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Error Bounds for *Discrete Geometric Approach*

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Abstract: Electromagnetic problems spatially discretized by the so called Discrete Geometric Approach are considered, where Discrete Counterparts of Constitutive Relations are discretized within an Energetic Approach. Pairs of oriented dual grids are considered in which the primal grid is composed of (oblique) parallelepipeds, (oblique) triangular prisms and tetrahedra and the dual grid is obtained according to the barycentric subdivision. The focus of the work is the evaluation of the constants bounding the approximation error of the electromagnetic field; the novelty is that such constants will be expressed in terms of the geometrical details of oriented dual grids. A numerical analysis will confirm the theory.

Keywords: Cell Method; Finite Integration Technique; Discrete Counterparts of Constitutive Equations; Convergence Analysis; Error Bounds

1 Introduction

The fundamental geometric structure on which electromagnetism is based, allows to formulate Maxwell's laws in a discrete manner with respect to a pair of oriented and interconnected grids, one dual to the other, leading to the so-called Discrete Geometric Approach (DGA) for computational electromagnetics. This idea has a solid physical and mathematical foundation, reflected in the scientific work of E. Tonti with the Cell Method [Tonti (2002); Tonti (1998); Tonti (2001); Heshmatzadeh and Bridges (2007); Cosmi (2001); Ferretti (2003); Ferretti (2004b); Ferretti (2004a); Ferretti (2005)],, the work of T. Weiland regarding the Finite Integration Technique [Weiland (1977); Weiland (1985)] and the work of A. Bossavit with the understanding of the geometric properties of the Finite Element Method Bossavit (1998).

Integrals – like circulations or fluxes – of the electromagnetic field quantities, referred to as Degrees of Freedom (DoFs), with respect to nodes, edges, faces and volumes of such a pair of oriented dual grids, are the working variables in DGA. The physical laws of electromagnetism can be straightforwardly written as *exact*

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Balance Equations in terms of the DoFs. This step, in the discretization process of Maxwell's laws, is quite clear and widely accepted within the electromagnetics community and it is a common *modus operandi* of other physical theories.

On the other hand, in recent years, most of the research in DGA has been focussed on the computation of the Discrete Counterparts of the Constitutive Equations; such discrete counterparts of the constitutive equations are *approximate* equations transforming the DoFs associated with geometric elements of one grid into the corresponding DoFs associated with geometric elements of the other grid. Discrete Counterparts of the Costitutive Equations have to guarantee the stability and consistency of the overall discrete formulated electromagnetic problem consisting of the pairing of the *exact* balance equations together with the *approximate* Discrete Counterparts of the Constitutive Equations.

It has been shown [Codecasa, Minerva, and Politi (2004); Codecasa and Trevisan (2006); Codecasa, Specogna, and Trevisan (2007); Codecasa and Trevisan (2007)] that specifically developed piece-wise uniform vector base functions can be profitably used within energetic functionals, yielding Discrete Counterparts of the Constitutive Equations ensuring stability and consistency of discrete equations for any pair of oriented polyhedral dual grids. This methodology has been referred to as *Energetic Approach*.

A further step has been recently performed, by demonstrating in [Codecasa and Trevisan (2010)] the *convergence* of an electromagnetic problem spatially discretized by DGA when the Discrete Counterparts of the Constitutive Equations are obtained by means of the Energetic Approach with respect to a pair of oriented dual grids.

Such a novel result provides to the DGA a sound mathematical background and constitutes the starting point of this paper. However the constants, derived in [Codecasa and Trevisan (2010)], bounding the approximation error of the electromagnetic field have simply a theoretical relevance, without any indication on their numerical extent. This is unlike what is provided by other discretization methods, in particular by the Finite Element Method (FEM), in which constants bounding the approximation error can be derived in terms of the geometric details of the meshes.

In this work, the significant case is considered in which the primal grid consists of (oblique) parallelepipeds, (oblique) triangular prisms and tetrahedra and the dual grid is obtained from the barycentric subdivision of the primal grid [Bossavit (1998)]. The novelty content of this work lies in the computation of the analytical values of the constants bounding the approximation error of the electromagnetic field by expressing them in terms of the geometrical properties of the pairs of oriented dual grids. A simple numerical example confirms the theoretical analysis.

The rest of this paper is organized as follows. In Section 2, the spatial discretization of an electromagnetic problem by the DGA is discussed briefly. The Energetic Approach for discretizing the Constitutive Equations is recalled in Section 3. The main convergence result of the discretized equations is reported in Section 4. Step-wise uniform vector functions attached to edges and faces for the Energetic Approach are introduced in Section 5, and their geometric properties are outlined. Constants bounding the approximation error of the electromagnetic field are derived in terms of the geometric properties of the pairs of oriented dual grids in Section 6. Finally, numerical results are given in Section 7.

2 Spatial discretization of electromagnetic problems by DGA

A time-domain electromagnetic boundary value problem is considered in a *bounded* spatial region Ω and in a time interval $[0,T]$. The electromagnetic field is described by the electric field $e(r,t)$, the electric displacement $d(r,t)$, the magnetic induction $b(r,t)$ and the magnetic field $h(r,t)$. These quantities are functions of the position vector r and of time instant t and are ruled by Faraday equation, Ampére-Maxwell equation, and constitutive equations. Linear, non-dispersive, in general, anisotropic electromagnetic media are considered. Thus the *electric* constitutive equation is

$$d(r,t) = \varepsilon(r)e(r,t) \tag{1}$$

in which the permittivity $\varepsilon(r)$ together with its inverse $\eta(r)$ is a double tensor, assumed to be symmetric, positive-definite and the *magnetic* constitutive equation is

$$h(r,t) = \nu(r)b(r,t), \tag{2}$$

in which the reluctivity $\nu(r)$ together with its inverse $\mu(r)$ is a double tensor, assumed to be symmetric, positive-definite.

For the sake of simplicity, magnetic walls boundary conditions are considered. Generic initial conditions for $d(r,t)$ and $b(r,t)$ are assumed.

The electromagnetic problem is spatially discretized by DGA as follows. Firstly the Ω spatial region is covered by a pair of oriented dual grids $\mathscr{G}$, $\tilde{\mathscr{G}}$ [Weiland (1996); Tonti (2002), Bossavit (1998)]. The primal grid $\mathscr{G}$ has n nodes, l edges, f faces and v volumes. Each of these geometrical elements is given an orientation. The dual grid $\tilde{\mathscr{G}}$ has $\tilde{n} = v$ nodes, $\tilde{l} = f$ edges, $\tilde{f} = l$ faces and $\tilde{v} = n$ volumes. Each of these geometrical elements has the orientation induced by the corresponding geometrical element of the primal grid $\mathscr{G}$ [Tonti (2002)].

Secondly, the electromagnetic field quantities are discretized into integral quantities associated with geometric elements of the pair of dual grids $\mathscr{G}$, $\tilde{\mathscr{G}}$ yielding the following arrays: the $l \times 1$ array $\mathbf{v}(t)$, which approximates the $l \times 1$ array[1] $\rho_e e(r,t)$ of the circulations of the electric field $e(r,t)$ along the edges of $\mathscr{G}$; the $f \times 1$ array $\varphi(t)$, which approximates the $f \times 1$ array[2] $\rho_f b(r,t)$ of the fluxes of the magnetic induction $b(r,t)$ through the faces of $\mathscr{G}$; the $\tilde{f} \times 1$ array $\tilde{\psi}(t)$, which approximates the $l \times 1$ array[3] $\rho_{\tilde{f}} d(r,t)$ of the fluxes of the electric displacement $d(r,t)$ through the faces of $\tilde{\mathscr{G}}$; the $\tilde{l} \times 1$ array $\tilde{\mathbf{f}}(t)$, which approximates the $f \times 1$ array[4] $\rho_{\tilde{e}} h(r,t)$ of the circulations of the magnetic field $h(r,t)$ along the edges of $\mathscr{G}$.

Thirdly, Faraday and Ampére-Maxwell equations are discretized [Tonti (2002)] substituting $\rho_e e(r,t)$, $\rho_f b(r,t)$, $\rho_{\tilde{f}} d(r,t)$, $\rho_{\tilde{e}} h(r,t)$ respectively with $v(t)$, $\varphi(t)$, $\tilde{\psi}(t)$, and $\tilde{f}(t)$ in the *exact* equations satisfied by $\rho_e e(r,t)$, $\rho_f b(r,t)$, $\rho_{\tilde{f}} d(r,t)$, $\rho_{\tilde{e}} h(r,t)$.

Boundary conditions, being magnetic wall boundary conditions, are included in a natural way in discretized Ampére-Maxwell equations [Tonti (2002)]. Initial conditions are written in terms of $\varphi(t)$ and $\tilde{\psi}(t)$.

Lastly, constitutive equations are discretized. *Electric* constitutive equation is discretized into matrix equation

$$\tilde{\psi}(t) = \mathbf{E}\mathbf{v}(t), \tag{3}$$

in which $\mathbf{E}$ is an $l \times l$ matrix, of inverse $\mathbf{H}$, representing the discrete counterpart of the $\varepsilon(r)$ tensor. *Magnetic* constitutive equation is discretized into matrix equation

$$\tilde{\mathbf{f}}(t) = \mathbf{N}\varphi(t), \tag{4}$$

in which $\mathbf{N}$ is an $f \times f$ matrix, of inverse $\mathbf{M}$, representing the discrete counterpart of the $\nu(r)$ tensor.

The discrete electric constitutive equation is only *approximately* satisfied by $\rho_e e(r,t)$ and $\rho_{\tilde{f}} d(r,t)$. In a similar way, the discrete magnetic constitutive equation is only *approximately* satisfied by $\rho_f b(r,t)$ and $\rho_{\tilde{e}} h(r,t)$. The problem of discretizing constitutive equations by approximate equations is crucial in DGA. It it here faced by the energetic approach previously proposed by the Authors [Codecasa, Minerva, and Politi (2004); Codecasa and Trevisan (2006); Codecasa, Specogna, and Trevisan (2007); Codecasa and Trevisan (2007)].

[1] Symbol ρ_e acts on a vector field yielding an array of circulations along the edges of $\mathscr{G}$.
[2] Symbol ρ_f acts on a vector field yielding an array of fluxes through the faces of $\mathscr{G}$.
[3] Symbol $\rho_{\tilde{f}}$ acts on a vector field yielding an array of fluxes through the faces of $\tilde{\mathscr{G}}$.
[4] Symbol $\rho_{\tilde{e}}$ acts on the vector field yielding an array of circulations along the edges of $\tilde{\mathscr{G}}$.

