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New Interpretation to Variational Iteration Method: Convolution Iteration Method Based on Duhamel's Principle for Dynamic System Analysis

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Abstract: Addressing the identification problem of the general Lagrange multiplier in the He's variational iteration method, this paper proposes a new kind of method based on Duhamel's principle for the dynamic system response analysis. In this method, we have constructed an analytical iteration formula in terms of the convolution for the residual error at the nth iteration, and have given a new interpretation to He's variational iteration method. The analysis illustrates that the computational result of this method is equal to that of He's variational iteration method on the assumption of considering the impulse response of the linear parts, or equal to that of Adomian's method on the assumption of considering the only the impulse response of the highest-ordered differential operator, respectively. However, new convolution iteration method doesn't need to solve the complicated Euler-Poisson variation equation. Some test examples for showing the application procedure of the convolution iteration method are provided.

Keywords: Adomian decomposition method, convolution, integral transformation, nonlinear differential equation, variational iteration method.

1 Introduction

Finding the analytical solution of the differential equation within a predefined accuracy by successive iterations has been being an important problem, and it makes a lot of mathematicians and engineers pay great attentions. In the past decades, several methods, e.g. the Adomian decomposition method(ADM) [Adomain (1994); Drof and Bishop (1998); Inokuti et al. (1978); and Wazwaz (2001)], and He's

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variational iteration method[He (1997-1999),(2007)], had been proposed and they had been applied to solve the ordinary differential equations(ODE) and the partial differential equations(PDE) in the vibration, thermodynamics, and fluid mechanics[Abdou and Soliman (2005); Inokuti et al. (1978); Lu (2007); Lai, Chen, and Hsu (2008);Tatari and Dehghan (2007); and Yıldırım (2008)].

The main problem existed in the ADM are that the analytical computation the high-order correction terms and the high-order Adomian's polynomials. When the order n of the Adomian's polynomials is great-equal than 3, their computations are with very difficult; especially, the computation explosive will take place for the more complicated source term function or the non-linear term. The other problem of the ADM is its slow convergent velocity. For simplifying the computation of the ADM, Tien and Chen(2007) proposed the ADM based on Legendre polynomials, and Hosseini(2006) put forward the ADM with Chebyshev polynomials. The above-mentioned two kinds of method can improve the global approaching precision when the solution component resulted from the source term is solved by the expand method of the function. Azreg-Aïnou (2009) expressed Adomian polynomials(AP's) in terms of new objects called reduced polynomials(RP's), so that RP's is more easily understand than AP's and the computation formulas of the ADM become dramatically simple and compact; but the convergent property of ADM is still needed to be improved.

In order to improve the convergent property of ADM, J.H.He (1997) proposed the variational iteration method (VIM). This method introduced the concept of the restricted variation, and converted the n-order iterating-correcting problem to the problem of a solution of the Lagrange multiplier and 1-order integral iteration. Because it considers the contribution of the overall linear components $L[u]+R[u]$in the nonlinear differential equation, it can increase the convergent velocity of the iteration. Since He's variational iteration method has been proposed, some of the progress has been made [see He (2007), Yıldırım (2008)]. At present, He's variational iteration method has become a kind of very effective method to solve the ODE and the PDE of the non-linear system [see He (2007), Inokuti et al. (1978); Lu (2007); Tatari and Dehghan (2007); Wu and He (2008); and Yıldırım (2008)].

Variation method is the mathematical fundamental of the domain decomposition methods, e.g. finite element method. It has been applied to solve the partial differential equations and a variety of boundary value problems [Martin (2008); Han (2007); Tsail (2010); and Vodička, Mantič, and París (2007)]. Generally speaking, to solve variational equation is not a easy task.

Addressing the determination of the Lagrange multiplier $\lambda(\tau)$ and the improvement of the iterating computation method in the He's VIM, this paper obtained the same iteration formula as the He's VIM from the Duhamel's principle for the system response analysis, thus a new method to find the $\lambda(\tau)$ and a new interpretation to He's variational iteration method were proposed, and the computation procedure was greatly simplified.

