichlet values on the boundaries of the computational domain $\Omega = \{ (x,y,z) \mid |x| \leq \frac{1}{2} \wedge |y| \leq \frac{1}{2} \wedge |z| \leq \frac{1}{2} \}$. Surprisingly, we find both the convergence and accuracy perform the same as the two-dimensional examples as depicted in Figs. 18 and 19. In Fig. 19, the RMSE reaches a excellent value, 6.54×10^{-12} , for only 386 boundary nodes and 512 Gauss-Lobatto nodes after 2000 fictitious time steps.

4.5 Example 5

Finally, we also demonstrate the flexibility of the present method to irregular domains. Therefore, we resolve example 2 in a peanut-shaped domain defined by

$$r(\theta) = 0.3\sqrt{\cos 2\theta + \sqrt{1.1 - \sin^2 2\theta}} \quad 0 \leq \theta \leq 2\pi \tag{39}$$

where (r, θ) is the usual polar coordinate. In this study, we perform Chebyshev interpolation in a rectangular domain defined by $[-0.45, 0.45] \times [-0.18, 0.18]$. Fig. 20 describes the history of the RMSE for two different spatial discretizations. The better one reaches a RMSE of 4.12×10^{-7} after 2000 fictitious time steps. These results have demonstrated the applicability of the proposed scheme for solving Poisson-type nonlinear PDE in an irregular domain.

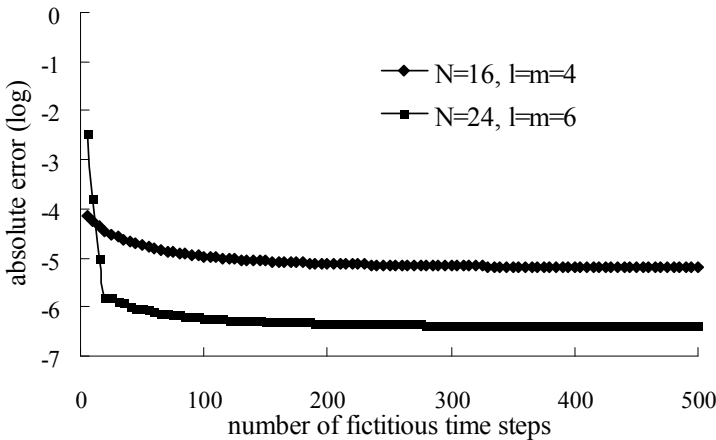


Figure 20: RMSEs of example 5 for different discretizations ($M = 10^{-3}$ and $\Delta t = 2^{-3}$).

5 Conclusions

In this study, the fictitious time integration method was combined with the method of fundamental solution with Chebyshev polynomials to form a meshless numerical method to solve the Poisson-type nonlinear PDEs. In the solution procedure, a Poisson-type nonlinear PDE is first converted into a sequence of linear inhomogeneous modified Helmholtz equations by the FTIM and then be formally solved by the MFS-CP. Several numerical experiments were carried to validate the proposed numerical method and accurate numerical solutions were found. Furthermore, a Poisson-type nonlinear PDE was computed in a peanut-shaped domain which demonstrated the flexibility of the present meshless numerical method to complex geometry.

Acknowledgement: The National Science Council of Taiwan is gratefully acknowledged for providing financial support for CC to carry out the present work under the Grant No. NSC 98-2221-E-022-019

