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Nonlinear Vibration of the Double-Beams Assembly Subjected to A.C. Electrostatic Force

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Abstract: In this study, the mathematical model of double-beams assembly subjected to the a.c. electrostatic force is established. This is helpful for designing sensors and actuators. The boundary condition of this system is nonlinear and time-dependent. Obviously, this system is very complicated. A new solution method is here developed to derive the analytical solution. Because the a.c. electrostatic force includes the static and harmonic forces, the system is divided into the nonlinear static and dynamic subsystems. The exact static solution is presented. In the other hand, the boundary conditions of the dynamic subsystem are nonlinear and time-dependent. First, using the balanced method the system with time-dependent coefficients is transformed into one with time-independent coefficients. Further, the analytical solution of the transformed dynamic subsystem is derived. It is found that there exists great difference between the linear and nonlinear spectrums. Moreover, the effects of several geometry and material parameters on the frequency spectrums of this system are significant.

Keywords: nonlinear; coupled beam; analytical solution; vibration; electrostatic force

1 Introduction

Advances in electromechanical systems are resulting in new applications ranging from mechanical mass or charge detectors to biological imaging [Kang et al., 2009]. Obviously, the investigation about the frequency shift is very interesting for designing sensors and actuators. Hassanpoura *et al.* (2007) investigated the influence of the concentrated mass on the natural frequency of a beam-type resonator. Kang *et al.* (2009) investigated the ultrahigh frequency nanomechanical resonators based on double-walled carbon nanotubes with different wall lengths. Prabhakar *et al.* (2009) studied the frequency shifts due to thermoelastic coupling in flexural-mode

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single beam as a resonator. The Galerkin technique was used to calculate the thermoelastically shifted frequencies. Forke *et al.* (2008) investigated the electrostatic force coupling of MEMS oscillators for spectral vibration measurements. For measuring vibration, the sensor output signal was designed to be linearly dependent on the amplitude of acceleration. In other words, the electrostatic coupling force was $F_e = 0.5V^2 dC/dz$ where V is the voltage and the spatial derivative of the capacitance, dC/dz, was proportional to the distance z. For this purpose, the comb electrodes were configured with linearly varying finger lengths but in a differential arrangement. Obviously, the above investigations are linear.

In addition to the above applications, the electromechanical system is also applied to the atomic force microscopy. The Kelvin probe force microscopy subjected to the a.c. electrostatic force is currently used to measure contact potential difference and topography of sample on a nanometer scale [Lin, 2007[a,b], Lin and Lin, 2009]. This probe is simulated by a cantilever and subjected to nonlinear time-dependent boundary condition. Moreover, some literatures investigated the linear system with the time-dependent boundary condition [Lee and Lin, 1996 and 1998; Huang and Shih, 2007; Lee *et al.*, 2008[a,b], Lin, 1998, 2002, and 2009[a,b]]. The above literatures are for a single beam.

Further, the double-beams assembly is broadly adopted in civil, mechanical, and aerospace engineering, such as cranes, resonators, spectrometers and interferometers. Some literatures are devoted to this field. Oniszczuk, Z. (2003) investigated the forced vibration of an elastically connected simply supported double-beam system. Gao and Cheng (2005) investigated the active vibration isolation of a two-beams assembly with a piezoelectric actuator. De Rosa and Lippiello (2007) investigated the free vibration of double-beams by using the differential quadrature method. Sadek *et al.* (2007) presented the computational method for solving optimal control of a system of parallel beams. Zhang *et al.* (2008) investigated the linear vibration and buckling of a double-beam system under compressive axial loading. The two coupled beams were simply supported and continuously joined by a Winkler elastic layer. Li and Hua (2008) studied the vibration of an elastically damped connected three-beam system. Obviously, these investigated systems are linear. So far, no literature is devoted to the investigation of the double-beams assembly subjected to the nonlinear a.c. electrostatic force.

In this study, the mathematical model of the double-beams assembly subjected to the a.c. electrostatic force is established. The analytical solution for this system is derived. Moreover, the effects of several geometry and material parameters on the frequency spectrums of this system are investigated.