3 Energetic approach for constructing discrete counterparts of constitutive equations

In the energetic approach, discrete counterparts of constitutive equations are constructed primal volume by primal volume. Precisely let $\mathscr{G}^k$, $\tilde{\mathscr{G}}^k$ be the pairs of dual grids obtained by restricting the pair of dual grids $\mathscr{G}$, $\tilde{\mathscr{G}}$ to the single volumes Ω^k of $\mathscr{G}$ with $k = 1,\ldots,v$. Let Γ_i^k, $\tilde{\Sigma}_i^k$ with $i = 1,\ldots,l^k$ be respectively the edges of $\mathscr{G}^k$ and the faces of $\tilde{\mathscr{G}}^k$, with $k = 1,\ldots,v$. Let Σ_i^k, $\tilde{\Gamma}_i^k$ with $i = 1,\ldots,f^k$ be respectively the faces of $\mathscr{G}^k$ and the edges of $\tilde{\mathscr{G}}^k$, with $k = 1,\ldots,v$. Let l_i^k be the edge vector of the edge Γ_i^k with $i = 1,\ldots,l^k$. Let s_i^k be the face vector of the face Σ_i^k with $i = 1,\ldots,f^k$. Similarly let $\tilde{l}_i^k$ be edge vector of edge $\tilde{\Gamma}_i^k$ with $i = 1,\ldots,f^k$ and let $\tilde{s}_i^k$ be face vector of face $\tilde{\Sigma}_i^k$ with $i = 1,\ldots,l^k$. Let r^k be the node of $\tilde{\mathscr{G}}^k$, with $k = 1,\ldots,v$.

3.1 Discrete counterpart of the permittivity tensor $\varepsilon(r)$

Let $\mathbf{v}^k(t)$ and $\rho_e^k e(r,t)$ be the $l^k \times 1$ arrays respectively of the approximate and of the exact circulations[5] of the electric field along the edges of $\mathscr{G}^k$. Let $\tilde{\psi}^k(t)$ and $\rho_{\tilde{f}}^k d(r,t)$ be the $l^k \times 1$ arrays respectively of the approximate and of the exact[6] fluxes of the electric displacement through the faces of $\tilde{\mathscr{G}}^k$.

Let $w_i^{ek}(r)$, with $i = 1,\ldots,l^k$, be *bounded* vector functions satisfying the following *geometric* properties

$$\int_{\Gamma_j^k} w_i^{ek}(r) \cdot t(r)\, d\Gamma = \delta_{ij}, \;\; i,j = 1,\ldots,l^k, \tag{5}$$

$$\sum_{1}^{l^k}{}_i\, w_i^{ek}(r) \otimes l_i^k = I, \tag{6}$$

$$\tilde{s}_i^k = \int_{\Omega^k} w_i^{ek}(r)\, d\Omega, \;\; i = 1,\ldots,l^k, \tag{7}$$

in which δ_{ij} is the Kronecker's delta symbol and I is the identity double tensor. Let matrix $\mathbf{E}^k$ be introduced, whose elements are

$$E_{ij}^k = \int_{\Omega^k} w_i^{ek}(r) \cdot \varepsilon(r^k) w_j^{ek}(r)\, d\Omega, \;\; i,j = 1,\ldots,l^k. \tag{8}$$

in which the permittivity tensor $\varepsilon(r)$ is evaluated at the node r^k of $\tilde{\mathscr{G}}^k$.

Matrix $\mathbf{E}$ in (3) can now generated from matrices $\mathbf{E}^k$, with $k = 1,\ldots,v$, as follows. Let $\mathbf{T}^k$ be the $l^k \times l$ matrix whose element t_{ij}^k is 1 if the i-th edge of $\mathscr{G}^k$ is the j-th

[5] Symbol ρ_e^k acts on a vector field yielding an array of circulations along the edges of $\mathscr{G}^k$.
[6] Symbol $\rho_{\tilde{f}}^k$ acts on a vector field yielding an array of fluxes through the faces of $\tilde{\mathscr{G}}^k$.

edge of $\mathscr{G}$ and is 0 otherwise. Then matrix $\mathbf{E}$ is constructed as

$$\mathbf{E} = \sum_{1}^{v}{}_k \mathbf{T}^{k^T} \mathbf{E}^k \mathbf{T}^k.$$

3.2 Discrete counterpart of the reluctivity tensor $\nu(r)$

Let $\varphi^k(t)$ and $\rho_f^k b(r,t)$ be the $f^k \times 1$ arrays respectively of the approximate and of the exact fluxes[7] of the magnetic induction through the faces of $\mathscr{G}^k$. Let $\tilde{\mathbf{f}}^k(t)$ and $\rho_{\tilde{e}}^k h(r,t)$ be the $f^k \times 1$ arrays respectively of the approximate and of the exact circulations[8] of the magnetic field along the edges of $\tilde{\mathscr{G}}^k$.

Let $w_i^{fk}(r)$, with $i = 1,\ldots,f^k$, be *bounded* vector functions satisfying the following geometric properties

$$\int_{\Sigma_j^k} w_i^{fk}(r) \cdot n(r)\, d\Sigma = \delta_{ij}, \;\; i,j = 1,\ldots,f^k, \tag{9}$$

$$\sum_{1}^{f^k}{}_i w_i^{fk}(r) \otimes s_i^k = I, \tag{10}$$

$$\tilde{l}_i^k = \int_{\Omega} w_i^{fk}(r)\, d\Omega, \;\; i = 1,\ldots,f^k. \tag{11}$$

Let matrix $\mathbf{N}^k$ have elements

$$N_{ij}^k = \int_{\Omega^k} w_i^{fk}(r) \cdot \nu(r^k) w_j^{fk}(r)\, d\Omega, \;\; i,j = 1,\ldots,f^k. \tag{12}$$

in which the reluctivity tensor $\nu(r)$ is evaluated at the node r^k of $\tilde{\mathscr{G}}^k$.

Matrix $\mathbf{N}$ in (4) can now generated from matrices $\mathbf{N}^k$, with $k = 1,\ldots,v$, as follows. Let $\mathbf{P}^k$ be the $f^k \times f$ matrix whose element p_{ij}^k is 1 if the i-th face of $\mathscr{G}^k$ is the j-th face of $\mathscr{G}$ and is 0 otherwise. Matrix $\mathbf{N}$ is constructed as

$$\mathbf{N} = \sum_{1}^{v}{}_k \mathbf{P}^{k^T} \mathbf{N}^k \mathbf{P}^k.$$

4 Convergence formulae for electromagnetic problems discretized by the DGA

Regularity conditions on material properties and electromagnetic field are assumed as in [Codecasa and Trevisan (2010)]. Thus it is assumed that the spatial domain Ω can be partitioned in a finite set of subdomains Ω_i, with $i = 1,\ldots,s$, in each of

[7] Symbol ρ_f^k acts on a vector field yielding an array of fluxes through the faces of $\mathscr{G}^k$.

[8] Symbol $\rho_{\tilde{e}}^k$ acts on a vector field yielding an array of circulations along the edges of $\tilde{\mathscr{G}}^k$.

which both the tensors $\varepsilon(r)$, $\nu(r)$ and their inverses $\eta(r)$, $\mu(r)$ are bounded and Lipschitz continuous. That is, if $A(r)$ is any of such tensors, constants M_A and L_A exist such that

$$
\begin{aligned}
&||A(r)||_2 \le M_A, \\
&||A(r_1) - A(r_2)||_2 \le L_A |r_1 - r_2|, \quad r_1, r_2 \in \Omega_i,\ i = 1, \ldots, s,
\end{aligned}
$$

hold, in which $||\cdot||_2$ is the spectral norm, recalled in Appendix A. Similarly, it is assumed that for all time instants $0 \le t \le T$, the fields $e(r,t)$, $h(r,t)$, $b(r,t)$, $d(r,t)$, together with their time derivatives are bounded and Lipschitz continuous within each subdomain Ω_i, with $i = 1, \ldots, s$. That is, if $a(r,t)$ is any of such fields, constants M_a and L_a exist such that

$$|a(r,t)| \le M_a, \tag{13}$$

$$|a(r_1,t) - a(r_2,t)| \le L_a |r_1 - r_2|, \quad r_1, r_2 \in \Omega_i,\ i = 1, \ldots, s, \tag{14}$$

hold.

Assumptions are also made on the pair of dual grids $\mathscr{G}$, $\tilde{\mathscr{G}}$. It is assumed that the Ω_i subdomains, with $i = 1, \ldots, s$, are exactly obtained as unions of volumes of the primal grid $\mathscr{G}$. Moreover any chosen pair of dual grids $\mathscr{G}$, $\tilde{\mathscr{G}}$ is such that the following disequalities

$$||\rho_e^k e(r,t)||_{\mathbf{E}^k} \le R_{\mathbf{E}} \sqrt{|\Omega^k|} \max_{r \in \Omega^k} \sqrt{||\varepsilon(r)||_2} \max_{r \in \Omega^k} |e(r,t)|, \quad k = 1, \ldots, \nu, \tag{15}$$

$$||\rho_{\tilde{f}}^k d(r,t)||_{\mathbf{H}^k} \le R_{\mathbf{H}} \sqrt{|\Omega^k|} \max_{r \in \Omega^k} \sqrt{||\eta(r)||_2} \max_{r \in \Omega^k} |d(r,t)|, \quad k = 1, \ldots, \nu, \tag{16}$$

$$||\rho_f^k b(r,t)||_{\mathbf{N}^k} \le R_{\mathbf{N}} \sqrt{|\Omega^k|} \max_{r \in \Omega^k} \sqrt{||\nu(r)||_2} \max_{r \in \Omega^k} |b(r,t)|, \quad k = 1, \ldots, \nu, \tag{17}$$

$$||\rho_{\tilde{e}}^k h(r,t)||_{\mathbf{M}^k} \le R_{\mathbf{M}} \sqrt{|\Omega^k|} \max_{r \in \Omega^k} \sqrt{||\mu(r)||_2} \max_{r \in \Omega^k} |h(r,t)|, \quad k = 1, \ldots, \nu, \tag{18}$$

hold, in which the notation in Appendix A is used and the constants $R_{\mathbf{E}}$, $R_{\mathbf{H}}$, $R_{\mathbf{N}}$ and $R_{\mathbf{M}}$ are independent of the pair of dual grids $\mathscr{G}$, $\tilde{\mathscr{G}}$. With these assumptions the following results are obtained in [Codecasa and Trevisan (2010)], in which h_M denotes the maximum diameter [Quarteroni and Valli (1994)] of the volumes of $\mathscr{G}$.

