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Green's Functions for Anisotropic/Piezoelectric Bimaterials and Their Applications to Boundary Element Analysis

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Abstract: The Green's function for anisotropic bimaterials has been investigated around three decades ago. Since the mathematical formulation of piezoelectric elasticity can be organized into the same form as that of anisotropic elasticity by just expanding the dimension of the corresponding matrix to include the piezoelectric effects, the extension of the Green's function to piezoelectric bimaterials can be obtained immediately through the associated anisotropic bimaterials. In this paper, the Green's function for the bimaterials bonded together with one anisotropic material and one piezoelectric material is derived by applying Stroh's complex variable formalism with the aid of analytical continuation method. For this problem, the interfacial condition of electric field depends on the electric conductivity of anisotropic elastic materials. Employing these Green's functions, a special boundary element satisfying the interfacial continuity conditions of anisotropic/piezoelectric bimaterials is developed. With the embedded Green's functions, this special boundary element preserves two special features: (1) the interface continuity conditions are satisfied exactly and no meshes are needed along the interface; (2) the materials below and above the interface can be any kinds of piezoelectric or anisotropic elastic materials. To show the advantages of the present special boundary element, several numerical examples such as orthotropic/isotropic bimaterials, PZT-7A/PZT-5H bimaterials and anisotropic/piezoelectric bimaterials are illustrated and compared with the solutions calculated by other numerical methods. The numerical results show that the present special boundary element is not only accurate but also efficient.

Keywords: Green's function, piezoelectric materials, anisotropic elastic materials, interfaces, Stroh formalism, boundary element method, fundamental solution,

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analytical continuation method

1 Introduction

A bimaterial bonded together with two dissimilar anisotropic materials is called *anisotropic bimaterials*. Similarly, a *piezoelectric bimaterial* is the one bonded by two dissimilar piezoelectric materials. Here, we use the name *anisotropic/piezoelectric bimaterials* to denote the bimaterials which can be any combination of anisotropic and piezoelectric materials. If an anisotropic/piezoelectric bimaterial is loaded by a *point force* (or called *concentrated force,* sometimes called *line force* for two-dimensional problems) applied at an internal point far from the boundary, the elasticity solution of this problem is known to be *Green's function* which can be used as a fundamental solution of boundary element method.

The Green's functions for anisotropic bimaterials have been investigated by [Barnett and Lothe (1973); Belov, Chamrov, Indenbom and Lothe (1983); Kirchner and Lothe (1987); Tewary, Wagoner and Hirth (1989); Ting (1992, 1996)]. Through the correspondence relations between anisotropic and piezoelectric materials [Kuo and Barnett (1991); Suo, Barnett and Willis (1992); Liang and Hwu (1996); Ting (1996); Hwu (2008)], the Green's functions for anisotropic bimaterials can easily be extended to piezoelectric bimaterials. It is known that the solution forms of the Green's functions for anisotropic bimaterials and piezoelectric bimaterials are exactly the same. The only difference is the dimension and content of the matrices used in the solutions. Due to this difference, when we consider the anisotropic/piezoelectric bimaterials the matrix dimension of anisotropic materials will not match that of piezoelectric materials. To deal with this problem, two different approaches are considered in this paper. One is re-deriving the Green's functions for two special combinations, and the other is specializing the Green's functions through general piezoelectric bimaterials. In the former we categorize the anisotropic elastic materials into two kinds, conductors and insulators. In the latter, the specialization is done by letting the piezoelectric stress tensor of anisotropic elastic materials be zero, and also letting the dielectric permittivity tensor be zero or infinity to represent, respectively, the insulators or conductors.

Using Green's functions of specific problems as basis, several special boundary elements have been developed in the literature, such as the anisotropic plates containing cracks, holes and inclusions [Hwu and Yen (1991); Hwu and Liao (1994); Berger and Tewary (1997); Pan and Amadei (1999); Shah, Tan and Wang (2006)], the particulate composite materials [Okada, Fukui and Kumazawa (2004)], the gradient elastic materials containing cracks [Karlis, Tsinopoulos, Polyzos and Beskos (2008)], and the piezoelectric plates containing defects [Liang and Hwu (1996); Qin and Lu (2000); Sanz, Solis and Dominguez (2007)]. With the success of these special boundary elements, in this paper we further apply the Green's functions for anisotropic/piezoelectric bimaterials to improve the traditional boundary elements. In this way, no meshes are needed along the interfaces and the materials below and above the interface can be any kinds of piezoelectric or anisotropic elastic materials. Since the continuity conditions have been satisfied exactly and no meshes are needed along the interfaces, for the interface problems the present special boundary element is much more accurate and efficient than the other numerical methods such as finite element method and traditional boundary element method. These advantages are confirmed through numerical examples for three different kinds of bimaterials - orthotropic/isotropic, PZT-7A/PZT-5H and anisotropic/piezoelectric bimaterials.

2 Basic Equations

In a fixed rectangular coordinate system x_i, $i = 1,2,3$, let $u_i, \sigma_{ij}, \varepsilon_{ij}, D_j$, and E_k be, respectively, the displacement, stress, strain, electric displacement (or called induction) and electric field. The constitutive laws, strain-displacement equations and the equilibrium equations for anisotropic elasticity are [Ting (1996)]

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl},\ \varepsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i}), \quad \sigma_{ij,j} = 0,\ i,j,k,l = 1,2,3 \tag{1}$$

where repeated indices imply summation, a comma stands for differentiation. In eqn.(1) the body forces are neglected and the elastic constants C_{ijkl} are assumed to be fully symmetric and positive definite.

For piezoelectric anisotropic elasticity, to include the piezoelectric effects the constitutive laws should be modified and the electrostatic equations should be considered, and hence the basic equations are modified as follows. [Rogacheva (1994)]

$$\begin{cases} \sigma_{ij} = C^E_{ijkl}\varepsilon_{kl} - e_{kij}E_k, \\ D_j = e_{jkl}\varepsilon_{kl} + \omega^\varepsilon_{jk}E_k, \end{cases} \quad \varepsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i}), \ \begin{cases} \sigma_{ij,j} = 0, \\ D_{i,i} = 0, \end{cases} \quad i,j,k,l = 1,2,3, \tag{2}$$

in which C^E_{ijkl},e_{kij} and ω^ε_{jk} are, respectively, the elastic stiffness tensor at constant electric field, piezoelectric stress tensor and dielectric permittivity tensor at constant strain. These tensors have the following symmetry properties

$$C^E_{ijkl} = C^E_{jikl} = C^E_{klij},\ e_{kij} = e_{kji},\ \omega^\varepsilon_{jk} = \omega^\varepsilon_{kj}. \tag{3}$$

By letting

$$\begin{aligned}&D_j = \sigma_{4j}, \quad -E_j = u_{4,j} = 2\varepsilon_{4j},\ j = 1,2,3,\\&C_{ijkl} = C^E_{ijkl},\ i,j,k,l = 1,2,3,\\&C_{ij4l} = e_{lij},\ i,j,l = 1,2,3,\\&C_{4jkl} = e_{jkl},\ j,k,l = 1,2,3,\\&C_{4j4l} = -\omega^\varepsilon_{jl},\ j,l = 1,2,3,\end{aligned} \tag{4}$$

the basic eqn. (2) can be rewritten in an *expanded tensor notation* as

$$\sigma_{IJ} = C_{IJKL}\varepsilon_{KL},\ \varepsilon_{IJ} = \frac{1}{2}(u_{I,J} + u_{J,I}),\ \sigma_{IJ,J} = 0,\ I,J,K,L = 1,2,3,4, \tag{5}$$

where expanded elastic stiffness tensor C_{IJKL} possesses the fully symmetric properties.