References

- Atluri, S.N.** (2004): The meshless method (MLPG) for domain & BIE discretization. 700 pages, *Tech Science Press*, Forsyth, GA, USA.
- Bogomolny, A.** (1985): Fundamental solutions method for elliptic boundary value problems. *SIAM Journal of Numerical Analysis*, vol. 22, pp. 644-669.
- Chang, C.W.; Liu, C.S.** (2009): A fictitious time integration method for backward advection-dispersion equation. *CMES: Computer Modeling in Engineering & Sciences*, Vol. 51(3), pp. 261-276.
- Chi, C.C.; Yeih, W.C; Liu, C.S.** (2009): A novel method for solving the Cauchy problem of Laplace equation using the fictitious time integration method. *CMES: Computer Modeling in Engineering & Sciences*, Vol. 47(2), pp. 167-190.
- Chen C.W.; Fan, C.M.; Young, D.L.; Murugesan, K.T; Tsai, C.C** (2005): Eigenanalysis for membranes with stringers using the methods of fundamental solutions and domain decomposition. *CMES: Computer Modeling in Engineering & Sciences*, vol. 8(1), pp. 29-44.
- Cozzano, B.S.; Rodríguez, B.S.** (2007): The Trefftz boundary method in viscoelasticity, *CMES: Computer Modeling in Engineering & Sciences*, vol. 20, pp. 21-34.
- Ding, J.; Chen, C.S.** (2007): Particular solutions of some elliptic partial differential equations via recursive formulas. *Journal of University of Science and Technology of China*, vol. 37, pp. 1-10.

Golberg, M.A.; Chen, C.S. (1999): The method of fundamental solutions for potential, Helmholtz and diffusion problems. M. A. Golberg, editor, *Boundary integral methods: numerical and mathematical aspects*. *WIT Press and Computational Mechanics Publications*, Boston, Southampton, pp. 103–176.

Golberg, M.A.; Muleshkov, A.S.; Chen, C.S.; Cheng, A.H.-D. (2003): Polynomial particular solutions for some partial differential operators. *Numerical Methods for Partial Differential Equations*, vol. 19, pp. 112–133.

Han, Z.D.; Atluri, S.N. (2004): A meshless local Petrov-Galerkin (MLPG) approach for 3-dimensional elasto-dynamics. *CMC: Computers, Materials & Continua*, vol. 1(2), pp. 129-140.

Hon, Y.C.; Wei, T. (2005): The method of fundamental solution for solving multidimensional inverse heat conduction problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 7(2), pp. 119-132.

Hu, S.P.; Young, D.L.; Fan, C.M. (2008): FDMFS for diffusion equation with unsteady forcing function, *CMES: Computer Modeling in Engineering & Sciences*, vol. 24, pp. 1-20.

Kansa, E.J. (1990a): Multiquadrics - a scattered data approximation scheme with applications to computational fluid dynamics-I. Surface approximations and partial derivative estimates. *Computers and Mathematics with Applications*, vol. 19, pp. 127-145.

Kansa, E.J. (1990b): Multiquadrics - a scattered data approximation scheme with applications to computational fluid dynamics-II. Solutions to parabolic, hyperbolic and elliptic partial differential equations. *Computers and Mathematics with Applications*, vol. 19, pp. 147-161.

Karageorghis, A.; Kyza, I. (2007): Efficient algorithms for approximating particular solutions of elliptic equations using Chebyshev polynomials. *Communications in Computational Physics*, vol. 2, pp.501–521.

Kim, H.G.; Atluri, S.N. (2000): Arbitrary placement of secondary nodes, and error control, in the meshless local Petrov-Galerkin (MLPG) method. *CMES: Computer Modeling in Engineering & Sciences*, vol. 1(3), pp. 11-32.

Ku, C.Y.; Yeih, W.C.; Liu, C-S; Chi, C.C. (2009): Applications of the Fictitious Time Integration Method Using a New Time-Like Function. *CMES: Computer Modeling in Engineering & Sciences*, vol. 43, pp. 173-190.

Kupradze, V.D.; Aleksidze, M.A. (1964): The method of functional equations for the approximate solution of certain boundary value problems. *USSR Computational Mathematics and Mathematical Physics*, vol. 4, pp. 82-126.

Lee, W.M.; Chen, J.T. (2009): Free vibration analysis of a circular plate with

multiple circular holes by using the multipole Trefftz method. *CMES: Computer Modeling in Engineering & Sciences*, vol. 50(2), pp. 141-160

Li, Z.C.; Lu, T.T.; Huang, H.T.; Cheng, A.H.-D. (2009): Error analysis of Trefftz methods for Laplace's equations and its applications. *CMES: Computer Modeling in Engineering & Sciences*, vol. 52(1), pp. 39-82

Lin, H.; Atluri S.N. (2000): Meshless local Petrov-Galerkin (MLPG) method for convection-diffusion problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 1(2), pp. 45-60.