2 Governing equations and associated conditions

In this study, the nonlinear dynamic model of the double-beams assembly subjected to the a.c. electrostatic force at the end of beam is established as follows:
The governing equation of the first beam is

$$(EI)_1 \frac{\partial^4 W_1}{\partial x_1^4} + (\rho A)_1 \frac{\partial^2 W_1}{\partial t^2} = 0 \tag{1}$$

The corresponding boundary conditions are
At $x_1 = 0$:

$$W_1(0,t) = 0 \tag{2}$$

$$\frac{\partial W_1(0,t)}{\partial x_1} = 0. \tag{3}$$

At $x_1 = L_1$:

$$\frac{\partial^2 W_1(L_1,t)}{\partial x_1^2} = 0, \tag{4}$$

$$(EI)_1 \frac{\partial^3 W_1}{\partial x_1^3} = F_e(t, W_1(L_1,t), W_2(L_2,t)), \tag{5}$$

where the electrostatic force is nonlinear and expressed as

$$F_e = \frac{1}{2}\frac{\partial C}{\partial z}\left[V_{ac}\sin(\Omega_a t)\right]^2 \triangleq F_s + F_a \tag{6}$$

where $\partial C/\partial z$ is the spatial derivative of the capacitance between the two beams in which C is the capacity and z is the distance between the two parallel plates and

$$F_s = \frac{\partial C}{\partial z}\frac{V_{ac}^2}{4}, \quad F_a = -\frac{\partial C}{\partial z}\frac{V_{ac}^2}{4}\cos(2\Omega_a t). \tag{7}$$

It should be noted that F_s is considered as the static external force and F_a is the harmonic excitation force with the frequency of $2\Omega_a$. For a parallel-plate capacitor at the ends of the two beams the spatial derivative of the capacitance is

$$\frac{\partial C}{\partial z} = \frac{-\varepsilon_0 A}{2\left[d_0 + (w_1(L_1,t) - w_2(L_2,t))\right]^2} \tag{8}$$

where A is the capacitor area, d_0 is the distance between the tips of two beams and ε_0 is the vacuum permittivity.

In the other hand, the governing equation of the second beam is

$$(EI)_2 \frac{\partial^4 W_2}{\partial x_2^4} + (\rho A)_2 \frac{\partial^2 W_2}{\partial t^2} = 0, \tag{9}$$

At $x_2 = 0$:

$$W_2(0,t) = 0, \tag{10}$$

$$\frac{\partial W_2(0,t)}{\partial x_2} = 0, \tag{11}$$

At $x_2 = L_2$:

$$\frac{\partial^2 W_2(L_2,t)}{\partial x_2^2} = 0, \tag{12}$$

$$(EI)_2 \frac{\partial^3 W_2}{\partial x_2^3} = -F_e(t, W_1(L_1,t), W_2(L_2,t)). \tag{13}$$

In terms of the following dimensionless parameters

$$b_{ij} = \frac{(EI)_i}{(EI)_j}, \quad c_{ac} = \frac{\varepsilon_0 A V_{ac}^2}{4(EI)_2}, \quad \bar{d}_0 = \frac{d_0}{L_2}$$

$$l_{ij} = \frac{L_i}{L_j}, \quad m_{ij} = \frac{(\rho A)_i}{(\rho A)_j}, \quad w(\xi,\tau) = \frac{W(x,t)}{L},$$

$$\xi = \frac{x}{L}, \quad \tau = \frac{t}{L_2^2}\sqrt{\frac{(EI)_2}{(\rho A)_2}}, \quad \omega_a = \Omega_a L_2^2 \sqrt{\frac{(\rho A)_2}{(EI)_2}}, \tag{14}$$

the dimensionless governing differential equation of the first beam is

$$\frac{\partial^4 w_1}{\partial \xi_1^4} + b_{21} m_{12} l_{12}^4 \frac{\partial^2 w_1}{\partial \tau^2} = 0 \tag{15}$$