Because the mathematical forms of basic equations (1) and (5) are exactly the same, the general solutions satisfying these equations can be organized into the same matrix form. For the convenience of following derivation, the general solution valid for both anisotropic and piezoelectric elasticity is now shown below. [Ting (1996)]

$$\mathbf{u} = 2\mathrm{Re}\{\mathbf{Af}(z)\},\pm\phi = 2\mathrm{Re}\{\mathbf{Bf}(z)\}, \tag{6}$$

where Re stands for the real part of a complex number, $\mathbf{u}$ and $\pm\phi$ are, respectively, the *displacement vector* and *stress function vector*; $\mathbf{f}(z)$ is a vector containing holomorphic functions of complex variables $z_\alpha(= x_1 + \mu_\alpha x_2)$; $\mathbf{A}$ and $\mathbf{B}$ are the material eigenvector matrices associated with the material eigenvalues μ_α. The range of the subscript α is 1 to 3 for general anisotropic materials and is 1 to 4 for piezoelectric materials, and hence the dimension of the associated vectors and matrices are as follows.

$$\begin{aligned}&\mathbf{u},\pm\phi,\mathbf{f} : 3\times 1,\ \mathbf{A},\mathbf{B} : 3\times 3,\ \text{for anisotropic elasticity;}\\&\mathbf{u},\pm\phi,\mathbf{f} : 4\times 1,\ \mathbf{A},\mathbf{B} : 4\times 4,\ \text{for piezoelectric elasticity.}\end{aligned} \tag{7}$$

The stresses and electric displacements are related to the stress function ϕ_i by

$$\sigma_{i1} = -\phi_{i,2},\ \sigma_{i2} = \phi_{i,1},\ i = 1,2,3,\ \text{and}\ D_1 = \sigma_{41} = -\phi_{4,2},\ D_2 = \sigma_{42} = \phi_{4,1}, \tag{8}$$

from which it has been proved that

$$\mathbf{t} = \frac{\partial \pm \phi}{\partial s} \tag{9}$$

where $\mathbf{t}$ is the *generalized surface traction vector* and s is the arc length measured along a curved boundary of the body.

3 Green's Functions

3.1 Anisotropic bimaterials

Consider a bimaterial that consists of two dissimilar anisotropic elastic half-spaces. Let the upper half-space $x_2 > 0$ be occupied by material 1 and the lower half-space $x_2 < 0$ be occupied by material 2 (see Fig. 1). Assume these two dissimilar materials are perfectly bonded along the interface $x_2 = 0$. The *Green's function* for bimaterials is the elasticity solution for a bimaterial subjected to a concentrated force $\hat{\mathbf{p}}$ applied at point $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2)$ of material 1. The boundary conditions of this problem can be expressed as

$$
\begin{aligned}
&\mathbf{u}_1 = \mathbf{u}_2,\ \pm\phi_1 = \pm\phi_2, \text{ along the interface } x_2 = 0,\\
&\int_C d\pm\phi_1 = \hat{\mathbf{p}} \text{ for any closed curve C enclosing the point } \hat{\mathbf{x}}, \qquad (10)\\
&\sigma_{ij} \to 0 \text{ at infinity,}
\end{aligned}
$$

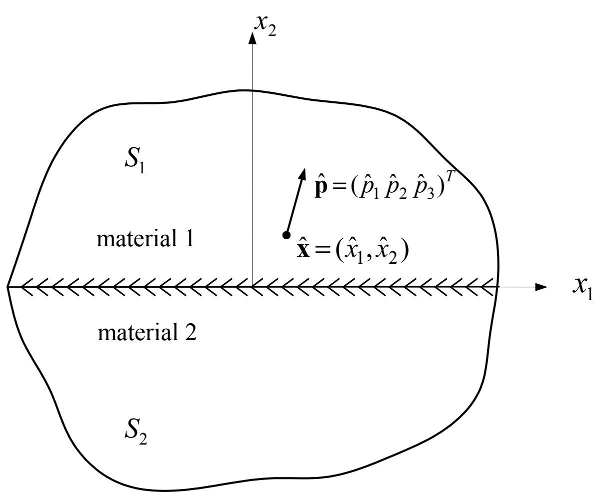


Figure 1: A bimaterial subjected to a point force $\hat{\mathbf{p}}$ on $\hat{\mathbf{x}}$.

in which the subscripts 1 and 2 denote materials 1 and 2, respectively. By employing the general solutions of Stroh formalism stated in eqn. (6), a solution satisfying the above boundary conditions has been obtained by [Ting (1992)] as

$$\mathbf{u}_1 = 2\mathrm{Re}\left\{\mathbf{A}_1[\mathbf{f}_0(z^{(1)})+\mathbf{f}_1(z^{(1)})]\right\}, \pm\phi_1 = 2\mathrm{Re}\left\{\mathbf{B}_1[\mathbf{f}_0(z^{(1)})+\mathbf{f}_1(z^{(1)})]\right\},$$
$$\mathbf{u}_2 = 2\mathrm{Re}\left\{\mathbf{A}_2\mathbf{f}_2(z^{(2)})\right\}, \pm\phi_2 = 2\mathrm{Re}\left\{\mathbf{B}_2\mathbf{f}_2(z^{(2)})\right\}, \tag{11}$$

where the subscripts 1 and 2 or the superscripts (1) and (2) denote materials 1 and 2, respectively.