The rest of the paper is organized as follows. In section 2, based on the Duhamel's principle for the system response analysis, the convolution iteration method is investigated, and we point out the Lagrange multiplier in He's VIM is similar to the weighted function or impulse response in the system response analysis, so that we can convert the solution of the ODE and the PDE of the non-linear system to a convolution iteration problem. Moreover, we studied the relationship between He's variational method and the convolution iteration method proposed by the author so as to reveal the physical meaning of the Lagrange multiplier and He's VIM. Section 3 gives some examples to illustrate how to apply the proposed method to solve the nonlinear ODE and PDE. In section 4, some significant conclusions are drawn.

2 Convolution iteration method

Convolution is an important integral operator in the differential equation [Edwards and Penny(2004)], signal and image processing [Gonzalez and Woods (2008)], and control theory [Drof and Bishop (1998)]. In the differential equation field, the convolution is mainly used to solve the linear system's response under the external forced excitation; and in the signal and image processing, the convolution is applied to the filter and the localization processing. Here, by means of the iteration, we introduce the convolution to solve the non-linear ODE and PDE.

2.1 Convolution iteration method based on Duhamel's principle

Considering the following operator equation of the non-linear system

$$H(u)+R(u)+N(u)=g(\tau), \tag{1}$$

where H, R and N are the highest order differential operator, the linear operator, and the non-linear operator, respectively, $g(\tau)$ is the known continuous function, and u is the solution to (1). Obviously, at the beginning of the n-th iteration, $H(u_n)+R(u_n)+N(u_n)\neq g(\tau)$, liked to the restricted variation method, a restricted correction is introduced to the (1) yields

$$H(u_n+c_n)+R(u_n+c_n)+N(u_n)=g(\tau), \tag{2}$$

where, c_n is the correction of $u$at the n-th iteration. Defining the residual error of u at the n-th iteration $r_n \stackrel{\Delta}{=} g(\tau)-H(u_n)-R(u_n)-N(u_n)$, and substituting it into (2), we can express r_n and c_n using the following formula

$$H(c_n)+R(c_n)=r_n, \tag{3}$$

at the n-th iteration respectively. According to the Duhamel's principle for the system response analysis, we can obtain the computational expression of c_n as follows

$$c_n = \int_0^t h(t-\tau)r_n(\tau)d\tau = \int_0^t h(t-\tau)(g(\tau) - H(u_n) - R(u_n) - N(u_n))d\tau, \quad (4)$$

where $h(\tau)$ is the weight function or impulse response function of the linear system $H(u) + R(u) = \delta(\tau)$, and $\delta(\tau)$ is the Dirac delta function. Therefore, the $(n+1)$-th iteration solution $u_{n+1}$of the non-linear system can be written as follows

$$u_{n+1} = u_n + \int_0^t h(t-\tau)(g(\tau) - H(u_n) - R(u_n) - N(u_n))d\tau. \quad (5)$$

Because the computation of the correction is actually a convolution of the residual error, we call (4) convolution iteration method. It converts the Lagrange's multiplier's computation in He's variational iteration method to solve the impulse response function of the linear system, and the latter is easier than the former. According to the definition of Laplace's transform, $h(\tau) = L^{-1}[1/(H(s) + R(s))])$, and this work can be easily completed by looking up Laplace's transform table.

Compared with He's variational iteration method, although the convolution iteration method expressed by (5) is equivalent from the viewpoint of the computational result, it gives the explicit physical meaning for the correction and the weighted function. This provides a new kind of the method for solving the non-linear differential equation.