Theorem 1 (Error bound for integral quantities) *For* $0 \le t \le T$*, it is*

$$\sqrt{||\mathbf{v}(t) - \rho_e e(r,t)||_{\mathbf{E}}^2 + ||\tilde{\mathbf{f}}(t) - \rho_{\tilde{e}} h(r,t)||_{\mathbf{M}}^2} \le \Big(\sqrt{S_e^2 + S_h^2} + T\sqrt{S_{\dot{d}}^2 + S_{\dot{b}}^2}\Big) h_M.$$

$$\sqrt{||\tilde{\psi}(t) - \rho_{\tilde{f}} d(r,t)||_{\mathbf{H}}^2 + ||\varphi(t) - \rho_f b(r,t)||_{\mathbf{N}}^2} \le \Big(2\sqrt{S_e^2 + S_h^2} + T\sqrt{S_{\dot{d}}^2 + S_{\dot{b}}^2}\Big) h_M.$$

in which

$$S_e = R_{\mathbf{H}}\sqrt{M_\eta|\Omega|}(L_\varepsilon M_e + M_\varepsilon L_e) + R_{\mathbf{E}}\sqrt{M_\varepsilon|\Omega|}L_e,$$
$$S_h = R_{\mathbf{N}}\sqrt{M_\nu|\Omega|}(L_\mu M_h + M_\mu L_h) + R_{\mathbf{M}}\sqrt{M_\mu|\Omega|}L_h,$$
$$S_{\dot{d}} = R_{\mathbf{E}}\sqrt{M_\varepsilon|\Omega|}(L_\eta M_{\dot{d}} + M_\eta L_{\dot{d}}) + R_{\mathbf{H}}\sqrt{M_\eta|\Omega|}L_{\dot{d}},$$
$$S_{\dot{b}} = R_{\mathbf{M}}\sqrt{M_\mu|\Omega|}(L_\nu M_{\dot{b}} + M_\nu L_{\dot{b}}) + R_{\mathbf{N}}\sqrt{M_\nu|\Omega|}L_{\dot{b}}.$$

These equations establish bounds for the approximation error of discrete quantities in DGA. Bounds for the approximation error of the electromagnetic field can also be derived. Introducing the fields

$$\pi_e(r)\mathbf{v}(t) = \sum_{1}^{l^k}{}_i\, v_i^k(t) w_i^{ek}(r), \quad r \in \Omega^k,\ k = 1,\ldots,v,$$
$$\pi_f^k(r)\varphi^k(t) = \sum_{1}^{f^k}{}_i\, \varphi_i^k(t) w_i^{fk}(r), \quad r \in \Omega^k,\ k = 1,\ldots,v,$$

and, using the notation of Appendix A, it results in

Theorem 2 (Error bound for field quantities) *For* $0 \le t \le T$*, it results in*

$$||\pi_e(r)\mathbf{v}(t) - e(r,t)||_\varepsilon \le \left(I_e + \sqrt{S_e^2 + S_h^2} + T\sqrt{S_{\dot{d}}^2 + S_{\dot{b}}^2}\right) h_M$$
$$||\pi_f(r)\varphi(t) - b(r,t)||_\nu \le \left(I_b + 2\sqrt{S_e^2 + S_h^2} + T\sqrt{S_{\dot{d}}^2 + S_{\dot{b}}^2}\right) h_M$$

in which

$$I_e = (R_{\mathbf{E}} + 1)\sqrt{M_\varepsilon|\Omega|}L_e,$$
$$I_b = (R_{\mathbf{N}} + 1)\sqrt{M_\nu|\Omega|}L_b.$$

5 Vector base functions for the considered pairs of oriented dual grids

As shown in [Codecasa and Trevisan (2010)] a general condition for which (15)-(18) are satisfied is to choose the pairs of dual grids in such a way that each primal volume is geometrically similar to a volume in a finite set. However the constants $R_{\mathbf{E}}$, $R_{\mathbf{H}}$, $R_{\mathbf{N}}$ and $R_{\mathbf{M}}$, obtained in this way, which bound the approximation errors of the electromagnetic field quantities have only a theoretic relevance, since they

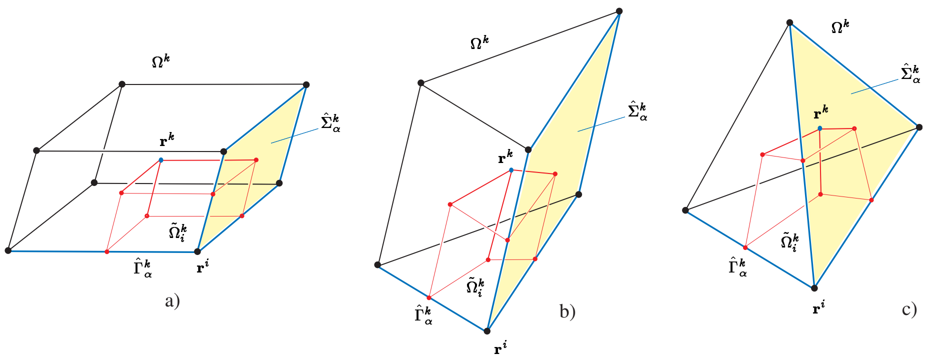

Figure 1: The three typologies of Ω^k volumes are shown: (oblique) parallelepiped (a), (oblique) triangular prism (b) and tetrahedron (c). The pair of edge $\hat{\Gamma}^k_\alpha$ and face $\hat{\Sigma}^k_\alpha$ corresponding to the same dual volume $\tilde{\Omega}^k_i$, with $i = n^k(\alpha)$ are also evidenced.

are *not* expressed in terms of the geometric properties of the pair of oriented dual grids.

Hereafter the significant case of primal grids $\mathscr{G}$ composed of tetrahedra, (oblique) triangular prisms and (oblique) parallelepipeds and dual grids $\tilde{\mathscr{G}}$ is considered in details. The dual grid $\tilde{\mathscr{G}}$ is obtained by the barycentric subdivision of $\mathscr{G}$ [Bossavit (1998)]. Error bounds are derived in terms of the geometrical details of such oriented dual grids $\mathscr{G}$, $\tilde{\mathscr{G}}$.

Let $\tilde{\Omega}^k_i$ be the volumes of $\tilde{\mathscr{G}}^k$, with $i = 1,\ldots,n^k$. For each volume $\tilde{\Omega}^k_i$ a triple of edges is introduced given by the edges of $\mathscr{G}^k$ which are parts of edges of $\tilde{\Omega}^k_i$. Such edges are named $\hat{\Gamma}^k_\alpha$, with $\alpha = 1,\ldots,3n^k$, and are independently oriented with respect to the edges of $\mathscr{G}^k$. Similarly for each volume $\tilde{\Omega}^k_i$ a triple of faces is introduced given by the faces of $\mathscr{G}^k$ which are parts of the faces of $\tilde{\Omega}^k_i$. Such faces are named $\hat{\Sigma}^k_\alpha$, with $\alpha = 1,\ldots,3n^k$. It is assumed that edge $\hat{\Gamma}^k_\alpha$ and face $\hat{\Sigma}^k_\alpha$, indexed by the same α, correspond to the same volume $\tilde{\Omega}^k_i$, are not coplanar. and are oriented in such a way that $\hat{l}^k_\alpha \cdot \hat{s}^k_\alpha > 0$, $\hat{l}^k_\alpha$ being the edge vector of $\hat{\Gamma}^k_\alpha$ and $\hat{s}^k_\alpha$ being the face vector of $\hat{\Sigma}^k_\alpha$, as shown in Fig. 1. Let $i = n^k(\alpha)$ be the function which associates to each edge $\hat{\Gamma}^k_\alpha$ and face $\hat{\Sigma}^k_\alpha$ the corresponding volume $\tilde{\Omega}^k_i$.

Let $\hat{K}^k$ be $1/8$, $1/12$, $1/24$ according to the volume Ω^k is respectively an (oblique) parallelepiped, an (oblique) triangular prism or a tetrahedron. Besides let $\hat{C}^k_\alpha$ be 1 or 2 according to the face $\hat{\Sigma}^k_\alpha$ is respectively a parallelogram or a triangle. Let $\hat{\mathbf{T}}^k$

be the $3n^k \times l^k$ rectangular matrix whose elements are

$$\hat{t}^k_{\alpha i} = \frac{\hat{l}^k_\alpha}{|\hat{l}^k_\alpha|} \cdot \frac{l^k_i}{|l^k_i|} \delta_{l^k(\alpha) i},$$

$l^k(\alpha)$ being the index of the edge $\Gamma^k_{l^k(\alpha)}$ of $\mathscr{G}^k$ corresponding to edge $\hat{\Gamma}^k_\alpha$, and let $\hat{\mathbf{P}}^k$ be the $3n^k \times f^k$ rectangular matrix whose elements are

$$\hat{p}^k_{\alpha i} = \frac{\hat{s}^k_\alpha}{|\hat{s}^k_\alpha|} \cdot \frac{s^k_i}{|s^k_i|} \delta_{f^k(\alpha) i},$$

$f^k(\alpha)$ being the index of the face $\Sigma^k_{f^k(\alpha)}$ of $\mathscr{G}^k$ corresponding to face $\hat{\Sigma}^k_\alpha$.

Lemma 1 *The following relations hold*

$$|\tilde{\Omega}^k_{n^k(\alpha)}| = \hat{K}^k \hat{C}^k_\alpha \hat{l}^k_\alpha \cdot \hat{s}^k_\alpha, \tag{19}$$

$$\sum_1^{3n^k}{}_\alpha \, \hat{t}^k_{\alpha i} \hat{K}^k \hat{C}^k_\alpha \hat{s}^k_\alpha = \tilde{s}^k_i \tag{20}$$

$$\sum_1^{3n^k}{}_\alpha \, \hat{p}^k_{\alpha i} \hat{K}^k \hat{C}^k_\alpha \hat{l}^k_\alpha = \tilde{l}^k_i. \tag{21}$$

Proof. For an (oblique) parallelepiped it is

$$(\hat{K}^k \hat{C}^k_\alpha)(\hat{l}^k_\alpha \cdot \hat{s}^k_\alpha) = \frac{1}{8} 8 |\tilde{\Omega}^k_{n^k(\alpha)}|,$$

$$\sum_1^{3n^k}{}_\alpha (\hat{K}^k \hat{C}^k_\alpha)(\hat{t}^k_{\alpha i} \hat{s}^k_\alpha) = \frac{1}{8} \sum_1^{3n^k}{}_\alpha \, \hat{t}^k_{\alpha i} \hat{s}^k_\alpha = \frac{1}{8} 8 \tilde{s}^k_i,$$

$$\sum_1^{3n^k}{}_\alpha (\hat{K}^k \hat{C}^k_\alpha)(\hat{p}^k_{\alpha i} \hat{l}^k_\alpha) = \frac{1}{8} \sum_1^{3n^k}{}_\alpha \, \hat{p}^k_{\alpha i} \hat{l}^k_\alpha = \frac{1}{8} 8 \tilde{l}^k_i$$