$$\mathbf{f}_0(z^{(1)}) = \frac{1}{2\pi i} < \ln(z_\alpha^{(1)} - \hat{z}_\alpha^{(1)}) > \mathbf{A}_1^T\hat{\mathbf{p}},$$
$$\mathbf{f}_1(z^{(1)}) = \frac{1}{2\pi i}\sum_{j=1}^{3} < ln(z_\alpha^{(1)} - \bar{\hat{z}}_j^{(1)}) > \mathbf{A}_1^{-1}(\bar{\mathbf{M}}_2+\mathbf{M}_1)^{-1}(\bar{\mathbf{M}}_2-\bar{\mathbf{M}}_1)\bar{\mathbf{A}}_1\mathbf{I}_j\bar{\mathbf{A}}_1^T\hat{\mathbf{p}}, \tag{12}$$
$$\mathbf{f}_2(z^{(2)}) = -\frac{1}{2\pi}\sum_{j=1}^{3} < ln(z_\alpha^{(2)} - \hat{z}_j^{(1)}) > \mathbf{A}_2^{-1}(\mathbf{M}_2+\bar{\mathbf{M}}_1)^{-1}\mathbf{A}_1^{-T}\mathbf{I}_j\mathbf{A}_1^T\hat{\mathbf{p}},$$

in which $\mathbf{M}_j, j=1,2$,is the impedance matrix defined by [Ting (1988)]

$$\mathbf{M}_j = -i\mathbf{B}_j\mathbf{A}_j^{-1}, \quad j=1,2. \tag{13}$$

In eqn. (12), the superscript T denotes the transpose of a matrix; the overbar denotes the complex conjugate; the angular bracket $< >$ stands for the diagonal matrix in which each component is varied according to its subscript α; $\mathbf{I}_j$ is a diagonal matrix with unit value at the *jj* component and all the others are zero.

3.2 Piezoelectric bimaterials

From the discussion provided in [Hwu and Ikeda (2008)] we know that the continuity of displacement and traction as well as the continuity of electric potential and electric displacement can be expressed by the first equation of (10) with the vectors $\mathbf{u}$ and $\pm\phi$ generalized from 3×1 to 4×1. Moreover, the point force $(\hat{p}_1, \hat{p}_2, \hat{p}_3)$ and point charge $\hat{p}_4 = \hat{q}$ equilibrium conditions can also be expressed by the second equation of (10) with $\pm\phi$ and $\hat{\mathbf{p}}$ generalized from 3×1 to 4×1, so is the infinity condition given in the last equation of (10). Because general solution and boundary conditions of this case can all be expressed by the same mathematical form as those of section 3.1, i.e. eqns. (6) and (10), the Green's function given in (11)-(13) should also be valid for the present case. The only difference is size and content of the matrices used in the Green's function, which are now expanded from 3×1 and 3×3 to 4×1 and 4×4, respectively. The number of summation terms of the last two equations of (12) should also be changed from 3 to 4.

3.3 Anisotropic/piezoelectric bimaterials

In this subsection the anisotropic elastic materials are categorized into two kinds, conductors and insulators. The electric field E_i of the conductors is considered to be zero in the entire body, whereas the electric displacement D_i of the insulators is zero. With the relations $u_{4,j} = -E_j$, $\phi_{4,1} = D_2$ and $\phi_{4,2} = -D_1$ given in (4) and (8), we know that the electric continuity conditions for the conductors and insulators can be represented, respectively, by $u_4 = 0$ and $\phi_4 = 0$ along the interface.

Case (i): the anisotropic material is a conductor

Consider a bimaterial whose upper half-space $x_2 > 0$ (region S_1) is occupied by a piezoelectric material and the lower half-space $x_2 < 0$ (region S_2) is occupied by an anisotropic elastic conductor. If a point force/charge $\hat{\mathbf{p}} = (\hat{p}_1,\, \hat{p}_2,\, \hat{p}_3,\, \hat{q})^T$ is applied at point $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2)$ of material 1, the boundary conditions along the interface can be written as

$$u_i^{(1)} = u_i^{(2)},\ \phi_i^{(1)} = \phi_i^{(2)},\ i = 1,2,3,\ u_4^{(1)} = 0,\ \text{along the interface } x_2 = 0. \tag{14}$$

The equilibrium and infinity conditions can still be expressed by using the last two equations of (10). To find the solution satisfying all the boundary conditions, we first assume the solution be expressed in the form of eqn.(11) whose complex function vectors $\mathbf{f}_1(z^{(1)})$ and $\mathbf{f}_2(z^{(2)})$ will be determined through the use of analytical continuation method. $\mathbf{f}_0(z^{(1)})$ that represents the Green's function of homogeneous materials is still the one given in the first equations of (12).

With the general solutions given in (11), the boundary conditions (14) may now be written as

$$\begin{aligned}
&\mathbf{A}_1^e[\mathbf{f}_0(x_1) + \mathbf{f}_1(x_1)] + \bar{\mathbf{A}}_1^e[\overline{\mathbf{f}_0(x_1)} + \overline{\mathbf{f}_1(x_1)}] = \mathbf{A}_2\mathbf{f}_2(x_1) + \bar{\mathbf{A}}_2\overline{\mathbf{f}_2(x_1)},\\
&\mathbf{B}_1^e[\mathbf{f}_0(x_1) + \mathbf{f}_1(x_1)] + \bar{\mathbf{B}}_1^e[\overline{\mathbf{f}_0(x_1)} + \overline{\mathbf{f}_1(x_1)}] = \mathbf{B}_2\mathbf{f}_2(x_1) + \bar{\mathbf{B}}_2\overline{\mathbf{f}_2(x_1)},\\
&\mathbf{A}_1^p[\mathbf{f}_0(x_1) + \mathbf{f}_1(x_1)] + \bar{\mathbf{A}}_1^p[\overline{\mathbf{f}_0(x_1)} + \overline{\mathbf{f}_1(x_1)}] = \mathbf{0},
\end{aligned} \tag{15}$$

where $\mathbf{A}_1^e$ and $\mathbf{B}_1^e$ are two $3{\times}4$ matrices, $\mathbf{A}_1^p$ and $\mathbf{B}_1^p$ are two $1{\times}4$ matrices, and they are the submatrices of $\mathbf{A}_1$ and $\mathbf{B}_1$ defined by

$$\mathbf{A}_1 = \begin{bmatrix} \mathbf{A}_1^e \\ \mathbf{A}_1^p \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} \mathbf{B}_1^e \\ \mathbf{B}_1^p \end{bmatrix}. \tag{16}$$

The dimensions of the other vectors and matrices are $\mathbf{f}_0, \mathbf{f}_1 : 4 \times 1$, $\mathbf{f}_2 : 3 \times 1$ and $\mathbf{A}_2, \mathbf{B}_2 : 3 \times 3$.