The dimensionless boundary conditions are
At ξ_1 =0:

$$w_1(0,\tau) = 0, \tag{16}$$

$$\frac{\partial w_1(0,\tau)}{\partial \xi_1} = 0. \tag{17}$$

At $\xi_1 = 1$:

$$\frac{\partial^2 w_1(1,\tau)}{\partial \xi_1^2} = 0, \tag{18}$$

$$b_{12}\frac{\partial^3 w_1(1,\tau)}{\partial \xi_1^3} = c_{ac}\left[-1+\cos(2\omega_a\tau)\right] f_e(\tau), \tag{19}$$

where $f_e(\tau) = 1/\left[l_{21}\bar{d}_0 + (\bar{w}_1(1,\tau) - l_{21}\bar{w}_2(1,\tau))\right]^2$.
Similarly, the dimensionless governing equation of the second beam is

$$\frac{\partial^4 w_2}{\partial \xi_2^4} + \frac{\partial^2 w_2}{\partial \tau^2} = 0 \tag{20}$$

The corresponding dimensionless boundary conditions are
At ξg$_2$= 0:

$$\frac{\partial \Psi}{\partial \xi} - \beta_1\Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \tag{21}$$

$$\frac{\partial \Psi}{\partial \xi} - \beta_1\Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \tag{22}$$

At ξ_yg 1:

$$\frac{\partial \Psi}{\partial \xi} - \beta_1\Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \tag{23}$$

$$\frac{\partial^3 w_2(1,\tau)}{\partial \xi_2^3} = -c_{ac}l_{21}^2\left[-1+\cos(2\omega_a\tau)\right] f_e(\tau) \tag{24}$$

3 Solution method

3.1 Change of variables

Based on Eqs. (6-8), the external electrostatic force is divided into the static and dynamic ones. Therefore, the responses of the double-beams assembly include the static and dynamic parts as follows:

$$w_1(\xi,\tau) = w_{1s}(\xi) + w_{1d}(\xi,\tau), \tag{25}$$

$$w_2(\xi,\tau) = w_{2s}(\xi) + w_{2d}(\xi,\tau), \tag{26}$$

where w_{is} is the static solution and w_{id} is the dynamic one, $w_{id0}\cos(2\omega_a\tau)$. Substituting Eq. (25) into Eqs. (15-24), one obtains

$$\frac{\partial^4 w_{1d}}{\partial \xi_1^4}+\frac{d^4 w_{1s}}{d\xi_1^4}+b_{21}m_{12}l_{12}^4\frac{\partial^2 w_{1d}}{\partial \tau^2}=0 \tag{27}$$

at ξg$_1$= 0:

$$\frac{\partial \Psi}{\partial \xi}-\beta_1\Psi=-\beta_1 f_1(\tau)-f_1^*(\tau), \tag{28}$$

$$\frac{\partial \Psi}{\partial \xi}-\beta_1\Psi=-\beta_1 f_1(\tau)-f_1^*(\tau), \tag{29}$$

At ξ_xg 1:

$$\frac{\partial \Psi}{\partial \xi}-\beta_1\Psi=-\beta_1 f_1(\tau)-f_1^*(\tau), \tag{30}$$

$$b_{12}\frac{\partial^3 w_{1d}(1,\tau)}{\partial \xi_1^3}+b_{12}\frac{d^3 w_{1s}(1)}{d\xi_1^3}=c_{ac}\left[-1+\cos(2\omega_a\tau)\right]f_e(\tau) \tag{31}$$

It should be noted that the solutions $\{\frac{\partial \Psi}{\partial \xi}-\beta_1\Psi=-\beta_1 f_1(\tau)-f_1^*(\tau)\}$ are coupled. Moreover, the system is nonlinear. Therefore, it is very difficult to solve. However, if the system is subjected to the static external force only, the corresponding boundary condition is

$$b_{12}\frac{d^3 w_{1s}(1)}{d\xi_1^3}=-c_{ac}f_{es} \tag{32}$$

where $f_{es}=1/\left[l_{21}\bar{d}_0+(\bar{w}_{1s}(1)-l_{21}\bar{w}_{2s}(1))\right]^2$. Based on this condition, the system composed of Eqs. (26-30) can be divided into the static and dynamic ones as follows:

The transformed static system of the first beam in terms of $\{w_{1s},w_{2s}\}$:

$$\frac{d^4 w_{1s}}{d\xi_1^4}=0. \tag{33}$$

The boundary conditions are

at ξg$_1$= 0:

$$\frac{\partial \Psi}{\partial \xi}-\beta_1\Psi=-\beta_1 f_1(\tau)-f_1^*(\tau), \tag{34}$$

$$\frac{\partial \Psi}{\partial \xi} - \beta_1 \Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \tag{35}$$