For an (oblique) triangular prism, limitedly to the parallelepipeds $\hat{\Sigma}^k_\alpha$ and the corresponding $\hat{\Gamma}^k_\alpha$, it is

$$(\hat{K}^k \hat{C}^k_\alpha)(\hat{l}^k_\alpha \cdot \hat{s}^k_\alpha) = \frac{1}{6} 6 |\tilde{\Omega}^k_{n^k(\alpha)}|,$$

$$\sum_1^{3n^k}{}_\alpha (\hat{K}^k \hat{C}^k_\alpha)(\hat{t}^k_{\alpha i} \hat{s}^k_\alpha) = \frac{1}{6} \sum_1^{3n^k}{}_\alpha \, \hat{t}^k_{\alpha i} \hat{s}^k_\alpha = \frac{1}{6} 6 \tilde{s}^k_i,$$

$$\sum_1^{3n^k}{}_\alpha (\hat{K}^k \hat{C}^k_\alpha)(\hat{p}^k_{\alpha i} \hat{l}^k_\alpha) = \frac{1}{6} \sum_1^{3n^k}{}_\alpha \, \hat{p}^k_{\alpha i} \hat{l}^k_\alpha = \frac{1}{6} 6 \tilde{l}^k_i.$$

For an (oblique) triangular prism, limitedly to the triangles $\hat{\Sigma}_{\alpha}^{k}$ and the corresponding $\hat{\Gamma}_{\alpha}^{k}$, and for a tetrahedron it is

$$(\hat{K}^k\hat{C}_{\alpha}^k)(\hat{l}_{\alpha}^k\cdot\hat{s}_{\alpha}^k)=\frac{1}{12}\,12|\tilde{\Omega}_{n^k(\alpha)}^k|$$

$$\sum_{1}^{3n^k}{}_{\alpha}(\hat{K}^k\hat{C}_{\alpha}^k)(\hat{t}_{\alpha i}^k\hat{s}_{\alpha}^k)=\frac{1}{12}\sum_{1}^{3n^k}{}_{\alpha}\,\hat{t}_{\alpha i}^k\hat{s}_{\alpha}^k=\frac{1}{12}\,12\tilde{s}_i^k$$

$$\sum_{1}^{3n^k}{}_{\alpha}(\hat{K}^k\hat{C}_{\alpha}^k)(\hat{p}_{\alpha i}^k\hat{l}_{\alpha}^k)=\frac{1}{12}\sum_{1}^{3n^k}{}_{\alpha}\,\hat{p}_{\alpha i}^k\hat{l}_{\alpha}^k=\frac{1}{12}\,12\tilde{l}_i^k$$

and the thesis follows. ∎

5.1 Discrete counterpart of the permittivity tensor $\varepsilon(r)$

The following *step-wise uniform* vector functions are introduced

$$\hat{w}_{\alpha}^{ek}(r)=\begin{cases}\dfrac{\hat{K}^k\hat{C}_{\alpha}^k\hat{s}_{\alpha}^k}{|\tilde{\Omega}_{n(\alpha)}^k|} & r\in\tilde{\Omega}_{n(\alpha)}^k\\ \mathbf{0} & r\notin\tilde{\Omega}_{n(\alpha)}^k.\end{cases}$$

Lemma 2 *The* $\hat{w}_{\alpha}^{ek}(r)$ *vector functions satisfy the following properties*

$$\int_{\hat{\Gamma}_{\alpha}^k\cap\tilde{\Omega}_{n(\alpha)}^k}\hat{w}_{\beta}^{ek}(r)\cdot t(r)\,d\Gamma=\frac{|\hat{\Gamma}_{\alpha}^k\cap\tilde{\Omega}_{n(\alpha)}^k|}{|\hat{\Gamma}_{\alpha}^k|}\,\delta_{\alpha\beta},\quad \alpha,\beta=1,\ldots,3n^k, \tag{22}$$

$$\sum_{1}^{3n^k}{}_{\alpha}\,\hat{w}_{\alpha}^{ek}(r)\otimes\hat{l}_{\alpha}^k=I, \tag{23}$$

$$\hat{K}^k\hat{C}_{\alpha}^k\hat{s}_{\alpha}^k=\int_{\Omega^k}\hat{w}_{\alpha}^{ek}(r)\,d\Omega,\quad \alpha=1,\ldots,3n^k, \tag{24}$$

Proof. If $n^k(\alpha)\neq n^k(\beta)$, since $\tilde{\Omega}_{n(\alpha)}^k$ and $\tilde{\Omega}_{n(\beta)}^k$ are disjoint, the left hand side of (22) is zero. Otherwise if $n^k(\alpha)=n^k(\beta)$ then

$$\int_{\hat{\Gamma}_{\alpha}^k\cap\tilde{\Omega}_{n^k(\alpha)}^k}\hat{w}_{\beta}^{ek}(r)\cdot t(r)\,d\Gamma=\frac{|\hat{\Gamma}_{\alpha}^k\cap\tilde{\Omega}_{n^k(\alpha)}^k|}{|\hat{\Gamma}_{\alpha}^k|}\,\hat{l}_{\alpha}^k\cdot\frac{\hat{K}^k\hat{C}_{\beta}^k\hat{s}_{\beta}^k}{|\tilde{\Omega}_{n^k(\beta)}^k|}$$

which is zero if $\alpha\neq\beta$, since $\hat{l}_{\alpha}^k\cdot\hat{s}_{\beta}^k=0$, and is $|\hat{\Gamma}_{\alpha}^k\cap\tilde{\Omega}_{n(\alpha)}^k|/|\hat{\Gamma}_{\alpha}^k|$, as a consequence of Lemma 1, if $\alpha=\beta$. Thus (22) is proved.

Let $r\in\tilde{\Omega}_i^k$. The sum in the left hand side of (23) has only three non-zero terms for the values α_1, α_2, α_3 of the index α such that $n^k(\alpha_1)=n^k(\alpha_2)=n^k(\alpha_3)=i$. Thus, from the definition of $\hat{w}_\alpha^{ek}(r)$, from Lemma 1, and since $\hat{s}_{\alpha_1}^k$, $\hat{s}_{\alpha_2}^k$, $\hat{s}_{\alpha_3}^k$ are parallel respectively to $\hat{l}_{\alpha_2}^k \times \hat{l}_{\alpha_3}^k$, $\hat{l}_{\alpha_3}^k \times \hat{l}_{\alpha_1}^k$, $\hat{l}_{\alpha_1}^k \times \hat{l}_{\alpha_2}^k$, it follows

$$\begin{aligned}\sum_{1}^{3n^k}{}_{\alpha} \hat{w}_\alpha^{ek}(r) \otimes \hat{l}_\alpha^k &= \frac{\hat{s}_{\alpha_1}^k \otimes \hat{l}_{\alpha_1}^k}{\hat{s}_{\alpha_1}^k \cdot \hat{l}_{\alpha_1}^k} + \frac{\hat{s}_{\alpha_2}^k \otimes \hat{l}_{\alpha_2}^k}{\hat{s}_{\alpha_1}^k \cdot \hat{l}_{\alpha_2}^k} + \frac{\hat{s}_{\alpha_3}^k \otimes \hat{l}_{\alpha_3}^k}{\hat{s}_{\alpha_3}^k \cdot \hat{l}_{\alpha_3}^k} \\ &= \frac{(\hat{l}_{\alpha_2}^k \times \hat{l}_{\alpha_3}^k) \otimes \hat{l}_{\alpha_1}^k}{(\hat{l}_{\alpha_2}^k \times \hat{l}_{\alpha_3}^k) \cdot \hat{l}_{\alpha_1}^k} + \frac{(\hat{l}_{\alpha_3}^k \times \hat{l}_{\alpha_1}^k) \otimes \hat{l}_{\alpha_2}^k}{(\hat{l}_{\alpha_3}^k \times \hat{l}_{\alpha_1}^k) \cdot \hat{l}_{\alpha_2}^k} + \frac{(\hat{l}_{\alpha_1}^k \times \hat{l}_{\alpha_2}^k) \otimes \hat{l}_{\alpha_3}^k}{(\hat{l}_{\alpha_1}^k \times \hat{l}_{\alpha_2}^k) \cdot \hat{l}_{\alpha_3}^k} = I\end{aligned}$$

in which the last equality for $\hat{l}_{\alpha_1}^k$, $\hat{l}_{\alpha_2}^k$, $\hat{l}_{\alpha_3}^k$ can be directly verified, and (23) is proved. Lastly, by recalling the definition of $\hat{w}_\alpha^{ek}(r)$, (24) straightforwardly descends. ■

Functions $w_i^{ek}(r)$ are now constructed in terms of functions $\hat{w}_\alpha^{ek}(r)$ as follows

$$w_i^{ek}(r) = \sum_{1}^{3n^k}{}_{\alpha} \hat{w}_\alpha^{ek}(r) \hat{t}_{\alpha i}^k, \quad i = 1, \dots, l^k.$$

From Lemma 2 it straightforwardly follows

Theorem 3 *Vector functions $w_i^{ek}(r)$, with $i = 1, \dots, l^k$, satisfy properties (5)-(7).*

Proof. From the definition of $w_i^{ek}(r)$ it follows

$$\begin{aligned}\int_{\Gamma_j^k} w_i^{ek}(r) \cdot t(r)\, d\Gamma &= \sum_{1}^{3n^k}{}_{\alpha} \hat{t}_{\alpha i}^k \int_{\Gamma_j^k} \hat{w}_\alpha^{ek}(r) \cdot t(r)\, d\Gamma \\ &= \sum_{1}^{3n^k}{}_{\alpha} \hat{t}_{\alpha i}^k \sum_{1}^{3n^k}{}_{\beta} \hat{t}_{\beta j}^k \int_{\hat{\Gamma}_\beta^k \cap \tilde{\Omega}_{n(\beta)}^k} \hat{w}_\alpha^{ek}(r) \cdot t(r)\, d\Gamma \\ &= \sum_{1}^{3n^k}{}_{\alpha\beta} \hat{t}_{\alpha i}^k \hat{t}_{\beta j}^k \frac{|\hat{\Gamma}_\alpha^k \cap \tilde{\Omega}_{n^k(\alpha)}^k|}{|\hat{\Gamma}_\alpha^k|} \delta_{\alpha\beta}, \\ &= \sum_{1}^{3n^k}{}_{\alpha} \frac{|\hat{\Gamma}_\alpha^k \cap \tilde{\Omega}_{n^k(\alpha)}^k|}{|\hat{\Gamma}_\alpha^k|} \hat{t}_{\alpha i}^k \hat{t}_{\alpha j}^k = \delta_{ij},\end{aligned} \tag{25}$$

Eq. (25) descending from (22). Eq. (5) is thus proved.