One of the important properties of the holomorphic functions is that $\overline{\mathbf{f}_1(\bar{z})}$ will be holomorphic in S_2 if $\mathbf{f}_1(z)$ is holomorphic in S_1. Similarly, $\overline{\mathbf{f}_0(\bar{z})}$ and $\overline{\mathbf{f}_2(\bar{z})}$ will be holomorphic in S_1 if $\mathbf{f}_0(z)$ and $\mathbf{f}_2(z)$ are holomorphic in S_2. With this understanding, through the relation (15) we can now introduce the following three functions which are continuous across the interface and holomorphic in the entire z-plane including the points at infinity.

$$\pm\theta_1(z) = \begin{cases} -\bar{\mathbf{A}}_1^e\overline{\mathbf{f}_0(\bar{z})} - \mathbf{A}_1^e\mathbf{f}_1(z) + \bar{\mathbf{A}}_2\overline{\mathbf{f}_2(\bar{z})}\,, & z \in S_1, \\ \mathbf{A}_1^e\mathbf{f}_0(z) + \bar{\mathbf{A}}_1^e\overline{\mathbf{f}_1(\bar{z})} - \mathbf{A}_2\mathbf{f}_2(z)\,, & z \in S_2, \end{cases} \tag{17a}$$

$$\pm\theta_2(z) = \begin{cases} -\bar{\mathbf{B}}_1^e\overline{\mathbf{f}_0(\bar{z})} - \mathbf{B}_1^e\mathbf{f}_1(z) + \bar{\mathbf{B}}_2\overline{\mathbf{f}_2(\bar{z})}\,, & z \in S_1, \\ \mathbf{B}_1^e\mathbf{f}_0(z) + \bar{\mathbf{B}}_1^e\overline{\mathbf{f}_1(\bar{z})} - \mathbf{B}_2\mathbf{f}_2(z)\,, & z \in S_2, \end{cases} \tag{17b}$$

$$\pm\theta_3(z) = \begin{cases} -\bar{\mathbf{A}}_1^p\overline{\mathbf{f}_0(\bar{z})} - \mathbf{A}_1^p\mathbf{f}_1(z)\,, & z \in S_1, \\ \mathbf{A}_1^p\mathbf{f}_0(z) + \bar{\mathbf{A}}_1^p\overline{\mathbf{f}_1(\bar{z})}\,, & z \in S_2. \end{cases} \tag{17c}$$

By Liouville's theorem we conclude that

$$\pm\theta_1(z) = \pm\theta_2(z) = \pm\theta_3(z) = \mathbf{0}. \tag{17d}$$

Combining the results of (17) and (18), we get

$$\begin{aligned} &\mathbf{f}_1(z) = -\mathbf{G}_1^{-1}\mathbf{G}_2\overline{\mathbf{f}_0(\bar{z})},\ z \in S_1, \\ &\mathbf{f}_2(z) = \mathbf{A}_2^{-1}(\mathbf{A}_1^e - \bar{\mathbf{A}}_1^e\mathbf{G}_3^{-1}\mathbf{G}_4)\mathbf{f}_0(z)\,,\ z \in S_2, \end{aligned} \tag{18a}$$

where

$$\mathbf{G}_1 = \begin{bmatrix} \mathbf{E} \\ \mathbf{A}_1^p \end{bmatrix},\ \mathbf{G}_2 = \begin{bmatrix} \bar{\mathbf{F}} \\ \bar{\mathbf{A}}_1^p \end{bmatrix},\ \mathbf{G}_3 = \begin{bmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{A}}_1^p \end{bmatrix},\ \mathbf{G}_4 = \begin{bmatrix} \mathbf{F} \\ \mathbf{A}_1^p \end{bmatrix}, \tag{18b}$$

and

$$\mathbf{E} = \mathbf{B}_1^e + i\bar{\mathbf{M}}_2\mathbf{A}_1^e,\ \mathbf{F} = \mathbf{B}_1^e - i\mathbf{M}_2\mathbf{A}_1^e. \tag{18c}$$

Note that when we derive the complex function vectors $\mathbf{f}(z)$ through the method of analytical continuation, all the functions are written with the argument $z = x_1 + \mu x_2$ without indicating the subscript of μ since the results are obtained based upon the conditions on $x_2 = 0$. Once the solution of $\mathbf{f}(z)$ is obtained from the condition of analytical continuation, a replacement of z_1, z_2, z_3 or z_4 should be made for each component function according to solution form required in the general solution (6). A translating technique based upon the above mathematical operation requirement has been introduced by [Hwu (1993)]. Thus, the explicit full field solutions of

$\mathbf{f}_1(z)$ and $\mathbf{f}_2(z)$ can be obtained by substituting $(12)_1$ into (19a) and employing the translation technique. The results are

$$\mathbf{f}_1(z^{(1)}) = \frac{1}{2\pi i}\sum_{j=1}^{4} < ln(z_\alpha^{(1)} - \bar{\hat{z}}_j^{(1)}) > \mathbf{G}_1^{-1}\,\mathbf{G}_2\mathbf{I}_j\bar{\mathbf{A}}_1^T\hat{\mathbf{p}},$$
$$\mathbf{f}_2(z^{(2)}) = \frac{1}{2\pi i}\sum_{j=1}^{4} < ln(z_\alpha^{(2)} - \hat{z}_j^{(1)}) > \mathbf{A}_2^{-1}(\mathbf{A}_1^e - \bar{\mathbf{A}}_1^e\,\mathbf{G}_3^{-1}\,\mathbf{G}_4)\mathbf{I}_j\mathbf{A}_1^T\hat{\mathbf{p}}. \quad (19)$$

Case (ii): the anisotropic material is an insulator

Same situation as case (i) is considered here except that the conductor of material 2 is now replaced by an insulator. With this replacement, the boundary conditions along the interface can be written as

$$u_i^{(1)} = u_i^{(2)},\ \phi_i^{(1)} = \phi_i^{(2)},\ i = 1,2,3,\ \phi_4^{(1)} = 0,\ \text{along the interface}\ x_2 = 0. \quad (20)$$

By a similar approach as case (i), the explicit full field solutions of $\mathbf{f}_1(z)$ and $\mathbf{f}_2(z)$ can be obtained as follows.

$$\mathbf{f}_1(z^{(1)}) = \frac{1}{2\pi i}\sum_{j=1}^{4} < ln(z_\alpha^{(1)} - \bar{\hat{z}}_j^{(1)}) > \mathbf{G}_1^{*-1}\,\mathbf{G}_2^*\mathbf{I}_j\vec{\mathbf{A}}_1^T\hat{\mathbf{p}},$$
$$\mathbf{f}_2(z^{(2)}) = \frac{1}{2\pi i}\sum_{j=1}^{4} < ln(z_\alpha^{(2)} - \hat{z}_j^{(1)}) > \mathbf{A}_2^{-1}(\mathbf{A}_1^e - \bar{\mathbf{A}}_1^e\,\mathbf{G}_3^{*-1}\,\mathbf{G}_4^*)\mathbf{I}_j\mathbf{A}_1^T\hat{\mathbf{p}}, \quad (21a)$$

where

$$\mathbf{G}_1^* = \begin{bmatrix}\mathbf{E}\\ \mathbf{B}_1^p\end{bmatrix},\ \mathbf{G}_2^* = \begin{bmatrix}\bar{\mathbf{F}}\\ \bar{\mathbf{B}}_1^p\end{bmatrix},\ \mathbf{G}_3^* = \begin{bmatrix}\bar{\mathbf{E}}\\ \bar{\mathbf{B}}_1^p\end{bmatrix},\ \mathbf{G}_4^* = \begin{bmatrix}\mathbf{F}\\ \mathbf{B}_1^p\end{bmatrix}. \quad (21b)$$