At ξ_xg 1:

$$\frac{\partial \Psi}{\partial \xi} - \beta_1 \Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \tag{36}$$

$$b_{12} \frac{d^3 \bar{w}_{1s}(1)}{d\xi_1^3} = -c_{ac} f_{es} \tag{37}$$

The transformed dynamic system of the first beam in terms of $\{w_{1s}, w_{2s}, w_{1d}, w_{2d}\}$:

$$\frac{d^4 w_{1d0}}{d\xi_1^4} - 4\omega_a^2 b_{21} m_{12} l_{12}^4 w_{1d0} = 0 \tag{38}$$

The dimensionless boundary conditions are
At ξg$_1$= 0:

$$\frac{\partial \Psi}{\partial \xi} - \beta_1 \Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \tag{39}$$

$$\frac{\partial \Psi}{\partial \xi} - \beta_1 \Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \tag{40}$$

At ξ_xg 1:

$$\frac{\partial \Psi}{\partial \xi} - \beta_1 \Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \tag{41}$$

$$b_{12} \frac{d^3 w_{1d0}}{d\xi_1^3} \cos(2\omega_a \tau) = c_{ac} f_{es} + c_{ac} [-1 + \cos(2\omega_a \tau)] f_e(\tau) \tag{42}$$

Multiplying Eq. (41)gy cosg$2\omega_a \tau$ and integrating it from 0 to the period T, 2πvyω_a, Eq. (41) becomes

$$b_{12} \frac{d^3 w_{1d0}}{d\xi_1^3} = c_{ac} f_{es} - \frac{c_{ac}}{\pi} \left(\tilde{f}_{e1} - \tilde{f}_{e2} \right), \tag{43}$$

where

$$f_e(\chi) = \frac{1}{\left[l_{21} \bar{d}_0 + ([w_{1s}(1) + w_{1d0}(1) \cos\chi] - l_{21} [w_{2s}(1) + w_{2d0}(1) \cos\chi]) \right]^2},$$

$$\tilde{f}_{e1} = \int_0^{2\pi} f_e(\chi)\cos\chi d\chi, \quad \tilde{f}_{e2} = \int_0^{2\pi} \cos^2(\chi) f_e(\chi) d\chi.$$

In the same way, the transformed static system of the second beam in terms of $\{w_{1s}, w_{2s}\}$:

$$\frac{d^4\bar{w}_{2s}}{d\xi_2^4} = 0. \tag{44}$$

The dimensionless boundary conditions are

at ξg$_2$= 0:

$$\frac{\partial\Psi}{\partial\xi} - \beta_1\Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \tag{45}$$

$$\frac{\partial\Psi}{\partial\xi} - \beta_1\Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \tag{46}$$

At $\xi_{\rm y}$g 1:

$$\frac{\partial\Psi}{\partial\xi} - \beta_1\Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \tag{47}$$

gg $$\frac{d^3\bar{w}_{2s}(1)}{d\xi_2^3} = c_{ac} l_{21}^2 f_{es}. \tag{48}$$

The transformed dynamic system of the second beam in terms of $\{w_{1s}, w_{2s}, w_{1d}, w_{2d}\}$:

$$\frac{d^4 w_{2d0}}{d\xi_2^4} - 4\omega_a^2 w_{2d0} = 0 \tag{49}$$

The dimensionless boundary conditions are

At ξg$_2$= 0:

$$\frac{\partial\Psi}{\partial\xi} - \beta_1\Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \tag{50}$$

$$\frac{\partial\Psi}{\partial\xi} - \beta_1\Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \tag{51}$$

At $\xi_{\rm y}$g 1:

$$\frac{\partial\Psi}{\partial\xi} - \beta_1\Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \tag{52}$$

$$\frac{d^3 w_{2d0}}{d\xi_2^3}\cos(2\omega_a\tau) = -c_{ac}l_{21}^2 f_{es} - c_{ac}l_{21}^2\left[-1+\cos(2\omega_a\tau)\right]f_e(\tau). \tag{53}$$

Multiplying Eq. (52)gy cosg2$\omega_a\tau$ and integrating it from 0 to the period T, 2πvyω_a, Eq. (52) becomes

$$\frac{d^3 w_{2d0}}{d\xi_2^3} = -c_{ac}l_{21}^2 f_{es} + \frac{c_{ac}l_{21}^2}{\pi}\left(\tilde{f}_{e1}-\tilde{f}_{e2}\right) \tag{54}$$