From the definition of $w_i^{ek}(r)$ and from (23) it follows

$$\begin{aligned}
\sum_{1}^{l^k}{}_i\, w_i^{ek}(r) \otimes l_i^k &= \sum_{1}^{l^k}{}_i \sum_{1}^{3n^k}{}_\alpha\, \hat{w}_\alpha^{ek}(r) \hat{t}_{\alpha i}^k \otimes l_i^k \\
&= \sum_{1}^{3n^k}{}_\alpha\, \hat{w}_\alpha^{ek}(r) \otimes \sum_{1}^{l^k}{}_i\, \hat{t}_{\alpha i}^k l_i^k \\
&= \sum_{1}^{3n^k}{}_\alpha\, \hat{w}_\alpha^{ek}(r) \otimes \hat{l}_\alpha^k = I.
\end{aligned}$$

Eq. (6) is thus proved.

Lastly from the definition of $w_i^{ek}(r)$, from (24) and from (20) it follows

$$\begin{aligned}
\int_{\Omega^k} w_i^{ek}(r)\, d\Omega &= \int_{\Omega^k} \sum_{1}^{3n^k}{}_\alpha\, \hat{w}_\alpha^{ek}(r) \hat{t}_{\alpha i}^k\, d\Omega \\
&= \sum_{1}^{3n^k}{}_\alpha\, \hat{t}_{\alpha i}^k \int_{\Omega^k} \hat{w}_\alpha^{ek}(r)\, d\Omega \\
&= \sum_{1}^{3n^k}{}_\alpha\, \hat{t}_{\alpha i}^k \hat{K}^k \hat{C}_\alpha^k \hat{s}_\alpha^k = \hat{s}_i^k, \;\; i = 1, \ldots, l^k.
\end{aligned}$$

Eq. (7) is thus proved. ∎

Functions $w_i^{ek}(r)$, with $i = 1, \ldots, l^k$, can thus be used in the energetic approach for constructing the $\mathbf{E}^k$ matrices. It is noted that

$$\mathbf{E}^k = \hat{\mathbf{T}}^{kT} \hat{\mathbf{E}}^k \hat{\mathbf{T}}^k \tag{26}$$

in which $\hat{\mathbf{E}}^k$ are $3n^k \times 3n^k$ matrices whose elements are

$$\hat{E}_{\alpha\beta}^k = \int_{\Omega^k} \hat{w}_\alpha^{ek}(r) \cdot \varepsilon(r^k) \hat{w}_\beta^{ek}(r)\, d\Omega, \;\; \alpha, \beta = 1, \ldots, 3n^k$$

5.2 Discrete counterpart of the reluctivity tensor $\nu(r)$

The following *step-wise uniform* vector functions are introduced

$$\hat{w}_\alpha^{fk}(r) = \begin{cases} \dfrac{\hat{K}^k \hat{C}_\alpha^k \hat{l}_\alpha^k}{|\tilde{\Omega}_{n^k(\alpha)}^k|} & r \in \tilde{\Omega}_{n^k(\alpha)}^k \\ \mathbf{0} & r \notin \tilde{\Omega}_{n^k(\alpha)}^k. \end{cases}$$

Lemma 3 *The $\hat{w}_{\alpha}^{fk}(r)$ vector functions satisfy the following properties*

$$\int_{\hat{\Sigma}_{\alpha}^{k}\cap\tilde{\Omega}_{n^{k}(\alpha)}^{k}} \hat{w}_{\beta}^{fk}(r)\cdot n(r)\, d\Sigma = \frac{|\hat{\Sigma}_{\alpha}^{k}\cap\tilde{\Omega}_{n^{k}(\alpha)}^{k}|}{|\hat{\Sigma}_{\alpha}^{k}|}\,\delta_{\alpha\beta},\ \ \alpha,\beta = 1,\dots,3n^{k}, \tag{27}$$

$$\sum_{1}^{3n^{k}}{}_{\alpha}\, \hat{w}_{\alpha}^{fk}(r)\otimes\hat{s}_{\alpha}^{k} = I, \tag{28}$$

$$\hat{K}^{k}\hat{C}_{\alpha}^{k}\hat{l}_{\alpha}^{k} = \int_{\Omega^{k}} \hat{w}_{\alpha}^{fk}(r)\, d\Omega,\ \ \alpha = 1,\dots,3n^{k}, \tag{29}$$

Proof. If $n^{k}(\alpha)\neq n^{k}(\beta)$, since $\tilde{\Omega}_{n^{k}(\alpha)}^{k}$ and $\tilde{\Omega}_{n^{k}(\beta)}^{k}$ are disjoint, the left hand side of (27) is zero. Otherwise if $n^{k}(\alpha)=n^{k}(\beta)$ then

$$\int_{\hat{\Sigma}_{\alpha}^{k}\cap\tilde{\Omega}_{n^{k}(\alpha)}^{k}} \hat{w}_{\beta}^{fk}(r)\cdot n(r)\, d\Gamma = \frac{|\hat{\Sigma}_{\alpha}^{k}\cap\tilde{\Omega}_{n^{k}(\alpha)}^{k}|}{|\hat{\Sigma}_{\alpha}^{k}|}\,\hat{s}_{\alpha}^{k}\cdot\frac{\hat{K}^{k}\hat{C}_{\beta}^{k}\hat{l}_{\beta}^{k}}{|\tilde{\Omega}_{n^{k}(\beta)}^{k}|}$$

which is zero if $\alpha\neq\beta$, since $\hat{l}_{\alpha}^{k}\cdot\hat{s}_{\beta}^{k}=0$, and is $|\hat{\Sigma}_{\alpha}^{k}\cap\tilde{\Omega}_{n^{k}(\alpha)}^{k}|/|\hat{\Sigma}_{\alpha}^{k}|$, as a consequence of Lemma 1, if $\alpha=\beta$. Thus (27) is proved.

Let $r\in\tilde{\Omega}_{i}^{k}$. The sum in the left hand side of (28) has only three non-zero terms for the values α_1, α_2, α_3 of the index α such that $n^{k}(\alpha_1)=n^{k}(\alpha_2)=n^{k}(\alpha_3)=i$. Thus, by recalling the definition of $\hat{w}_{\alpha}^{fk}(r)$ and Lemma 1,

$$\sum_{1}^{3n^{k}}{}_{\alpha}\, \hat{w}_{\alpha}^{fk}(r)\otimes\hat{s}_{\alpha}^{k} = \frac{\hat{s}_{\alpha_1}^{k}\otimes\hat{l}_{\alpha_1}^{k}}{\hat{s}_{\alpha_1}^{k}\cdot\hat{l}_{\alpha_1}^{k}} + \frac{\hat{s}_{\alpha_2}^{k}\otimes\hat{l}_{\alpha_2}^{k}}{\hat{s}_{\alpha_1}^{k}\cdot\hat{l}_{\alpha_2}^{k}} + \frac{\hat{s}_{\alpha_3}^{k}\otimes\hat{l}_{\alpha_3}^{k}}{\hat{s}_{\alpha_3}^{k}\cdot\hat{l}_{\alpha_3}^{k}} = I$$

and, proceeding as in the proof of Lemma 2, (28) is proved.

Lastly, by recalling the definition of $\hat{w}_{\alpha}^{fk}(r)$, (29) straightforwardly descends. ∎

Vector functions $\hat{w}_{i}^{fk}(r)$ are now constructed in terms of functions $\hat{w}_{\alpha}^{fk}(r)$ as follows

$$w_{i}^{fk}(r) = \sum_{1}^{3n^{k}}{}_{\alpha}\, \hat{w}_{\alpha}^{fk}(r)\hat{p}_{\alpha i}^{k},\quad i=1,\dots,f^{k}.$$

From Lemma 3 it straightforwardly follows

Theorem 4 *Vector functions $w_{i}^{fk}(r)$, with $i=1,\dots,f^{k}$, satisfy properties (9)-(11).*

Proof. From the definition of $w_i^{fk}(r)$ it follows

$$\begin{aligned}
\int_{\Sigma_j^k} w_i^{fk}(r)\cdot n(r)\,d\Sigma &= \sum_{\alpha\,1}^{3n^k} \hat{p}_{\alpha i}^k \int_{\Sigma_j^k} \hat{w}_\alpha^{fk}(r)\cdot n(r)\,d\Sigma \\
&= \sum_{\alpha\,1}^{3n^k} \hat{p}_{\alpha i}^k \sum_{\beta\,1}^{3n^k} \hat{p}_{\beta j}^k \int_{\hat{\Sigma}_\beta^k \cap \tilde{\Omega}_{n^k(\beta)}^k} \hat{w}_\alpha^{fk}(r)\cdot n(r)\,d\Sigma \\
&= \sum_{\alpha\beta\,1}^{3n^k} \hat{p}_{\alpha i}^k \hat{p}_{\beta j}^k \frac{|\hat{\Sigma}_\alpha^k \cap \tilde{\Omega}_{n^k(\alpha)}^k|}{|\hat{\Sigma}_\alpha^k|}\,\delta_{\alpha\beta}, \\
&= \sum_{\alpha\,1}^{3n^k} \frac{|\hat{\Sigma}_\alpha^k \cap \tilde{\Omega}_{n^k(\alpha)}^k|}{|\hat{\Sigma}_\alpha^k|} \hat{p}_{\alpha i}^k \hat{p}_{\alpha j}^k = \delta_{ij},
\end{aligned} \tag{30}$$

Eq. (30) descending from (27). Eq. (9) is thus proved.

From the definition of $w_i^{fk}(r)$ and from (28) it follows

$$\begin{aligned}
\sum_{i\,1}^{f^k} w_i^{fk}(r)\otimes s_i^k &= \sum_{i\,1}^{f^k} \sum_{\alpha\,1}^{3n^k} \hat{w}_\alpha^{fk}(r)\hat{p}_{\alpha i}^k \otimes s_i^k \\
&= \sum_{\alpha\,1}^{3n^k} \hat{w}_\alpha^{fk}(r)\otimes \sum_{i\,1}^{f^k} \hat{p}_{\alpha i}^k s_i^k \\
&= \sum_{\alpha\,1}^{3n^k} \hat{w}_\alpha^{fk}(r)\otimes \hat{s}_\alpha^k = I.
\end{aligned}$$

Eq. (10) is thus proved.

Lastly from the definition of $w_i^{fk}(r)$, from (29) and from (20) it follows

$$\begin{aligned}
\int_\Omega w_i^{fk}(r)\,d\Omega &= \int_\Omega \sum_{\alpha\,1}^{3n^k} \hat{w}_\alpha^{fk}(r)\hat{p}_{\alpha i}^k\,d\Omega \\
&= \sum_{\alpha\,1}^{3n^k} \hat{p}_{\alpha i}^k \int_\Omega \hat{w}_\alpha^{fk}(r)\,d\Omega \\
&= \sum_{\alpha\,1}^{3n^k} \hat{p}_{\alpha i}^k \hat{K}^k \hat{C}_\alpha^k \hat{l}_\alpha^k = \tilde{l}_i^k,\;\; i=1,\ldots,f^k.
\end{aligned}$$

Eq. (11) is thus proved. ■

Functions $w_i^{fk}(r)$, with $i=1,\ldots,f^k$, can thus be used in the energetic approach for constructing the $\mathbf{N}^k$ matrices.