Liu, C.S. (2006): The Lie-group shooting method for nonlinear two-point boundary value problems exhibiting multiple solutions. *CMES: Computer Modeling in Engineering & Sciences*, vol. 13, pp. 149-163.

Liu, C.S. (2007a): A modified Trefftz method for two-dimensional Laplace equation considering the domain's characteristic length. *CMES: Computer Modeling in Engineering & Sciences*, vol. 21, pp. 53-65.

Liu, C.S. (2007b): An effectively modified direct Trefftz method for 2D potential problems considering the domain's characteristic length. *Eng. Anal. Bound. Elem.*, vol. 31, pp. 983-993.

Liu, C.S. (2008a): A Lie-group shooting method for simultaneously estimating the time-dependent damping and stiffness coefficients. *CMES: Computer Modeling in Engineering & Sciences*, vol. 27, pp. 137-149.

Liu, C.S. (2008b): Improving the ill-conditioning of the method of fundamental solutions for 2D Laplace equation. *CMES: Computer Modeling in Engineering & Sciences*, vol. 28, pp. 77-93.

Liu, C.S. (2008c): A time-marching algorithm for solving non-linear obstacle problems with the aid of an NCP-function. *CMC: Computers, Materials & Continua*, vol. 8, pp. 53-65.

Liu, C.S. (2008d): A fictitious time integration method for two-dimensional quasi-linear elliptic boundary value problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 33, pp. 179-198.

Liu, C.S. (2008e): A highly accurate MCTM for inverse Cauchy problems of Laplace equation in arbitrary plane domains. *CMES: Computer Modeling in Engineering & Sciences*, vol. 35, pp. 91-111.

Liu, C.S. (2008f): A fictitious time integration method for solving the discretized inverse Sturm-Liouville problems, for specified eigenvalues. *CMES: Computer Modeling in Engineering & Sciences* vol. 36, pp. 261-286.

Liu, C.S. (2009a): A fictitious time integration method for solving m-point boundary value problems. *CMES: Computer Modeling in Engineering & Sciences*, vol.

39, pp. 125-154.

Liu, C.S. (2009b): A fictitious time integration method for a quasilinear elliptic boundary value problem, defined in an arbitrary plane domain. *CMC: Computers, Materials & Continua*, vol. 11, pp. 15-32.

Liu, C.S.; Atluri, S. N. (2008a): A novel time integration method for solving a large system of non-linear algebraic equations. *CMES: Computer Modeling in Engineering & Sciences*, vol. 31, pp. 71-83.

Liu, C.S.; Atluri, S. N. (2008b): A fictitious time integration method (FTIM) for solving mixed complementarity problems with applications to non-linear optimization. *CMES: Computer Modeling in Engineering & Sciences*, vol. 34, pp. 155-178.

Liu, C.S.; Atluri, S. N. (2009): A fictitious time integration method for the numerical solution of the Fredholm integral equation and for numerical differentiation of noisy data, and its relation to the filter theory. *CMES: Computer Modeling in Engineering & Sciences*, vol. 41, pp. 243-261.

Marin, L. (2008): Stable MFS solution to singular direct and inverse problems associated with the Laplace equation subjected to noisy data, *CMES: Computer Modeling in Engineering & Sciences*, vol. 37, pp. 203-242.

Marin, L. (2009a): Boundary reconstruction in two-dimensional functionally graded materials using a regularized MFS. *CMES: Computer Modeling in Engineering & Sciences*, Vol. 46(3), pp. 221-254

Marin, L. (2009b): An iterative MFS algorithm for the Cauchy problem associated with the Laplace equation. *CMES: Computer Modeling in Engineering & Sciences*, Vol. 48(2), pp. 121-154.