2.2 Convolution iteration method derived from He's iteration method

For the operator equation of the non-linear system described by (1), we apply the He's variational iteration method to (1) yields

$$u_{n+1} = u_n + \int_0^t \lambda(\tau)\{g - Hu_n - Ru_n - N\tilde{u}_n\}d\tau, \quad (6)$$

where $\lambda(\tau)$ is a general Lagrange multiplier [Abdou and Soliman (2005); and He (1997)], and $\tilde{u}_n$ is considered as a restricted variation to satisfy the condition$\delta\tilde{u}_n =$ 0. Because the optimal correction at the n-th iteration meets the variation condition $\delta u_{n+1} = 0$, and to substitute it into (6) we can obtain

$$\delta u_n - \int_0^t \lambda(\tau)\delta((H+R)u_n)d\tau = 0. \quad (7)$$

From the Euler-Poisson variation equation and through a series of the variation calculation to (7), we can write the following stationary conditions:

$$\begin{cases} \delta y_n: & (H_1+R_1)\lambda(\tau)=0, \\ \delta y_n^{i+1}: & \lambda^{(i)}(\tau)|_{\tau=t}=0, 0\leq i<n-1, \\ \delta y_n: & 1-(-1)^n\lambda^{(n-1)}(\tau)|_{\tau=t}=0, \end{cases} \tag{8}$$

where $H=D^n, H_1=(-1)^nD^n, R=\sum_{i=0}^{n-1}a_iD^i$, $R_1=\sum_{i=0}^{n-1}a_i(-1)^iD^i$, $D=\frac{\mathrm{d}}{\mathrm{d}\tau}$. Let $t-\tau=v$, then $\tau=t\Rightarrow v=0$; and $\lambda(\tau)\Rightarrow h(v)$, we substitute these results into (8) and change the self variable v to τ and then we can get the following equation:

$$\begin{cases} (H+R)h(\tau)=0, \\ h^{(i)}(\tau)|_{\tau=0}=0, 0\leq i<n-1, \\ h^{(n-1)}(\tau)|_{\tau=0}=1. \end{cases} \tag{9}$$

Applying Laplace's transform to (9) and considering $L(h^{(n)}(\tau))=s^nH(s)-s^{n-1}h(0)-\ldots-sh^{(n-2)}(0)-h^{(n-1)}(0)$ yield

$$(s^n+R(s))H(s)=1. \tag{10}$$

Obviously seen from (10), $H(s)$ is the Laplace transform of unit impulse response with the transfer function $(s^n+R(s))^{-1}$, it's inverse Laplace transform $h(\tau)$ is the unit impulse response of the linear system without containing nonlinear term.

In this way, we can convert to solve $\lambda(\tau)$ using Euler-Poisson equation into a problem to determine unit impulse response $h(\tau)$, and the latter can be obtained by the simple reading up Laplace transforms table. Thus, we can get

$$u_{n+1}=u_n+\int_0^t h(t-\tau)\{g-Hu_n-Ru_n-Nu_n\}d\tau. \tag{11}$$

In (11), because we don't need to solve the Lagrange multiplier, the restriction $N\tilde{u}_n$to nonlinear item Nu_n is not needed. Obviously, (11) is the expression of the convolution iteration method.

If we derive the iteration formula (11) only to consider the highest term $H[u]$for solving $h(\tau)$, we have

$$h(\tau)=\frac{\tau^{n-1}}{(n-1)!}, \tag{12}$$

and

$$u_{n+1}=u_n+\int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!}\{g-Hu_n-Ru_n-Nu_n\}d\tau. \tag{13}$$

After a series of the operation of the integration by parts for (13), we can obtain

$$\int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} g d\tau = H^{-1} g,$$

$$\int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} [R(u_n)+N(u_n)] d\tau = H^{-1}R(u_n)+H^{-1}N(u_n),$$

$$\int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} H(u_n) d\tau = u_n - u_n(0) - tu_n'(0) - \cdots - t^{n-1}u_n^{(n-1)}(0) = u_n - \phi_n(t).$$