3.2 Exact static solutions

The coupled static system is composed of Eqs. (32-36) and (43-47). One assumes the static solutions are

$$\frac{\partial\Psi}{\partial\xi} - \beta_1\Psi = -\beta_1 f_1(\tau) - f_1^*(\tau) \tag{55}$$

$$\frac{\partial\Psi}{\partial\xi} - \beta_1\Psi = -\beta_1 f_1(\tau) - f_1^*(\tau) \tag{56}$$

Substituting the solution (54a) into the boundary conditions (33-35), one obtains

$$\frac{\partial\Psi}{\partial\xi} - \beta_1\Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \quad C_2 = -3C_3 \text{ and } w_{1s}(1) = -2C_3 \tag{57}$$

Similarly, substituting the solution (52b) into the boundary conditions (44-46), the coefficients are

$$\frac{\partial\Psi}{\partial\xi} - \beta_1\Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \quad \bar{C}_2 = -3\bar{C}_3 \text{ and } w_{2s}(1) = -2\bar{C}_3 \tag{58}$$

Further, substituting the solutions (54-56) back into the boundary conditions (36) and (47), two implicit relations are found

$$6b_{12}C_3 = \frac{-c_{ac}}{\left[l_{21}\bar{d}_0 - 2\left(C_3 - l_{21}\bar{C}_3\right)\right]^2}, \tag{59}$$

and

$$6\bar{C}_3 = \frac{c_{ac}l_{21}^2}{\left[l_{21}\bar{d}_0 - 2\left(C_3 - l_{21}\bar{C}_3\right)\right]^2}. \tag{60}$$

Based on these relations, the coefficients $\{C_3, \bar{C}_3\}$ will be determined. So far, the static solutions $\{w_{1s}, w_{2s}\}$ have been derived.

3.3 Dynamic solutions

The dynamic solution of Eq. (37) is expressed as

$$w_{1d0}\left(\xi_1\right)=\alpha_1 V_1\left(\xi_1\right)+\alpha_2 V_2\left(\xi_1\right)+\alpha_3 V_3\left(\xi_1\right)+\alpha_4 V_4\left(\xi_1\right) \tag{61}$$

where

$$V_1\left(\xi_1\right)=\frac{1}{2}\left(\cosh\varepsilon\xi_1+\cos\varepsilon\xi_1\right),\quad V_2\left(\xi_1\right)=\frac{1}{2\varepsilon}\left(\sinh\varepsilon\xi_1+\sin\varepsilon\xi_1\right),$$

$$V_3\left(\xi_1\right)=\frac{1}{2\varepsilon^2}\left(\cosh\varepsilon\xi_1-\cos\varepsilon\xi_1\right),\quad V_4\left(\xi_1\right)=\frac{1}{2\varepsilon^3}\left(\sinh\varepsilon\xi_1-\sin\varepsilon\xi_1\right), \tag{62}$$

$$\varepsilon=l_{12}\left(4\omega_a^2 b_{21} m_{12}\right)^{1/4}.$$

V_i are the fundamental solutions of equation (37) and satisfy the following normalized condition

$$\begin{bmatrix} V_1 & V_2 & V_3 & V_4 \\ V_1' & V_2' & V_3' & V_4' \\ V_1'' & V_2'' & V_3'' & V_4'' \\ V_1''' & V_2''' & V_3''' & V_4''' \end{bmatrix}_{\xi_1=0}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{63}$$

Substituting the solution (58) into the boundary conditions (38-40), the coefficients are

$$\frac{\partial\Psi}{\partial\xi}-\beta_1\Psi=-\beta_1 f_1(\tau)-f_1^*(\tau),\quad \alpha_3=-\frac{V_4''(1)}{V_3''(1)}\alpha_4,$$

$$\frac{\partial\Psi}{\partial\xi}-\beta_1\Psi=-\beta_1 f_1(\tau)-f_1^*(\tau). \tag{64}$$