3.4 Discussions

From the above solutions we see that Green's functions of the anisotropic bimaterials and piezoelectric bimaterials have exactly the same mathematical forms, whereas those of the anisotropic/piezoelectric bimaterials are different. Since piezoelectric materials can be specialized to anisotropic materials by neglecting their piezoelectric constants, it is expected that Green's functions of anisotropic /piezoelectric bimaterials may be reduced from those of piezoelectric bimaterials through the specialization of piezoelectric tensor and dielectric permittivity tensor. From the constitutive relations shown in (2), we observe that the alternative solutions of (20) and (22) can be obtained from the Green's function of piezoelectric bimaterials by letting $e_{kij} = 0$ and $\omega_{ij}^{\varepsilon} \to \infty$ for conductors, and letting $e_{kij} = 0$ and $\omega_{ij}^{\varepsilon} \to 0$ for insulators.

4 Boundary Integral Equations

If body forces are omitted, the *boundary integral equations* for boundary value problems in anisotropic elasticity can be written as [Brebbia, Telles and Wrobel (1984)]

$$c_{ij}(\boldsymbol{\xi})u_j(\boldsymbol{\xi})+\int_\Gamma t^*_{ij}(\boldsymbol{\xi},\mathbf{x})\,u_j(\mathbf{x})d\Gamma(\mathbf{x})=\int_\Gamma u^*_{ij}(\boldsymbol{\xi},\mathbf{x})\,t_j(\mathbf{x})d\Gamma(\mathbf{x}), \tag{22}$$

in which the subscripts i and j range from 1 to 3 for anisotropic elastic materials and range from 1 to 4 for piezoelectric materials. Γ denotes the boundary of the elastic solid;$u_j(\mathbf{x})$ and $t_j(\mathbf{x})$ are the displacements and surface tractions along the boundaries; $u^*_{ij}(\boldsymbol{\xi},\mathbf{x})$ and $t^*_{ij}(\boldsymbol{\xi},\mathbf{x})$ are, respectively, the displacements and tractions in the x_j direction at point $\mathbf{x}=(x_1,x_2)$ corresponding to a unit point force acting in the x_i direction applied at point $\boldsymbol{\xi}=(\hat{x}_1,\hat{x}_2)$; $c_{ij}(\boldsymbol{\xi})$ is a coefficient dependent on the location of $\boldsymbol{\xi}$, which equals to $\delta_{ij}/2$ for a smooth boundary and $c_{ij}=\delta_{ij}$ for an internal point. The symbol δ_{ij} is Kronecker delta. In practical applications, $c_{ij}(\boldsymbol{\xi})$ can be computed by considering rigid body motion. In other words, if we let a unit rigid body movement in the direction of x_j, $u_j=1$ which will not induce any stresses and hence $t_j=0$. Substituting this condition into (23), we get

$$c_{ij}(\boldsymbol{\xi})=-\int_\Gamma t^*_{ij}(\boldsymbol{\xi},\mathbf{x})\,d\Gamma(\mathbf{x}). \tag{23}$$

The boundary integral equations given in (23) have now three unknown functions, i.e., u_j or t_j, $j=1,2,3$ if the point $\mathbf{x}$ is located on the anisotropic body, and have four unknown functions if the point $\mathbf{x}$ is located on the piezoelectric body. Similarly, equation (23) contains three equations if point $\boldsymbol{\xi}$ is located on the anisotropic body, and four equations if point $\boldsymbol{\xi}$ is located on the piezoelectric body. For the convenience of boundary element formulation, the boundary integral equations (23) are usually written in matrix form as

$$\mathbf{C}(\boldsymbol{\xi})\mathbf{u}(\boldsymbol{\xi})+\int_\Gamma \mathbf{T}^*(\boldsymbol{\xi},\mathbf{x})\mathbf{u}(\mathbf{x})d\Gamma(\mathbf{x})=\int_\Gamma \mathbf{U}^*(\boldsymbol{\xi},\mathbf{x})\mathbf{t}(\mathbf{x})d\Gamma(\mathbf{x}), \tag{24}$$

in which $\mathbf{C},\mathbf{T}^*,\mathbf{U}^*,\mathbf{u}$ and $\mathbf{t}$ are, respectively, the matrix symbols of $c_{ij},t^*_{ij},u^*_{ij},u_i$ and t_i. To make equation (25) work for the programming of boundary element codes, we need to find the fundamental solutions $t^*_{ij}(\boldsymbol{\xi},\mathbf{x})$ and $u^*_{ij}(\boldsymbol{\xi},\mathbf{x})$ which are related to the Green's functions obtained in previous section. With the aid of relation (9), we can now write down the results of $\mathbf{T}^*$ and $\mathbf{U}^*$ as

$$\mathbf{T}^* = \begin{cases} 2\,\mathrm{Re}\{[\mathbf{B}_\alpha(\mathbf{F}_{0,s}+\mathbf{F}_{1,s})]^T\}, & \text{if } \boldsymbol{\xi}\in S_1, \mathbf{x}\in S_1, \text{ or } \boldsymbol{\xi}\in S_2, \mathbf{x}\in S_2, \\ 2\,\mathrm{Re}\{[\mathbf{B}_\alpha \mathbf{F}_{2,s}]^T\}, & \text{if } \boldsymbol{\xi}\in S_1, \mathbf{x}\in S_2, \text{ or } \boldsymbol{\xi}\in S_2, \mathbf{x}\in S_1, \end{cases}$$
$$\mathbf{U}^* = \begin{cases} 2\,\mathrm{Re}\{[\mathbf{A}_\alpha(\mathbf{F}_0+\mathbf{F}_1)]^T\}, & \text{if } \hat{\mathbf{x}}\in S_1, \mathbf{x}\in S_1, \text{ or } \boldsymbol{\xi}\in S_2, \mathbf{x}\in S_2, \\ 2\,\mathrm{Re}\{[\mathbf{A}_\alpha \mathbf{F}_2]^T\}, & \text{if } \hat{\boldsymbol{\xi}}\in S_1, \boldsymbol{\xi}\in S_2, \text{ or } \boldsymbol{\xi}\in S_2, \mathbf{x}\in S_1, \end{cases} \tag{25}$$

in which the subscript $\alpha = 1$, if $\mathbf{x}\in S_1$ and $\alpha = 2$, if $\mathbf{x}\in S_2.\mathbf{F}_0, \mathbf{F}_1$and $\mathbf{F}_2$ are related to $\mathbf{f}_0(z^{(\alpha)}), \mathbf{f}_1(z^{(\alpha)})$ and $\mathbf{f}_2(z^{(\alpha)})$ obtained in Section 3 by

$$\mathbf{f}_0(z^{(\alpha)}) = \mathbf{F}_0\hat{\mathbf{p}},\ \mathbf{f}_1(z^{(\alpha)}) = \mathbf{F}_1\hat{\mathbf{p}},\ \mathbf{f}_2(z^{(\alpha)}) = \mathbf{F}_2\hat{\mathbf{p}}. \tag{26}$$