Note that $u_n(0)=u_0(0),\cdots,u_n^{(n-1)}(0)=u_0^{(n-1)}(0)$; so $u_n^{(n-1)}(0)=u_0^{(n-1)}(0)$; $\phi_n(t)=\phi_0(t)=\phi(t)$. Substituting these results into (13) yields

$$u_{n+1} = \phi(t)+H^{-1}g - H^{-1}Ru_n - H^{-1}Nu_n, \tag{14}$$

where $H^{-1}=\int_0^t\cdots\int_0^t(\cdot)d\tau$ is an inverse of operator H. Let (14) subtract $u_n = \phi(t)+H^{-1}g-H^{-1}Ru_{n-1}-H^{-1}Nu_{n-1}$ results in

$$u_{n+1}-u_n = -H^{-1}R(u_n-u_{n-1}) - H^{-1}(Nu_n - Nu_{n-1}). \tag{15}$$

2.3 Revised Adomian Method derived from Convolution iteration method

According to the Adomian decomposition method, we can express the $u$as

$$u = u_0+\bar{u}_1+\bar{u}_2+\ldots = u_0+\sum_{i=1}^{+\infty}\bar{u}_i, \tag{16}$$

where u_0 is the initial solution to consider the contribution of the initial condition, terminal condition and the source function,$u_0=\phi(\tau)+\int_0^t h(t-\tau)g(\tau)d\tau$. Let $n=0$ and substituting the expression of the u_0 into (11) yields $\bar{u}_1 = u_1-u_0 = \int_0^t h(t-\tau)[g(\tau)-H(u_0)-R(u_0)-N(u_0)]d\tau$, and note $H(u_0)-R(u_0)=g(\tau)$, so we have

$$\bar{u}_1 = -\int_0^t h(t-\tau)N(u_0)d\tau, \tag{17}$$

Defining $\bar{u}_0=0$ and $u_n=u_{n-1}+\bar{u}_n$, by the same manipulation, we can obtain the correction $\bar{u}_2$ and $\bar{u}_n$ as follows

$$\bar{u}_2 = \bar{u}_1 - \int_0^t h(t-\tau)(H(\bar{u}_1)+R(\bar{u}_1)+A_1)d\tau, \tag{18}$$

and

$$\bar{u}_{n+1} = \bar{u}_n - \int_0^t h(t-\tau)(H(\bar{u}_n)+R(\bar{u}_n)+A_n)d\tau, \tag{19}$$

where the expressions of the Adomain polynomials A_n are as follows

$$A_0 = N(u_0),$$

$$A_1 = \bar{u}_1 \frac{d}{du_0} N(u_0),$$

$$A_2 = \bar{u}_2 \frac{d}{du_1} N(u_1),$$

$$\dots$$

$$A_n = \bar{u}_n \frac{d}{du_{n-1}} N(u_{n-1}).$$

Thus, we have obtained the expression of the revised Adomian's decomposition method in terms of the convolution. Note $\bar{u}_i$ in the above-equations are not same as the one of Adomian's decomposition method, so it results in the computational expressions of the Adomain's polynomials A_n different to ADM. Compared the revised Adomian polynomials with the traditional Adomian's decomposition method, it has simplified the computation of the Adomian's polynomials and the correction terms.

2.4 Relationship among of three kinds of the iteration methods

Through the above discussions, we can deduce the relationship among of the convolution iteration method, the He's variational iteration method and the Adomian's decomposition method as follows:

1) He's variational iteration method is same as the convolution iteration method from the view point of the computational result, but the latter simplifies the computation of the weight function and gives the more significant physical meaning.

2) If the weight function is only determined by the highest differential term, there are the same computational results and the convergent speed in three kinds of the iteration methods.

3 Test examples

Here, we select some examples used in [see He (1999, 2007); and Tatari and Dehghan(2007)] to illustrate the effectiveness of the convolution iteration method.