Further, substituting Eq. (61) into (42), the implicit relation is found

$$b_{12}\alpha_4\left[-\frac{V_4''(1)}{V_3''(1)}V_3'''(1)+V_4'''(1)\right]=c_{ac}f_{es}-\frac{c_{ac}}{\pi}\left(\tilde{f}_{e1}-\tilde{f}_{e2}\right) \tag{65}$$

In the same way, the dynamic solution of Eq. (48) is expressed as

$$w_{2d0}\left(\xi_2\right)=\beta_1\bar{V}_1\left(\xi_2\right)+\beta_2\bar{V}_2\left(\xi_2\right)+\beta_3\bar{V}_3\left(\xi_2\right)+\beta_4\bar{V}_4\left(\xi_2\right) \tag{66}$$

where

$$\bar{V}_1\left(\xi_2\right)=\frac{1}{2}\left(\cosh\bar{\varepsilon}\xi_2+\cos\bar{\varepsilon}\xi_2\right),\quad \bar{V}_2\left(\xi_2\right)=\frac{1}{2\bar{\varepsilon}}\left(\sinh\bar{\varepsilon}\xi_2+\sin\bar{\varepsilon}\xi_2\right),$$

$$\bar{V}_3(\xi_2) = \frac{1}{2\bar{\varepsilon}^2}(\cosh\bar{\varepsilon}\xi_2 - \cos\bar{\varepsilon}\xi_2), \quad \bar{V}_4(\xi_2) = \frac{1}{2\bar{\varepsilon}^3}(\sinh\bar{\varepsilon}\xi_2 - \sin\bar{\varepsilon}\xi_2), \tag{67}$$

$\bar{\varepsilon} = \sqrt{2\omega_a}$.

$\bar{V}_i$ are the fundamental solution of equation (48) and satisfy the following normalized condition

$$\begin{bmatrix} \bar{V}_1 & \bar{V}_2 & \bar{V}_3 & \bar{V}_4 \\ \bar{V}'_1 & \bar{V}'_2 & \bar{V}'_3 & \bar{V}'_4 \\ \bar{V}''_1 & \bar{V}''_2 & \bar{V}''_3 & \bar{V}''_4 \\ \bar{V}'''_1 & \bar{V}'''_2 & \bar{V}'''_3 & \bar{V}'''_4 \end{bmatrix}_{\xi_2=0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{68}$$

Substituting the solution (63) into the boundary conditions (49-51), the coefficients are

$$\frac{\partial\Psi}{\partial\xi} - \beta_1\Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \quad \beta_3 = -\frac{\bar{V}''_4(1)}{\bar{V}''_3(1)}\beta_4,$$

$$\frac{\partial\Psi}{\partial\xi} - \beta_1\Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \tag{69}$$

Further, substituting Eq. (66) into Eq. (53), another implicit relation is found

$$\beta_4\left[-\frac{\bar{V}''_4(1)}{\bar{V}''_3(1)}\bar{V}'''_3(1) + \bar{V}'''_4(1)\right] = -c_{ac}l_{21}^2 f_{es} + \frac{c_{ac}l_{21}^2}{\pi}\left(\tilde{f}_{e1} - \tilde{f}_{e2}\right) \tag{70}$$

Multiplying (62) by l_{21}^2 and adding it to (67), one gets the relation between the coefficients

$$\beta_4 = \delta\alpha_4 \tag{71}$$

where $\delta = -b_{12}l_{21}^2\left[-\frac{V''_4(1)}{V''_3(1)}V'''_3(1) + V'''_4(1)\right] / \left[-\frac{\bar{V}''_4(1)}{\bar{V}''_3(1)}\bar{V}'''_3(1) + \bar{V}'''_4(1)\right]$.
Substituting the relation (68) back into (62), the characteristic equation in terms of α only is derived

$$b_{12}\alpha_4\left[-\frac{V''_4(1)}{V''_3(1)}V'''_3(1) + V'''_4(1)\right] = c_{ac}f_{es} - \frac{c_{ac}}{\pi}\left(\tilde{f}_{e1} - \tilde{f}_{e2}\right) \tag{72}$$

where the coefficients become

$$\tilde{f}_{e1} = \int_0^{2\pi}\cos\chi f_e(\chi)d\chi, \quad \tilde{f}_{e2} = \int_0^{2\pi}\cos^2(\chi) f_e(\chi)d\chi,$$

$$f_e(\chi) = \frac{1}{\left[l_{21}\bar{d}_0 + \left(\begin{array}{l} \left[w_{1s}(1) + \alpha_4 \left[-\frac{V_4''(1)}{V_3''(1)} V_3(1) + V_4(1) \right] \cos\chi \right] \\ -l_{21} \left[w_{2s}(1) + \delta\alpha_4 \left[-\frac{\bar{V}''_4(1)}{\bar{V}''_3(1)} \bar{V}_3(1) + \bar{V}_4(1) \right] \cos\chi \right] \end{array} \right) \right]^2}. \tag{73}$$