It is noted that

$$\mathbf{N}^k = \hat{\mathbf{P}}^{kT}\hat{\mathbf{N}}^k\hat{\mathbf{P}}^k, \tag{31}$$

in which $\hat{\mathbf{N}}^k$ are $3n^k \times 3n^k$ matrices whose elements are

$$\hat{N}^k_{\alpha\beta} = \int_{\Omega^k} \hat{w}^{fk}_{\alpha}(r) \cdot \nu(r^k) \hat{w}^{fk}_{\beta}(r)\, d\Omega, \quad \alpha, \beta = 1, \ldots, 3n^k.$$

6 Derivation of constants in error bounds

Assumption for choosing the pairs of dual grids are now introduced. For each primal volume, each triple of edges incident into one node, is such that the modulus of the angle formed by any of the edges of the triple with the normal to the plane formed by the other two edges is less than a chosen angle $\vartheta_M < \pi/2$. Such condition can be obtained using common grid generators. Hereafter it is shown that for such a pair of dual grids the disequalities (15)-(18) hold, $R_{\mathbf{E}}$, $R_{\mathbf{H}}$, $R_{\mathbf{N}}$ and $R_{\mathbf{M}}$ being estimated using the results in Section 3.

6.1 Evaluation of the $R_{\mathbf{E}}$ and $R_{\mathbf{H}}$ constants

The constant $R_{\mathbf{E}}$, defined in (15) is evaluated as follows.

Theorem 5 *It results in*

$$R_{\mathbf{E}} = \frac{3}{\cos \vartheta_M}.$$

Proof. Let $\mathbf{v}^k(t)$ be the array of elements $v^k_i(t)$ of the circulations of $e(r,t)$ along the edges Γ^k_i, with $i = 1, \ldots, l^k$, and let $\hat{\mathbf{v}}^k(t)$ be the array of elements $\hat{v}^k_\alpha(t)$, of the circulations of $e(r,t)$ along the edges $\hat{\Gamma}^k_\alpha$, with $\alpha = 1, \ldots, 3n^k$. Then it is

$$\hat{\mathbf{v}}^k(t) = \hat{\mathbf{T}}^k \mathbf{v}^k(t)$$

so that

$$||\mathbf{v}^k(t)||_{\mathbf{E}^k} = ||\hat{\mathbf{v}}^k(t)||_{\hat{\mathbf{E}}^k} = \sqrt{\int_{\Omega^k} \left(\sum_{1}^{3n^k}{}_{\alpha}\, \hat{v}^k_\alpha(t) \hat{w}^{ek}_\alpha(r) \right) \cdot \varepsilon(r^k) \left(\sum_{1}^{3n^k}{}_{\beta}\, \hat{v}^k_\beta(t) \hat{w}^{ek}_\beta(r) \right) d\Omega}.$$

From Lemma 1 and from the assumption on the pair of dual grids it is

$$\begin{aligned}
|\hat{v}^k_\alpha(t) \hat{w}^{ek}_\alpha(r)| &\leq \frac{\hat{K}^k \hat{C}^k_\alpha |\hat{s}^k_\alpha|}{|\tilde{\Omega}^k_{n^k(\alpha)}|} |\hat{l}^k_\alpha| \max_{r \in \Omega^k} |e(r,t)| \\
&= \frac{|\hat{l}^k_\alpha||\hat{s}^k_\alpha|}{\hat{l}^k_\alpha \cdot \hat{s}^k_\alpha} \max_{r \in \Omega^k} |e(r,t)| \\
&\leq \frac{1}{\cos \vartheta_M} \max_{r \in \Omega^k} |e(r,t)|.
\end{aligned}$$

Thus it is

$$\left|\sum_{1}^{3n^k}{}_{\alpha}\hat{v}_{\alpha}^{k}(t)\hat{w}_{\alpha}^{ek}(r)\right| \leq \frac{3}{\cos\vartheta_M}\max_{r\in\Omega^k}|e(r,t)|.$$

so that

$$||\mathbf{v}^k(t)||_{\mathbf{E}^k} \leq \frac{3}{\cos\vartheta_M}\sqrt{|\Omega^k|}\max_{r\in\Omega^k}\sqrt{||\varepsilon(r)||_2}\max_{r\in\Omega^k}|e(r,t)|.$$

from which the thesis follows. ∎

The constant $\hat{R}_{\mathbf{H}}$ in (16) is estimated as follows.

Lemma 4 *Let $\hat{\mathbf{H}}^k$ be the inverse of matrix $\hat{\mathbf{E}}^k$, with $k = 1,\ldots,\nu$. The elements of $\hat{\mathbf{H}}^k$ are*

$$\hat{H}_{\alpha\beta}^{k} = \int_{\Omega^k}\left(\frac{\hat{w}_{\alpha}^{fk}(r)}{\hat{K}^k\hat{C}_{\alpha}^{k}}\right)\cdot\eta(r^k)\left(\frac{\hat{w}_{\beta}^{fk}(r)}{\hat{K}^k\hat{C}_{\beta}^{k}}\right)d\Omega,\quad \alpha,\beta = 1,\ldots,3n^k.$$

Proof. The product $\hat{\mathbf{E}}^k\hat{\mathbf{H}}^k$ has elements

$$\hat{I}_{\alpha\beta} = \sum_{1}^{3n^k}{}_{\gamma}\hat{E}_{\alpha\gamma}^{k}\hat{H}_{\gamma\beta}^{k},\quad \alpha,\beta = 1,\ldots,3n^k.$$

If $n^k(\alpha) \neq n^k(\beta)$ then no γ exists such that $n^k(\alpha) = n^k(\gamma)$ and $n^k(\beta) = n^k(\gamma)$. Thus it cannot be $\hat{E}_{\alpha\gamma}^{k} \neq 0$ and $\hat{H}_{\gamma\beta}^{k} \neq 0$ and $\hat{I}_{\alpha\beta} = 0$ holds. If $n^k(\alpha) = n^k(\beta)$, then from (23), (24) it is

$$\begin{aligned}
\hat{I}_{\alpha\beta} &= \sum_{1}^{3n^k}{}_{\gamma}\hat{K}^k\hat{C}_{\alpha}^{k}\hat{s}_{\alpha}^{k}\cdot\varepsilon(r^k)\left(\frac{\hat{K}^k\hat{C}_{\gamma}^{k}\hat{s}_{\gamma}^{k}}{|\tilde{\Omega}_{n^k(\gamma)}^{k}|}\otimes\hat{l}_{\gamma}^{k}\right)\eta(r^k)\frac{\hat{l}_{\beta}^{k}}{|\tilde{\Omega}_{n^k(\beta)}^{k}|}\\
&= \hat{K}^k\hat{C}_{\alpha}^{k}\hat{s}_{\alpha}^{k}\cdot\varepsilon(r^k)\left(\sum_{1}^{3n^k}{}_{\gamma}\frac{1}{|\Omega^k|}\int_{\Omega^k}\hat{w}_{\gamma}^{ek}(r)\otimes\hat{l}_{\gamma}^{k}d\Omega\right)\eta(r^k)\frac{\hat{l}_{\beta}^{k}}{|\tilde{\Omega}_{n^k(\beta)}^{k}|}\\
&= \hat{K}^k\hat{C}_{\alpha}^{k}\hat{s}_{\alpha}^{k}\cdot\varepsilon(r^k)\frac{1}{|\Omega^k|}\int_{\Omega^k}\left(\sum_{1}^{3n^k}{}_{\gamma}\hat{w}_{\gamma}^{ek}(r)\otimes\hat{l}_{\gamma}^{k}\right)d\Omega\;\eta(r^k)\frac{\hat{l}_{\beta}^{k}}{|\tilde{\Omega}_{n^k(\beta)}^{k}|}\\
&= \hat{K}^k\hat{C}_{\alpha}^{k}\hat{s}_{\alpha}^{k}\cdot\varepsilon(r^k)\eta(r^k)\frac{\hat{l}_{\beta}^{k}}{|\tilde{\Omega}_{n^k(\beta)}^{k}|}\\
&= \frac{\hat{K}^k\hat{C}_{\alpha}^{k}\hat{s}_{\alpha}^{k}\cdot\hat{l}_{\beta}^{k}}{|\tilde{\Omega}_{n^k(\beta)}^{k}|}.
\end{aligned}$$

Thus if $\alpha \neq \beta$, since $\hat{s}_{\alpha}^{k}\cdot\hat{l}_{\beta}^{k} = 0$, then $\hat{I}_{\alpha\beta} = 0$ holds. Otherwise if $\alpha = \beta$, then from Lemma 1, $\hat{I}_{\alpha\beta} = 1$. ∎

Theorem 6 *It results in*

$$R_{\mathbf{H}} = \frac{3}{\cos\vartheta_M}.$$

Proof. Let $\tilde{\psi}^k(t)$ be the array of elements $\tilde{\psi}_i^k(t)$ of the fluxes of $d(r,t)$ through the faces $\tilde{\Sigma}_i^k$, with $i = 1,\dots,l^k$. Let $\hat{\psi}^k(t)$ be the array of elements

$$\hat{\psi}_\alpha^k(t) = \frac{\tilde{s}^k_{l^k(\alpha)} \cdot \hat{K}^k \hat{C}^k_\alpha \hat{s}^k_\alpha}{|\tilde{s}^k_{lk(\alpha)}|^2}\, \tilde{\psi}^k_{l^k(\alpha)}(t), \quad \alpha = 1,\dots,3n^k.$$

Then, from Lemma 1 it results in

$$\begin{aligned}
\sum_{1}^{3n^k}{}_{\alpha}\, \hat{t}_{\alpha i} \hat{\psi}^k_\alpha(t) &= \sum_{1}^{3n^k}{}_{\alpha}\, \hat{t}_{\alpha i} \frac{\tilde{s}^k_i \cdot \hat{K}^k \hat{C}^k_\alpha \hat{s}^k_\alpha}{|\tilde{s}^k_i|^2}\, \tilde{\psi}^k_i(t) \\
&= \frac{\tilde{s}^k_i}{|\tilde{s}^k_i|^2} \cdot \left(\sum_{1}^{3n^k}{}_{\alpha}\, \hat{t}_{\alpha i} \hat{K}^k \hat{C}^k_\alpha \hat{s}^k_\alpha \right) \tilde{\psi}^k_i(t) \\
&= \tilde{\psi}^k_i(t),
\end{aligned}$$

or equivalently

$$\tilde{\psi}^k(t) = \hat{\mathbf{T}}^{kT}\, \hat{\psi}^k(t).$$

Then, by applying Theorem 9 in Appendix A, with $\hat{\mathbf{A}} = \hat{\mathbf{E}}^k$ and $\hat{\mathbf{Q}} = \hat{\mathbf{T}}^k$, it follows

$$||\tilde{\psi}^k(t)||_{\mathbf{H}^k} \le ||\hat{\psi}^k(t)||_{\hat{\mathbf{H}}^k} = \sqrt{\int_{\Omega^k} \left(\sum_{1}^{3n^k}{}_{\alpha}\, \frac{\hat{w}^{fk}_\alpha(r)\hat{\psi}^k_\alpha(t)}{\hat{K}^k \hat{C}^k_\alpha} \right) \cdot \eta(r^k) \left(\sum_{1}^{3n^k}{}_{\beta}\, \frac{\hat{w}^{fk}_\beta(r)\hat{\psi}^k_\beta(t)}{\hat{K}^k \hat{C}^k_\beta} \right) d\Omega}.$$