5 Boundary Element Formulation

After getting the fundamental solutions in (26), the unknowns remained in the boundary integral equations (25) are **u** and **t** over the boundary Γ. In boundary element formulation, the boundary Γ is approximated by a series of elements, and the points **x**, displacements **u** and tractions **t** on the boundary are approximated by the nodal points $\mathbf{x}_n$, nodal displacement $\mathbf{u}_n$ and nodal traction $\mathbf{t}_n$ through different interpolation functions. In this paper, we assume the same linear variation within each element for the boundary points **x**, displacements **u** and tractions **t**. Thus, the values of **x**, **u** and **t** at any point on the *m*th element can be defined in terms of their nodal values and two linear interpolation functions ϖ_1 and ϖ_2 of the dimensionless coordinate ς, such that

$$\mathbf{x} = \varpi_1\mathbf{x}_m^{(1)} + \varpi_2\mathbf{x}_m^{(2)},\ \mathbf{u} = \varpi_1\mathbf{u}_m^{(1)} + \varpi_2\mathbf{u}_m^{(2)},\ \mathbf{t} = \varpi_1\mathbf{t}_m^{(1)} + \varpi_2\mathbf{t}_m^{(2)}, \tag{27}$$

where a symbol with subscript *m* and superscript (1) or (2) denotes the value of node 1 or 2 of the *m*th element. The interpolation functions ϖ_1 and ϖ_2 are given by

$$\varpi_1 = \frac{1}{2}(1-\varsigma), \quad \varpi_2 = \frac{1}{2}(1+\varsigma), \tag{28}$$

where ς is the dimensionless coordinate defined by $\varsigma = 2s/\ell_m$ in which ℓ_m is the length of the *m*th element and *s* is the coordinate lying along the linear element and directed from the first node to the second node of element *m*.

If the boundary Γ is discretized into *M* segments with *N* nodes, substitution of (28) and (29) into (25) yields

$$\mathbf{C}(\boldsymbol{\xi})\mathbf{u}(\boldsymbol{\xi}) + \sum_{m=1}^{M} \left\{ \hat{\mathbf{Y}}_m^{(1)}(\boldsymbol{\xi})\mathbf{u}_m^{(1)} + \hat{\mathbf{Y}}_m^{(2)}(\boldsymbol{\xi})\mathbf{u}_m^{(2)} \right\} = \sum_{m=1}^{M} \left\{ \mathbf{G}_m^{(1)}(\boldsymbol{\xi})\mathbf{t}_m^{(1)} + \mathbf{G}_m^{(2)}(\boldsymbol{\xi})\mathbf{t}_m^{(2)} \right\}, \quad (29)$$

in which $\hat{\mathbf{Y}}_m^{(i)}(\boldsymbol{\xi})$ and $\mathbf{G}_m^{(i)}(\boldsymbol{\xi})$, i=1,2, are the matrices of influence coefficients defining the interaction between the point $\boldsymbol{\xi}$ and the particular node (1 or 2) on element m, and are defined as

$$\begin{aligned} \hat{\mathbf{Y}}_m^{(i)}(\boldsymbol{\xi}) &= \int_{\Gamma_m} \mathbf{T}^*(\boldsymbol{\xi}, \mathbf{x}_m^{(1)}, \mathbf{x}_m^{(2)}, \varsigma)\varpi_i(\varsigma)d\Gamma_m(\varsigma), \\ \mathbf{G}_m^{(i)}(\boldsymbol{\xi}) &= \int_{\Gamma_m} \mathbf{U}^*(\boldsymbol{\xi}, \mathbf{x}_m^{(1)}, \mathbf{x}_m^{(2)}, \varsigma)\varpi_i(\varsigma)d\Gamma_m(\varsigma), \ i = 1,2. \end{aligned} \quad (30)$$

Γ_m denotes the mth segment of the discretized boundary. To evaluate the integrals along Γ_m, $\mathbf{T}^*$ and $\mathbf{U}^*$ are expressed in terms of the dimensionless coordinate ς and the differential $d\Gamma_m(\varsigma)$ is transformed to $d\varsigma$ multiplied by the Jacobian $|J_m| = \ell_m/2$. Substituting the results of (26) into (31), $\hat{\mathbf{Y}}_m^{(i)}(\boldsymbol{\xi})$ and $\mathbf{G}_m^{(i)}(\boldsymbol{\xi})$ can be evaluated numerically by employing a numerical integration scheme such as Gaussian quadrature rule.

If the connecting elements, for example the $(m-1)$th and the mth element, are continuous at the connecting nodal points, the second node of the $(m-1)$th element will be the first node of the mth element and can be named as the nth node of the whole boundary element. We let

$$\mathbf{u}_{m-1}^{(2)} = \mathbf{u}_m^{(1)} = \mathbf{u}_n, \ \mathbf{t}_{m-1}^{(2)} = \mathbf{t}_m^{(1)} = \mathbf{t}_n, ..., \text{ etc.} \quad (31)$$

To write (25) corresponding to point $\boldsymbol{\xi}$ in discrete form, we need to add the contribution from two adjoining elements, m and m-1, into one term. Hence, we let

$$\hat{\mathbf{Y}}_{m-1}^{(2)} + \hat{\mathbf{Y}}_m^{(1)} = \hat{\mathbf{Y}}_n, \ \mathbf{G}_{m-1}^{(2)} + \mathbf{G}_m^{(1)} = \mathbf{G}_n, ..., \text{ etc.} \quad (32)$$

Equation (30) can then be rewritten as

$$\mathbf{C}(\boldsymbol{\xi})\mathbf{u}(\boldsymbol{\xi}) + \sum_{n=1}^{N} \hat{\mathbf{Y}}_n(\boldsymbol{\xi})\mathbf{u}_n = \sum_{n=1}^{N} \mathbf{G}_n(\boldsymbol{\xi})\mathbf{t}_n. \quad (33)$$