In conclusion, via Eq. (69), the coefficient can be easily calculated by using the method given by Lin [ ]. Substituting these coefficients back into Eqs. (58) and (63), the analytical dynamic solutions are found.

4 Numerical results and discussion

The effects of several parameters on the frequency spectrums of the double-beams assembly are investigated here.

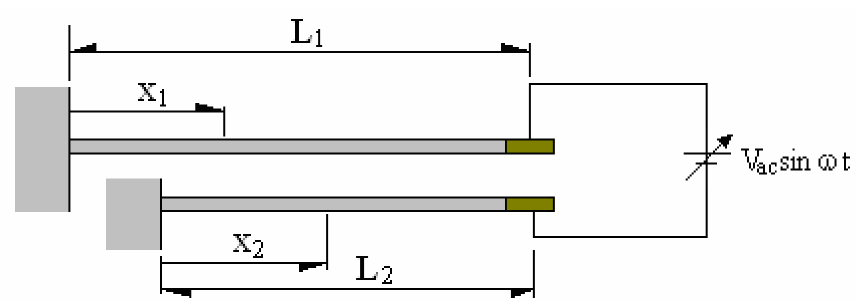


Figure 1: Geometry and coordinate system of the double- beams assembly subjected to a.c. electrostatic force.

Table 1:

		$b_{21}=0.5,$	$l_{12}=m_{12}=1$	$b_{21}=1,$	$l_{12}=2,\ m_{12}=1$
mode	ω_{2j}(#)	ω_{1j}(#)	ω_{1j}(*)	ω_{1j}(#)	ω_{1j}(*)
1	3.5160	4.9724	4.9724	0.8790	0.8790
2	22.0345	31.1615	31.1615	5.5086	5.5086
3	61.6972	87.2530	87.2530	15.4243	15.4243
4	120.9019	170.9811	170.9811	30.2254	30.2254
5	199.8595	282.6441	282.6441	49.9649	49.9649

(*): determined via the relation (A-11)
(#): the method given by Lin (2005)

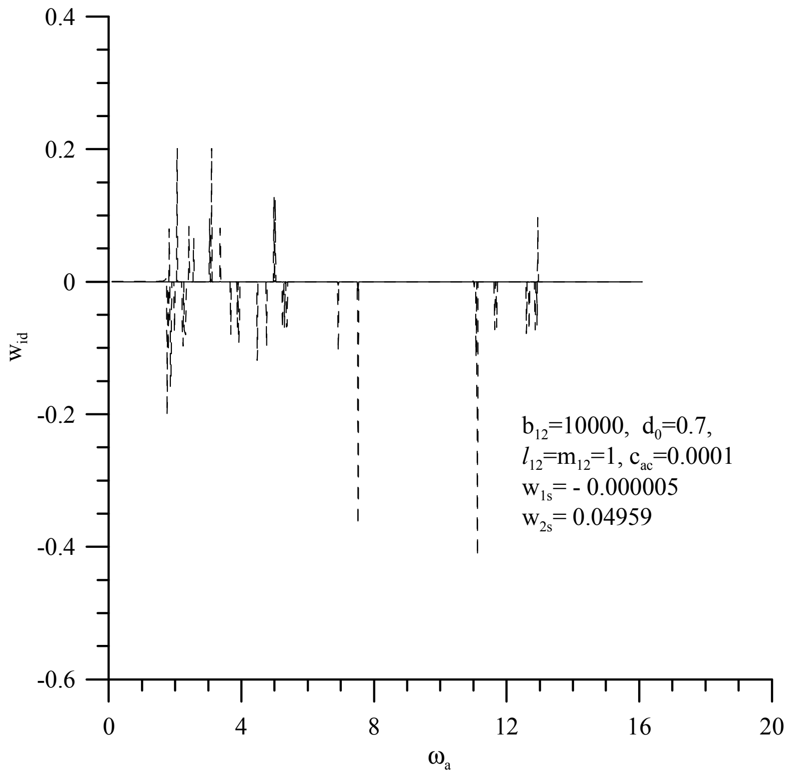


Figure 2: Influence of the a.c. frequency ω_{ac} on the dynamic tip amplitudes of the double- beams assembly with very large bending rigidity ratio [real line: for the first beam; dashed line: for the second beam].