From Lemma 1 ad from the assumption on the pair of dual grids it is

$$\begin{aligned}
\left| \frac{\hat{w}^{fk}_\alpha(r)}{\hat{K}^k \hat{C}^k_\alpha}\, \hat{\psi}^k_\alpha(t) \right| &\le \frac{|\hat{l}^k_\alpha|}{|\tilde{\Omega}^k_{n^k(\alpha)}|} \frac{|\tilde{s}^k_{l^k(\alpha)}| \hat{C}^k_\alpha \hat{K}^k |\hat{s}^k_\alpha|}{|\tilde{s}^k_{l^k(\alpha)}|^2} |\tilde{s}^k_{l^k(\alpha)}| \max_{r\in\Omega^k} |d(r,t)| \\
&\le \frac{|\hat{l}^k_\alpha||\hat{s}^k_\alpha|}{\hat{l}^k_\alpha \cdot \hat{s}^k_\alpha} \max_{r\in\Omega^k} |d(r,t)| \\
&\le \frac{1}{\cos\vartheta_M} \max_{r\in\Omega^k} |d(r,t)|.
\end{aligned}$$

Thus it is

$$\left| \sum_{1}^{3n^k}{}_{\alpha}\, \frac{\hat{w}^{fk}_\alpha(r)}{\hat{K}^k \hat{C}^k_\alpha}\, \hat{\psi}^k_\alpha(t) \right| \le \frac{3}{\cos\vartheta_M} \max_{r\in\Omega^k} |d(r,t)|$$

so that

$$||\tilde{\psi}^k(t)||_{\mathbf{H}^k} \leq \frac{3}{\cos\vartheta_M}\sqrt{|\Omega^k|}\max_{r\in\Omega^k}\sqrt{||\eta(r)||_2}\max_{r\in\Omega^k}|d(r,t)|$$

from which the thesis follows. ∎

6.2 Evaluation of the $R_{\mathbf{N}}$ and $R_{\mathbf{M}}$ constants

The constant $R_{\mathbf{N}}$, defined in (17) is evaluated as follows.

Theorem 7 *It results in*

$$R_{\mathbf{N}} = \frac{3}{\cos\vartheta_M}.$$

Proof. Let $\varphi^k(t)$ be the array of elements $\varphi_i^k(t)$ of the fluxes of $b(r,t)$ through the faces Σ_i^k, with $i = 1,\ldots,f^k$, and let $\hat{\varphi}^k(t)$ be the array of elements $\hat{\varphi}_\alpha^k(t)$, of the fluxes of $b(r,t)$ through the faces $\hat{\Sigma}_\alpha^k$, with $\alpha = 1,\ldots,3n^k$. Then it is

$$\hat{\varphi}^k(t) = \hat{\mathbf{P}}^k\varphi^k(t)$$

so that

$$||\varphi^k(t)||_{\mathbf{N}^k} = ||\hat{\varphi}^k(t)||_{\hat{\mathbf{N}}^k} = \sqrt{\int_{\Omega^k}\left(\sum_{1}^{3n^k}{}_\alpha\, \hat{\varphi}_\alpha^k(t)\hat{w}_\alpha^{fk}(r)\right)\cdot\nu(r^k)\left(\sum_{1}^{3n^k}{}_\beta\, \hat{\varphi}_\beta^k(t)\hat{w}_\beta^{fk}(r)\right)d\Omega}.$$

From Lemma 1 and from the assumption on the pair of dual grids it is

$$\begin{aligned}
|\hat{\varphi}_\alpha^k(t)\hat{w}_\alpha^{fk}(r)| &\leq \frac{\hat{K}^k\hat{C}_\alpha^k|\hat{l}_\alpha^k|}{|\tilde{\Omega}_{n^k(\alpha)}^k|}|\hat{s}_\alpha^k|\max_{r\in\Omega^k}|b(r,t)| \\
&= \frac{|\hat{l}_\alpha^k||\hat{s}_\alpha^k|}{\hat{l}_\alpha^k\cdot\hat{s}_\alpha^k}\max_{r\in\Omega^k}|b(r,t)| \\
&\leq \frac{1}{\cos\vartheta_M}\max_{r\in\Omega^k}|b(r,t)|.
\end{aligned}$$

Thus it is

$$\left|\sum_{1}^{3n^k}{}_\alpha\, \hat{\varphi}_\alpha^k(t)\hat{w}_\alpha^{fk}(r)\right| \leq \frac{3}{\cos\vartheta_M}\max_{r\in\Omega^k}|b(r,t)|.$$

so that

$$||\varphi^k(t)||_{\mathbf{N}^k} \leq \frac{3}{\cos\vartheta_M}\sqrt{|\Omega^k|}\max_{r\in\Omega^k}\sqrt{||\nu(r)||_2}\max_{r\in\Omega^k}|b(r,t)|.$$

from which the thesis follows. ∎

The constant $\hat{R}_{\mathbf{M}}$, in (18), is estimated as follows.

Lemma 5 *Let $\hat{\mathbf{M}}^k$ be the inverse of matrix $\hat{\mathbf{N}}^k$, with $k = 1,\ldots,\nu$. The elements of $\hat{\mathbf{M}}^k$ are*

$$\hat{M}^k_{\alpha\beta} = \int_{\Omega^k} \left(\frac{\hat{w}^{ek}_{\alpha}(r)}{\hat{K}^k\hat{C}^k_{\alpha}}\right) \cdot \mu(r^k) \left(\frac{\hat{w}^{ek}_{\beta}(r)}{\hat{K}^k\hat{C}^k_{\beta}}\right) d\Omega, \quad \alpha,\beta = 1,\ldots,3n^k.$$

Proof. The product $\hat{\mathbf{N}}^k\hat{\mathbf{M}}^k$ has elements

$$\hat{I}_{\alpha\beta} = \sum_{1}^{3n^k}{}_{\gamma} \hat{N}^k_{\alpha\gamma}\hat{M}^k_{\gamma\beta}, \quad \alpha,\beta = 1,\ldots,3n^k.$$

If $n^k(\alpha) \neq n^k(\beta)$ then no γ exists such that $n^k(\alpha) = n^k(\gamma)$ and $n^k(\beta) = n^k(\gamma)$. Thus it cannot be $\hat{N}^k_{\alpha\gamma} \neq 0$ and $\hat{M}^k_{\gamma\beta} \neq 0$ and $\hat{I}_{\alpha\beta} = 0$ holds. If $n^k(\alpha) = n^k(\beta)$, then from (28), (29) it is

$$\begin{aligned}
\hat{I}_{\alpha\beta} &= \sum_{1}^{3n^k}{}_{\gamma} \hat{K}^k\hat{C}^k_{\alpha}\hat{l}^k_{\alpha} \cdot \nu(r^k) \left(\frac{\hat{K}^k\hat{C}^k_{\gamma}\hat{l}^k_{\gamma}}{|\tilde{\Omega}^k_{n^k(\gamma)}|} \otimes \hat{s}^k_{\gamma}\right) \mu(r^k)\frac{\hat{s}^k_{\beta}}{|\tilde{\Omega}^k_{n^k(\beta)}|} \\
&= \hat{K}^k\hat{C}^k_{\alpha}\hat{l}^k_{\alpha} \cdot \nu(r^k) \left(\sum_{1}^{3n^k}{}_{\gamma} \frac{1}{|\Omega^k|}\int_{\Omega^k} \hat{w}^{fk}_{\gamma}(r) \otimes \hat{s}^k_{\gamma} d\Omega\right) \mu(r^k)\frac{\hat{s}^k_{\beta}}{|\tilde{\Omega}^k_{n^k(\beta)}|} \\
&= \hat{K}^k\hat{C}^k_{\alpha}\hat{l}^k_{\alpha} \cdot \nu(r^k) \frac{1}{|\Omega^k|}\int_{\Omega^k} \left(\sum_{1}^{3n^k}{}_{\gamma} \hat{w}^{fk}_{\gamma}(r) \otimes \hat{s}^k_{\gamma}\right) d\Omega\, \mu(r^k)\frac{\hat{s}^k_{\beta}}{|\tilde{\Omega}^k_{n^k(\beta)}|} \\
&= \hat{K}^k\hat{C}^k_{\alpha}\hat{l}^k_{\alpha} \cdot \nu^k(r^k)\mu(r^k)\frac{\hat{s}^k_{\beta}}{|\tilde{\Omega}^k_{n^k(\beta)}|} \\
&= \frac{\hat{K}^k\hat{C}^k_{\alpha}\hat{l}^k_{\alpha} \cdot \hat{s}^k_{\beta}}{|\tilde{\Omega}^k_{n^k(\beta)}|}.
\end{aligned}$$

Thus if $\alpha \neq \beta$, since $\hat{l}^k_{\alpha} \cdot \hat{s}^k_{\beta} = 0$, then $\hat{I}_{\alpha\beta} = 0$ holds. Otherwise if $\alpha = \beta$, then from Lemma 1, $\hat{I}_{\alpha\beta} = 1$. ■

Theorem 8 *It results in*

$$R_{\mathbf{M}} = \frac{3}{\cos\vartheta_M}.$$

Proof. Let $\tilde{\mathbf{f}}^k(t)$ be the array whose elements $\tilde{f}^k_i(t)$ are the circulations of $h(r,t)$ along the edges $\tilde{\Gamma}^k_i$, with $i = 1,\ldots,f^k$. Let $\hat{\mathbf{f}}^k(t)$ be the array of elements

$$\hat{f}^k_{\alpha}(t) = \frac{\tilde{l}^k_{f^k(\alpha)} \cdot \hat{K}^k\hat{C}^k_{\alpha}\hat{l}^k_{\alpha}}{|\tilde{l}^k_{f^k(\alpha)}|^2}\tilde{f}^k_{f^k(\alpha)}(t), \quad \alpha = 1,\ldots,3n^k.$$