Example 1. Consider $\frac{\partial}{\partial t}u(x,t) + Ru(x,t) + Nu(x,t) = 0,\ u(x,0) = f(x)$, Abassy et al. constructed an iteration formula as

$$u_{n+1} = u_n - \int_0^t \{R(u_n - u_{n-1}) + G_n - G_{n-1}\} d\tau, \tag{20}$$

where $G_n(x,t)+O(t^{n+1})=Nu_n(x,t)$, $u_{-1}=0, u_0=f(x)$. Using the convolution iteration method, we can give the proof of (20).

According to (15), obviously $h(\tau)=1$, so we can derive

$$u_{n+1}-u_n=-\int_0^t \{R(u_n-u_{n-1})-(Nu_n-Nu_{n-1})\}d\tau,$$

it is exactly (20).

Example 2. He (2007) summarized some of useful iteration formulas to direct to the common non-linear differential equations. Where, we list the following convolution iteration formula as

$$\begin{cases} u'+f(u,u')=0 \\ h(t)=1 \\ u_{n+1}(t)=u_{n+1}(t)-\int_0^t \{u_n'+f(u_n,u_n')\}d\tau \end{cases}, \tag{21}$$

$$\begin{cases} u'+\alpha u+f(u,u')=0 \\ h(t)=e^{-\alpha t} \\ u_{n+1}(t)=u_{n+1}(t)-\int_0^t e^{-\alpha(t-\tau)}\{u_n'+\alpha u_n+f(u_n,u_n')\}d\tau \end{cases}, \tag{22}$$

$$\begin{cases} u''+f(u,u',u'')=0 \\ h(t)=t \\ u_{n+1}(t)=u_{n+1}(t)+\int_0^t (t-\tau)\{u_n''+f(u_n,u_n',u_n'')\}d\tau \end{cases}, \tag{23}$$

$$\begin{cases} u''+\omega^2 u+f(u,u',u'')=0 \\ h(t)=\frac{\sin\omega t}{\omega} \\ u_{n+1}(t)=u_{n+1}(t)-\int_0^t \frac{\sin\omega(t-\tau)}{\omega}\{u_n''+\omega^2 u_n+f(u_n,u_n',u_n'')\}d\tau \end{cases}, \tag{24}$$

$$\begin{cases} u''-\alpha^2 u+f(u,u',u'')=0 \\ h(t)=\frac{1}{2\alpha}(e^{\alpha t}-e^{-\alpha t}) \\ u_{n+1}(t)=u_{n+1}(t)-\int_0^t \frac{[e^{\alpha(t-\tau)}-e^{-\alpha(t-\tau)}]}{2\alpha}\{u_n''-\alpha^2 u_n+f(u_n,u_n',u_n'')\}d\tau \end{cases}, \tag{25}$$

$$\begin{cases} u^{(n)}+f(u,u',u'',\cdots,u^{(n-1)})=0 \\ h(t)=\frac{1}{(n-1)!}t^{n-1} \\ u_{n+1}(t)=u_{n+1}(t)-\int_0^t \frac{1}{(n-1)!}(t-\tau)^{n-1}\ \{u^{(n)}+f(u_n,u_n',u_n'',\cdots,u_n^{(n-1)})\}d\tau \end{cases}. \tag{26}$$

Seen easily, the above results are same as He's results.

Example 3. Consider the telegraph equation [see Tatari and Dehghan(2007)]

$$\begin{aligned} &u_{tt} + 2du_t - u_{xx} = 0 \quad (x,t) \in \mathrm{R} \times (0,\infty), \\ &u = g,\ u_t = h \quad (x,t) \in \mathrm{R} \times \{t = 0\}, \end{aligned} \tag{27}$$

for $d > 0$, the term $2du_t$ representing a physical damping of wave propagation. Using the convolution iteration method, we can write $(s^2 + 2ds)H(s) = 1$, and $h(\tau) = \frac{1}{2d}(1 - e^{-2d\tau})$, so

$$u_{n+1} = u_n + \int_0^t h(t-\tau)\{0 - \frac{\partial^2}{\partial t^2}u_n - 2d\frac{\partial}{\partial t}u_n + \frac{\partial^2}{\partial x^2}u_n\}d\tau. \tag{28}$$