Consider $\boldsymbol{\xi}$ to be the location of node i and use $\mathbf{C}_i, \mathbf{u}_i, \hat{\mathbf{Y}}_{in}, \mathbf{G}_{in}$ to denote the values of $\mathbf{C}, \mathbf{u}, \hat{\mathbf{Y}}_n, \mathbf{G}_n$ at node i. Equation (34) can now be expressed as

$$\sum_{n=1}^{N} \mathbf{Y}_{in}\mathbf{u}_n = \sum_{n=1}^{N} \mathbf{G}_{in}\mathbf{t}_n, \; i = 1, 2, ..., N, \tag{34a}$$

in which

$$\begin{aligned} \mathbf{Y}_{in} &= \hat{\mathbf{Y}}_{in}, \text{ for } i \neq n, \\ \mathbf{Y}_{in} &= \hat{\mathbf{Y}}_{in} + \mathbf{C}_i, \text{ for } i = n. \end{aligned} \tag{34b}$$

When all the nodes are taken into consideration, equation (35a) produces a $3N \times 3N$ system of equations. By applying the boundary condition such that either u_i or t_i at each node is prescribed, the system of equations (35a) can be reordered in such a way that the final system of equations can be expressed as **Kv=p** where **K** is a fully populated matrix, **v** is a vector containing all the boundary unknowns and **p** is a vector containing all the prescribed values given on the boundary. Once (35a) has been solved, all the values of tractions and displacements on the boundary are determined. With this result, the values of stresses and displacements at any interior point can be calculated through the strain-displacement relation and the stress-strain law shown in eqns. (1) and (2).

6 Numerical Examples

To show the advantages of the present special boundary element method (SBEM), in this section several numerical examples such as orthotropic/isotropic, PZT-7A/PZT-5H and anisotropic/piezoelectric bimaterials are illustrated.

Example 1. orthotropic/isotropic bimaterials

An orthotropic/isotropic bimaterial subjected to uniform tension $\hat{\sigma} = 1\,\text{MPa}$ is considered in this example. The loading, geometry and boundary element meshes of this problem are shown in Fig. 2. The material above the interface is orthotropic whose mechanical properties are:

$$\begin{aligned} &E_{11} = 134.45\,\text{GPa}, E_{22} = E_{33} = 11.03\,\text{GPa}, \nu_{12} = \nu_{13} = 0.301, \; \nu_{23} = 0.49, \\ &G_{12} = G_{13} = 5.84\,\text{GPa}, G_{23} = 2.98\,\text{GPa}, \end{aligned}$$

and the material below the interface is isotropic whose properties are $E = 10\,\text{GPa}$ and $\nu = 0.2$. Fig. 3 shows the results of stresses along the interface. From this figure we see that the stresses calculated from the finite element software ANSYS are discontinuous across the interface, which will approach to a continuous value

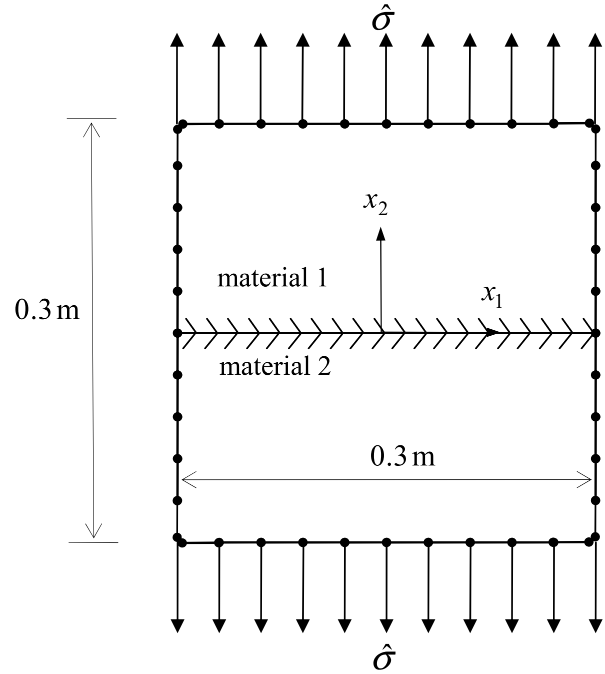
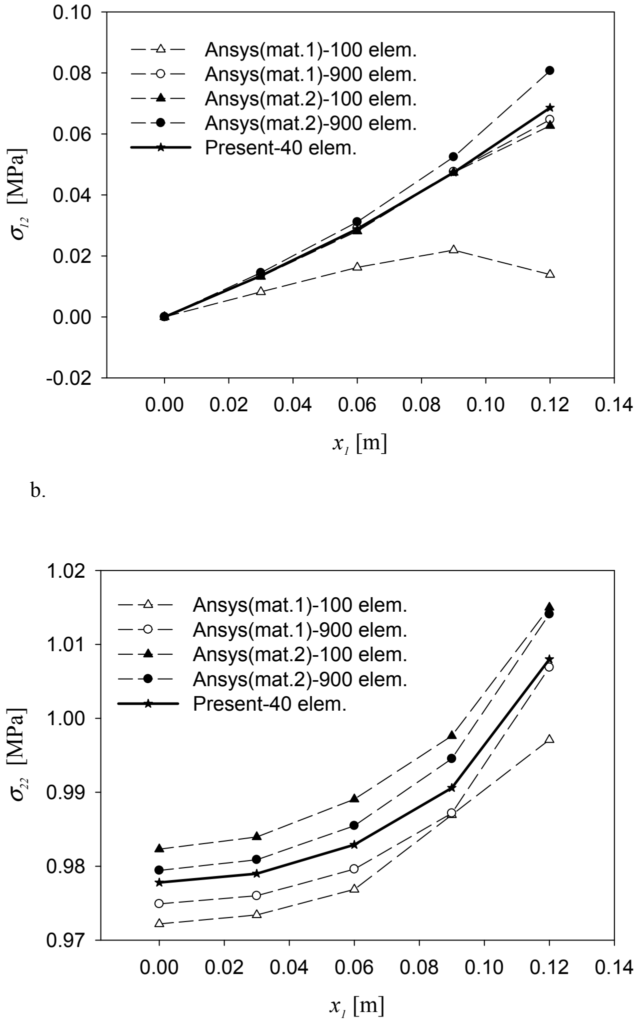


Figure 2: A mesh diagram for a bimaterial plate subjected to uniform tension $\hat{\sigma}$.

calculated by the present SBEM if finer meshes are used for ANSYS. In other words, with much less elements (40 of SBEM compared to 900 of ANSYS) the present SBEM can provide more accurate results than ANSYS for the interface problems. This result is expected since the continuity condition of SBEM has been satisfied exactly through the Green's functions obtained in Section 3, and no meshes are needed along the interface for SBEM.

Example 2. PZT-7A/PZT-5H bimaterials

In order to know the applicability of SBEM on piezoelectric materials, same problem of example 1 is reconsidered in this example by replacing the materials above and below the interface to PZT-7A and PZT-5H and imposing the electric displacement $\hat{D}_2 = -0.001\text{C / M}$ on the lower edge and $\hat{D}_2 = 0.001\text{C / M}$ on the upper edge of the plate. The material properties of PZT-7A and PZT-5H are given in Tab. 1. Fig. 4 shows the results of electric displacements along the interface. Like the results of previous example, this figure also shows that the electric displacements calculated from ANSYS are discontinuous across the interface, and will approach to a continuous value calculated by the present SBEM if finer meshes are used for a.



b.