Then, from Lemma 1 it results in

$$\begin{aligned}
\sum_{1}^{3n^k}{}_{\alpha}\, \hat{p}_{\alpha i} \hat{f}^k_{\alpha}(t) &= \sum_{1}^{3n^k}{}_{\alpha}\, \hat{p}_{\alpha i} \frac{\tilde{l}^k_i \cdot \hat{K}^k \hat{C}^k_{\alpha} \hat{l}^k_{\alpha}}{|\tilde{l}^k_i|^2} \tilde{f}^k_i(t) \\
&= \frac{\tilde{l}^k_i}{|\tilde{l}^k_i|^2} \cdot \left(\sum_{1}^{3n^k}{}_{\alpha}\, \hat{p}_{\alpha i} \hat{K}^k \hat{C}^k_{\alpha} \hat{l}^k_{\alpha} \right) \tilde{f}^k_i(t) \\
&= \tilde{f}^k_i(t),
\end{aligned}$$

or equivalently

$$\tilde{\mathbf{f}}^k(t) = \hat{\mathbf{P}}^{kT} \hat{\mathbf{f}}^k(t).$$

Then, by applying Theorem 9 in Appendix A, with $\hat{\mathbf{A}} = \hat{\mathbf{N}}^k$ and $\hat{\mathbf{Q}} = \hat{\mathbf{P}}^k$, it follows

$$||\tilde{\mathbf{f}}^k(t)||_{\mathbf{M}^k} \leq ||\hat{\mathbf{f}}^k(t)||_{\hat{\mathbf{M}}^{\mathbf{k}}} = \sqrt{\int_{\Omega^k} \sum_{1}^{3n^k}{}_{\alpha} \left(\frac{\hat{w}^{ek}_{\alpha}(r) \hat{f}^k_{\alpha}(t)}{\hat{K}^k \hat{C}^k_{\alpha}} \right) \cdot \mu(r^k) \left(\sum_{1}^{3n^k}{}_{\beta} \frac{\hat{w}^{ek}_{\beta}(r) \hat{f}^k_{\beta}(t)}{\hat{K}^k \hat{C}^k_{\beta}} \right) d\Omega}.$$

From Lemma 1 it is

$$\begin{aligned}
\left| \frac{\hat{w}^{ek}_{\alpha}(r)}{\hat{K}^k \hat{C}^k_{\alpha}} \hat{f}^k_{\alpha}(t) \right| &\leq \frac{|\hat{s}^k_{\alpha}|}{|\tilde{\Omega}^k_{n^k(\alpha)}|} \frac{|\tilde{l}^k_{f^k(\alpha)}| \hat{C}^k_{\alpha} \hat{K}^k |\hat{l}^k_{\alpha}|}{|\tilde{l}^k_{f^k(\alpha)}|^2} |\tilde{l}^k_{f^k(\alpha)}| \max_{r \in \Omega^k} |h(r,t)| \\
&\leq \frac{|\hat{s}^k_{\alpha}||\hat{l}^k_{\alpha}|}{\hat{s}^k_{\alpha} \cdot \hat{l}^k_{\alpha}} \max_{r \in \Omega^k} |h(r,t)| \\
&\leq \frac{1}{\cos \vartheta_M} \max_{r \in \Omega^k} |h(r,t)|.
\end{aligned}$$

Thus it is

$$\left| \sum_{1}^{3n^k}{}_{\alpha} \frac{\hat{w}^{ek}_{\alpha}(r)}{\hat{K}^k \hat{C}^k_{\alpha}} \hat{f}^k_{\alpha}(t) \right| \leq \frac{3}{\cos \vartheta_M} \max_{r \in \Omega^k} |h(r,t)|$$

so that

$$||\tilde{\mathbf{f}}^k(t)||_{\mathbf{M}^k} \leq \frac{3}{\cos \vartheta_M} \sqrt{|\Omega^k|} \max_{r \in \Omega^k} \sqrt{||\mu(r)||_2} \max_{r \in \Omega^k} |h(r,t)|$$

from which the thesis follows. ∎

7 Numerical results

A rectangular waveguide of section 5cm × 2.5cm and length 10cm is considered. At one end a TE_{10} electric field is applied. At the other end a PEC termination is applied. The corresponding time domain electromagnetic boundary value problem

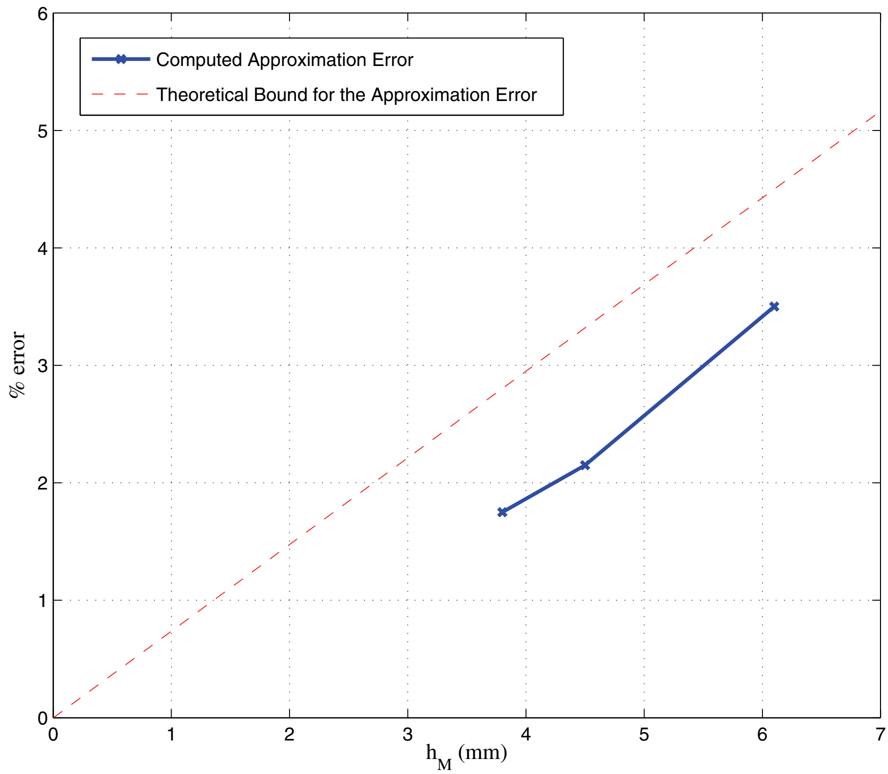

Figure 2: Percent error of the electric field in the energy norm versus the maximum grid diameter.

has been spatially discretized by means of DGA, the oriented primal grid being tetrahedral, the oriented dual grid being its barycentric subdivision and constitutive relations being discretized as in Section 3. The resulting semi-discrete equations have been discretized with respect to time by means of the FD-TD scheme, in the time interval 0ns $\leq t \leq$ 0.95ns. The time step has been chosen in such a way that its effect on the approximate electromagnetic field is negligible. The approximation error in the energy norm for the electromagnetic field has been evaluated at $t =$ 0.95ns for primal grids having different maximum diameters h_M and $\vartheta_M = 1.2$rad. The evaluated percent error, for the electric field, is compared in Fig. 2 with the theoretical error bound estimated by Theorem 2.

8 Conclusions

The paper has proposed a novel and original computation of the analytical values of the constants bounding the approximation error of the electromagnetic field by expressing them in terms of the geometrical properties of the pairs of oriented dual grids. A numerical example confirmed the theoretical analysis.

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Appendix A

If $\mathbf{A}$ is a symmetric, positive definite matrix of order n and $\mathbf{x}_1$, $\mathbf{x}_2$ are a pair of column arrays of n rows, then a scalar product and its corresponding norm in the space of column arrays of n rows are defined by

$$(\mathbf{x}_1, \mathbf{x}_2)_\mathbf{A} = \mathbf{x}_1^T \mathbf{A} \mathbf{x}_2$$
$$||\mathbf{x}_1||_\mathbf{A} = \sqrt{(\mathbf{x}_1, \mathbf{x}_1)_\mathbf{A}} = \sqrt{\mathbf{x}_1^T \mathbf{A} \mathbf{x}_1}.$$

Theorem 9 *Let $\hat{\mathbf{A}}$ be a symmetric, positive definite matrix of order m and let $\hat{\mathbf{Q}}$ be a real, full rank, $m \times n$ matrix with $m \geq n$. Let it be $\mathbf{A} = \hat{\mathbf{Q}}^T \hat{\mathbf{A}} \hat{\mathbf{Q}}$. Then for each real column vector $\hat{\mathbf{x}}$ of m rows*

$$||\mathbf{x}||_{\mathbf{A}^{-1}} \leq ||\hat{\mathbf{x}}||_{\hat{\mathbf{A}}^{-1}} \tag{32}$$

holds, being $\mathbf{x} = \hat{\mathbf{Q}}^T \hat{\mathbf{x}}$.

Proof. For each real column vector $\mathbf{c}$ of n rows it results in

$$\mathscr{H} = (\hat{\mathbf{A}}^{-\frac{1}{2}}\hat{\mathbf{x}} - \hat{\mathbf{A}}^{\frac{1}{2}}\hat{\mathbf{Q}}\mathbf{c})^T(\hat{\mathbf{A}}^{-\frac{1}{2}}\hat{\mathbf{x}} - \hat{\mathbf{A}}^{\frac{1}{2}}\hat{\mathbf{Q}}\mathbf{c}) \geq 0. \tag{33}$$

By expanding the terms in (33) it results in

$$\mathscr{H} = \hat{\mathbf{x}}^T\hat{\mathbf{A}}^{-1}\hat{\mathbf{x}} - 2\hat{\mathbf{x}}^T\hat{\mathbf{Q}}\mathbf{c} + \mathbf{c}^T\hat{\mathbf{Q}}^T\hat{\mathbf{A}}\mathbf{Q}\mathbf{c} \geq 0.$$

In particular, by choosing

$$\mathbf{c} = \mathbf{A}^{-1}\hat{\mathbf{Q}}^T\hat{\mathbf{x}}.$$

it results in

$$\mathscr{H} = \hat{\mathbf{x}}^T\hat{\mathbf{A}}^{-1}\hat{\mathbf{x}} - \mathbf{x}^T\mathbf{A}^{-1}\mathbf{x} \geq 0$$

from which (32) descends. ∎

Let now $A(r)$ be a symmetric, positive definite double tensor defined in a spatial region Ω. If both $||A(r)||_2$ and $||A^{-1}(r)||_2$ are bounded in Ω, then in the space of vector functions square integrable in Ω, a scalar product and its corresponding norm are defined as

$$(x_1(r), x_2(r))_{A(r)} = \int_\Omega x_1(r) \cdot A(r) x_2(r)\, d\Omega$$

$$||x_1||_{A(r)} = \sqrt{(x_1, x_1)_{A(r)}} = \sqrt{\int_\Omega x_1(r) \cdot A(r) x_1(r)\, d\Omega}.$$