Substituting the expression of $h(\tau)$ into (28) yields

$$u_{n+1} = u_n + \int_0^t \frac{1}{2d}[1 - e^{-2d(t-\tau)}]\{\frac{\partial^2}{\partial x^2}u_n - \frac{\partial^2}{\partial t^2}u_n - 2d\frac{\partial}{\partial t}u_n\}d\tau. \tag{29}$$

Example 4. Consider the Duffing equation with non-linearity of the fifth order [see He (2007)]

$$\begin{aligned} &u'' + u + \varepsilon u^5 = 0 \\ &u(0) = A, \quad u'(0) = 0. \end{aligned} \tag{30}$$

We apply the convolution iteration method for (30), then we can write the Laplace transformation equation of the weight function as $(s^2 + 1)H(s) = 1$, and then we can solve the weight function $h(\tau) = \sin\tau$. According to (5), the iteration formula can be written down as follows

$$\begin{aligned} u_{n+1} &= u_n + \int_0^t \sin(t-\tau)\{0 - u_n'' - u_n - \varepsilon u_n^5\}d\tau \\ &= u_n - \int_0^t \sin(t-\tau)\{u_n'' + u_n + \varepsilon u_n^5\}d\tau. \end{aligned} \tag{31}$$

Obviously, equation (31) is same as the result derived by example 1 in He's paper[He (2007)], so the follow-up computational procedure is no longer discussed.

Example 5. Consider a non-linear wave equation [He (2007)]

$$u_{tt} - c^2u_{xx} + f(u) = 0, \tag{32}$$

with initial conditions

$$\begin{aligned} u(x,0) &= F(x), \\ u_t(x,0) &= G(x). \end{aligned} \tag{33}$$

In case $f=0$ we have the D'Alembert exact solution to be expressed as

$$u(x,t)\big|_{f=0} = \frac{F(x+ct)+F(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\tau)\, d\tau. \tag{34}$$

Obviously, while the n-th iteration, the differential equation between the correction and the residual error is as follows

$$c_n'' = r_n = c^2 u_{nxx} - f(u_n) - u_{ntt}. \tag{35}$$

So, we have $h(\tau)=\tau$. Let $u_0(x,t) = u(x,0)+tu_t(x,0) = F(x)+tG(x)$, from (11), we can obtain

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (t-\tau)(c^2 u_{nxx} - f(u_n) - u_{n\tau\tau})d\tau. \tag{36}$$

To apply the integration by parts to $-\int_0^t (t-\tau)u_{n\tau\tau}d\tau$ in (36) yields

$$u_{n+1}(x,t) = u_n(x,0) + tu_{nt}(x,0) + \int_0^t (t-\tau)(c^2 u_{nxx} - f(u_n))d\tau, \tag{37}$$

where $u_n(x,0) = u_0(x,0) = F(x), u_{nt}(x,0) = u_{0t}(x,0) = G(x)$. Note that (34) can also meet the initial condition (33) while $t=0$, so we can select

$$u_0(x,t) = \frac{F(x+ct)+F(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\tau)\, d\tau,$$

substituting it into (36) can get the same result as (37), i.e.

$$u_{n+1}(x,t) = F(x) + tG(x) + \int_0^t (t-\tau)(c^2 u_{nxx} - f(u_n))d\tau. \tag{38}$$

If $f(u)=0$, by the iterating manipulation for (38), we can obtain

$$u_{n+1}(x,t) = F(x) + \frac{c^2t^2}{2}F''(x) + \cdots + \frac{c^{2n}t^{2n}}{(2n)!}F^{(2n)}(x) + tG(x) + \frac{c^2t^3}{2}G''(x) + \cdots + t\frac{c^{2n}t^{2n}}{(2n)!}G^{(2n)}(x), \tag{39}$$

$$u(x,t) = \lim_{n\to\infty} u_{n+1}(x,t) = \frac{F(x+ct)+F(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\tau)d\tau. \tag{40}$$