Figure 3: Stresses along the interface of anisotropic bimaterials. (a) σ_{12}; (b) σ_{22}.

ANSYS.

Example 3. anisotropic/piezoelectric bimaterials

As shown in Section 3, the Green's functions for anisotropic/piezoelectric bimaterials are categorized into two kinds, conductors and insulators. As discussed in Section 3.4, both of these two cases can be specialized from the piezoelectric bimaterials. In order to prove this specialization numerically, in this example the material above the interface is selected to be PZT-7A, whereas the material below the interface is selected to be the one made up by PZT-5H and is called *p-elastic* whose properties is given in Tab. 1. The scaling factor k appearing in the properties of p-elastic is used to approximate the conductors or insulators. From the discussion of Section 3.4, we know that p-elastic will behave like a conductor when $k \rightarrow \infty$, and like an insulator when $k \rightarrow 0$.

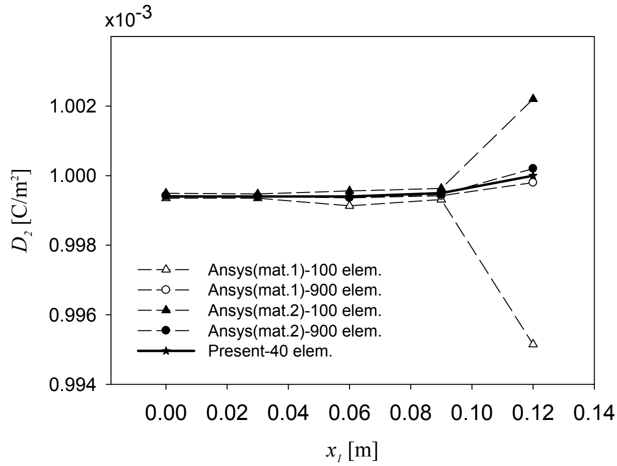


Figure 4: Electric displacement along the interface of piezoelectric bimaterials.

Consider a point force/charge $\hat{\mathbf{p}} = (1, 1, 0, 0.0001) \times 10^6$ applied at $\hat{\mathbf{x}} = (0, 1)$. The numerical results of u_4 and ϕ_4 of Green's functions along the axis $x_1 = 1$ are shown in Fig. 5. The solid line denotes the results calculated by the solutions presented in Section 3.3 for the bimaterials with conductors or insulators. The dotline with symbols are the results calculated by the solutions presented in Section 3.2 for piezoelectric bimaterials. Fig. 5(a) shows that the electric potential u_4 approximates to zero for the entire p-elastic material if its $k = 10^3$, which is exactly the boundary condition set for the conductors. On the other hand, Fig. 5(b) shows that the generalized stress function ϕ_4 approximates to zero for the entire p-elastic material if its $k = 10^{-3}$, which is exactly the boundary condition set for the insulators.

7 Conclusions

The Green's functions for any combination of anisotropic/piezoelectric bimaterials are derived in this paper. With these Green's functions, a special boundary

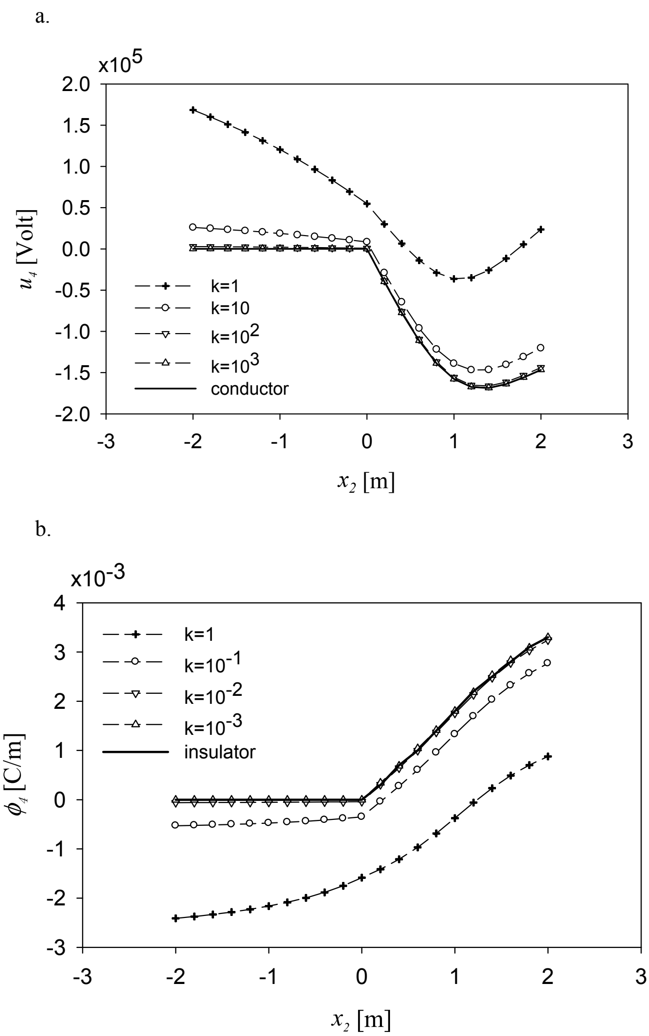


Figure 5: Electric potential and generalized stress function along the axis $x_1 = 1$ of anisotropic/piezoelectric bimaterials. (a) u_4; (b) ϕ_4.

element for the interface problems is developed. Through two simple numerical examples presented in Section 6, it has been shown that the present special boundary element is more accurate and efficient than the finite element software ANSYS. Because similar special boundary elements have also been developed for the anisotropic plates containing holes, cracks and inclusions, by employing the subregion technique it can be expected that more practical problems including multiple

Table 1: Material constants of PZT-5H, PZT-7A and p-elastic.

	PZT-7A	PZT-5H	p-elastic
C_{11}, C_{33}[GPa]	148	126	126
C_{12}, C_{23}[GPa]	74.2	53	53
C_{13}[GPa]	76.2	55	55
C_{22}[GPa]	131	117	117
C_{44}, C_{66}[GPa]	25.4	35.3	35.3
C_{55}[GPa]	55.9	35.5	35.5
e_{21}[C/m^2]	-2.1	-6.5	0
e_{22}[C/m^2]	9.5	23.3	0
e_{23}[C/m^2]	-2.1	-6.5	0
e_{16}, e_{34}[C/m^2]	9.7	17	0
ω_{11}, ω_{33}[10^{-9} C/(V m)]	8.11	15.1	15.1k
ω_{22}[10^{-9} C/(V m))	7.35	13	13k

holes, cracks, inclusions and interfaces can be analyzed accurately and efficiently by combining these special boundary elements.

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