Mathon, R.; Johnston, R.L. (1977): The approximate solution of elliptic boundary-value problems by fundamental solutions. *SIAM Journal on Numerical Analysis*, vol. 14, pp. 638-650.

Mason, J.C.; Handscomb, D.C. (2003): Chebyshev Polynomials. *Chapman & Hall/CRC*, Boca Raton.

Nardini, D.; Brebbia, C.A. (1982): A new approach to free vibration analysis using boundary elements, C. A. Brebbia, editor, *Boundary element methods in engineering*, Springer-Verlag, New York, pp. 312-326.

Press, W.H.; Teukolsky, S.A.; Vetterling, W.T.; Flannery, B.P. (2007): *Numerical Recipes, The Art of Scientific Computing*, Cambridge University Press.

Reutskiy, S.Y.; Chen, C.S. (2006): Approximation of multivariate functions and evaluation of particular solutions using Chebyshev polynomial and trigonometric basis functions. *International Journal for Numerical Methods in Engineering*, vol. 67, pp. 1811-1829.

Smyrlis, Y.S.; Karageorghis, A. (2001): Some aspects of the method of fundamental solutions for certain harmonic problems. *Journal of Scientific Computing*, vol. 16, pp. 341-371.

Smyrlis, Y.S.; Karageorghis, A. (2004): A linear least square MFS for certain elliptic problems. *Numerical Algorithms*, vol. 35, pp. 29-44.

Tsai, C.C. (2008): Particular solutions of Chebyshev polynomials for polyharmonic and poly-Helmholtz equations. *CMES: Computer Modeling in Engineering & Sciences*, vol. 27, pp. 151-162.

Tsai, C.C. (2009): The particular solutions of Chebyshev polynomials for Reissner plates under arbitrary loadings. *CMES: Computer Modeling in Engineering & Sciences*, vol. 45, pp. 249-271.

Tsai, C.C.; Lin, Y.C.; Young, D.L.; Atluri, S.N. (2006): Investigations on the accuracy and condition number for the method of fundamental solutions, *CMES: Computer Modeling in Engineering & Sciences*, vol. 16, pp. 103-114.

Tsai, C.C.; Young, D.L.; and Cheng, A.H.-D. (2002): Meshless BEM for three-dimensional Stokes flows, *CMES: Computer Modeling in Engineering & Sciences*, vol. 3, pp. 117-128.

Wordelman, C.J.; Aluru, N.R.; Ravaioli, U. (2000): A meshless method for the numerical solution of the 2- and 3-D Semiconductor Poisson equation. *CMES: Computer Modeling in Engineering & Sciences*, vol. 1(1), pp. 121-126.

Young, D.L.; Chen, K.H.; Chen, J.T.; Kao, J.H. (2007): A modified method of fundamental solutions with source on the boundary for solving Laplace equations with Circular and arbitrary domains, *CMES: Computer Modeling in Engineering & Sciences*, vol. 19, pp. 197-222.

Young, D.L.; Chen, C.S.; Wong, T.K. (2005): Solution of Maxwell's equations using the MQ method. *CMC: Computers, Materials & Continua*, Vol. 2(4), pp. 267-276.

Young, D.L.; Jane, S.C.; Lin, C.Y.; Chiu, C.L.; Chen, K.C. (2004): Solutions of 2D and 3D Stokes laws using multiquadrics method. *Engineering Analysis with Boundary Elements*, vol. 28, pp. 1233-1243.

Young, D.L.; Ruan J.W. (2005): Method of fundamental solutions for scattering problems of electromagnetic waves. *CMES: Computer Modeling in Engineering & Sciences*, Vol. 7(2), pp. 223-232

Young, D.L.; Tsai, C.C.; Lin, Y.C.; Chen, C.S. (2006): The method of fundamental solutions for eigenfrequencies of plate vibrations. *CMC: Computers, Materials & Continua*, Vol. 4, No. 1, pp. 1-10.