Obviously, this is exactly the exact solution to be given by (34). For non-homogeneous case of (32), $f(u) = f(x,t)$, we can have

$$u_{n+1}(x,t) = F(x) + tG(x) - \int_0^t (t-\tau)f(x,\tau)d\tau + \int_0^t (t-\tau)c^2 u_{nxx}d\tau. \tag{41}$$

Note that

$$\int_0^t (t-\tau)\int_0^\tau (\tau-\tau_1) f_{xx}(x,\tau_1)d\tau_1 d\tau = \int_0^t \frac{c^2}{3!}(t-\tau)^3 f_{xx}(x,\tau)d\tau,$$

$$\ldots$$

$$\int_0^t (t-\tau)\int_0^\tau \frac{1}{(2n-1)!}(\tau-\tau_1)^{2n-1} f_x^{(2(n-1))}(x,\tau_1)d\tau_1 d\tau = \int_0^t \frac{c^2}{(2n+1)!}(t-\tau)^{2n+1} f_x^{(2(n-1))}(x,\tau_1)d\tau_1 d\tau. \tag{42}$$

After a similar the operation of the integration to (39) for (41) and substituting (42) into (41), we can obtain the exact solution of the non-homogenous equation as follows

$$u(x,t) = \lim_{n\to\infty} u_{n+1}(x,t) = \frac{F(x+ct)+F(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\tau)d\tau - \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)d\xi d\tau. \tag{43}$$

Equation 43 is exactly the exact solution of the non-homogenous wave equation.

4 Conclusions

Based on the discussions of the iteration solution of the non-linear differential equation and the Duhamel's principle for the system dynamic response analysis, this paper has constructed a new kind of the method to solve the Lagrange multiplier in He's variational iteration method and provided a new interpretation for He's variational iteration method. The theoretical analysis in the paper has revealed the relationship among of the convolution iteration method, J.H. He's variation method and Adomian's decomposition method. Some test examples illustrate the effectiveness of the method proposed in the paper.

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References

Abdou, M.A.; Soliman, A.A. (2005): New application of variational iteration method. *Physica D* 211(1-2), pp. 1–8.

Adomain,G.A. (1994): *Solving Frontier Problems of Physics: The Decomposition Method*. Kluwer Academic Publishers, Boston, MA.

Atluri, S. N. (2004): *The Meshless Local Petrov-Galerkin (MLPG) Method*. Tech. Science Press.

Atluri, S. N.; Han, Z.; Shen, S. (2003): Meshless Local Patrov-Galerkin (MLPG) approaches for weaklysingular traction & displacement boundary integral equations. *CMES: Computer Modeling in Engineering & Sciences*, vol. 4, no. 5, pp. 507–517.

Atluri, S. N.; Zhu, T. (1998): A new meshless local Petrov-Galerkin (MLPG) approach in computational mechanics. *Comput. Mech.*, vol. 22, pp. 117–127.

Azreg-Aïnou, M. (2009): A Developed New Algorithm for Evaluating Adomian Polynomials. *CMES: Computer Modeling in Engineering & Sciences*, vol.42, no.1, pp.1–18.

Drof, R.C.; Bishop, R.H.(1998): *Modern Control Systems*, Addison Wesley Longman Inc., Reading, Mass.

Edwards, C.H.; Penny,D.E. (2004): *Differential Equations and Boundary Value Problems*. Pearson Education Inc. and Tsinghua University Press, Beijing.

Gonzalez, R.C.; Woods, R.E. (2008): *Digital image processing*. Third Edition, Pearson Prentice Hall Pearson Education Inc., pp. 712–724

Han, Z.D.; Aluri, S.N. (2007): A Systematic Approach for the Development of Weakly-Singular BIEs. *CMES: Computer Modeling in Engineering & Sciences*, vol.21, no.1, pp.41–52.

He, J.H. (1997): Variational iteration method for delay differential equations. *Comm. Non-Linear. Sci. Numer. Simulation*, no2, pp. 25–29

He, J.H. (1997): A new approach to nonlinear partial differential equations, *Comm. Non-Linear. Sci. Numer. Simulation*, no2, pp 230–235

He, J.H. (1998): Approximate solution of nonlinear differential equation with convolution product nonlinearities. *Comput. Methods Appl. Mech. Eng.*, vol.167, pp. 69–73

He, J. H. (1999): Variational iteration method — a kind of non-linear analytical technique: some examples. *International Journal of Non-Linear Mechanics,* vol.34, pp 699–708

He, J.H. (2007): Variational iteration method—some recent results and new interpretations. *Journal of Computational and applied mathematics*, vol. 207, pp 3–17.

He, J.H.; Wu, X.H. (2007): Variational iteration method: New development and applications. *Computers and Mathematics with Applications*, vol. 54, pp 881–894.

Hosseini, M. M. (2006): Adomian decomposition method with Chebyshev polynomials. *Applied Mathematics and Computation*, vol. 175, pp 1685–1693.

Inokuti, M. et al. (1978): General use of the Lagrange multiplier in non-linear mathematical physics, in: S. Nemat-Nasser(Ed.). *Variation Method in the Mechanics of Solids*, Pergamon Press, Oxford, pp. 156–162.

Lai, H.–Y.; Chen, C. K.; Hsu, J. –C. (2008): Free Vibration of Non-Uniform Euler-Bernoulli Beams by the Adomain Modified Decomposition Method. *CMES: Computer Modeling in Engineering & Sciences*, vol.34, no.1, pp.87–116.

Lu, J. F. (2007) Variational iteration method for solving a nonlinear system of second-order boundary value problems. *Computers and Mathematics with Applications*, vol.54, pp 1133–1138.

Martin, R.; Komatitsch, D.; Gedney, S. D. (2008): A Variational Formulation of a Stabilized Unsplit Convolutional Perfectly Matched Layer for The Isotropic or Anisotropic Seismic Wave Equation. *CMES: Computer Modeling in Engineering & Sciences*, vol.37, no.3, pp.274-304.

Tatari, M.; Dehghan, M. (2007): On the convergence of He's variational iteration method. *Journal of Computational and applied mathematics,* vol. 207, pp 121–128.

Tien, W.C.; Chen, C.K. (2007): Adomain decomposition method by Legendre polynomials. *Chaos, Solitons & Fractals,* doi:10.1016/j.chaos.2007.06.066.

Tsail, C. C.; Liu, C. S.; Yeih, W. C. (2010): Fictitious Time Integration Method of Fundamental Solutions with Chebyshev Polynomials for Solving Poisson-type Nonlinear PDEs. *CMES: Computer Modeling in Engineering & Sciences*, vol.56, no.2, pp.131-151.

Vodička, R.; Mantič, V.; París, F. (2007): Symmetric Variation Formulation of BIE for Domain Decomposition Problem in Elasticity – An SGBEM Approach for Nonconforming Discretizations of Curved Interfaces. *CMES: Computer Modeling in Engineering & Sciences*, vol.17, no.3, pp.173–204.

Wazwaz, A.M.; El-Sayed, S.M. (2001): A new modification of the Adomain decomposition method for linear and nonlinear operators. *Appl. Math. Comput.,* vol. 122, pp. 393–405.

Wu, X.H.; He, J.H. (2008): EXP-function method and its application to nonlinear equations. *Chaos, Solitons and Fractals,* vol. 38, pp 903–910.

Yıldırım, A. (2008): Applying He's Variational Iteration Method for Solving Differential-Difference Equation. *Mathematical Problems in Engineering*, doi:10.-1155/2008/869614.