At first, the free vibration of the independent beam is discussed in the Appendix. Considering the definitions of the dimensionless parameters (14), the relation between the dimensionless natural frequencies of these two beams is expressed as Eq. (A-11). This relation is verified in Table 1. Figure 2 shows the spectrum of two beams subjected to the a.c. electrostatic force. The bending rigidity of the first beam is ten thousand times of that of the second one. It is implied that the first one is almost rigid and the second beam becomes an independent one subjected to the a.c. electrostatic force. Figure 2 demonstrates that the static and dynamic deflections of the first beam are negligible because it is almost rigid. Moreover, there exist many resonant points of the second beam. According to the natural frequencies of a beam listed in Table 1, if the second independent beam is excited by the external harmonic linear force, there should exists the resonant point at its natural frequency only. It should be noted that because the harmonic electrostatic force is $F_a = -0.25\,(\partial C/\partial z)\,V_{ac}^2 \cos(2\Omega_a t)$, the a.c frequency Ω_a is equal to $0.5\omega_{1j}$ or $0.5\omega_{2j}$, in the linear viewpoint the resonant phenomena will occur. Figure 2 shows that in addition to these linear resonant phenomena there exist many resonant points due to the nonlinear a.c. electrostatic force.

Figure 3 shows the spectrum of two coupled beams with the same geometry and material properties and subjected to the a.c. electrostatic force. Obviously, there is great difference between the spectrums shown in Figures 2 and 3. For the case with the distance d_0 = 0.5 in Figure 3 there are a large resonant domain (1.740, 2.820) and a little one. There does not exist other resonant points which are different to that shown in Figure 2. If the distance d_0 = 0.8, the first resonant domain significantly contracts.

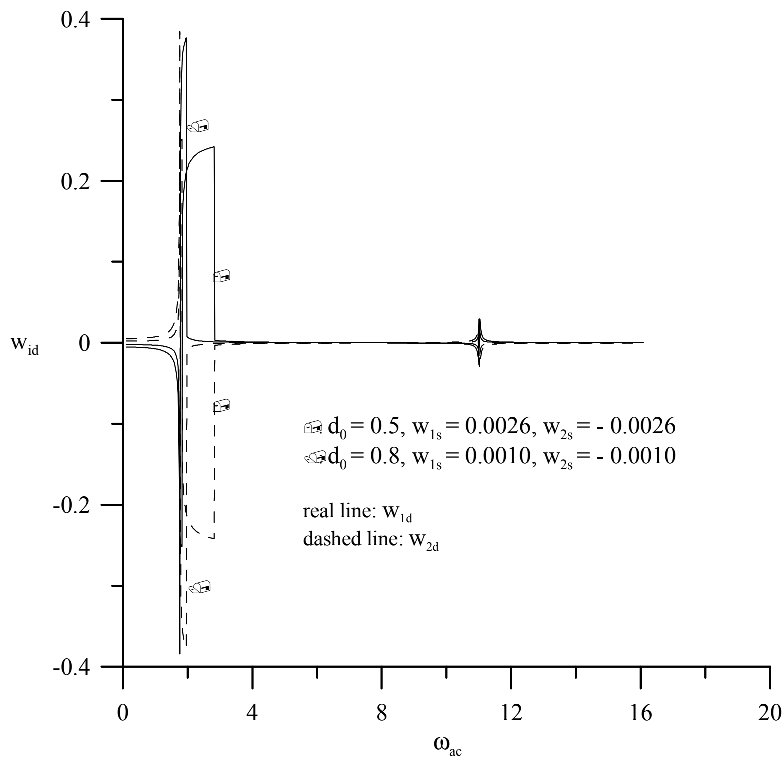
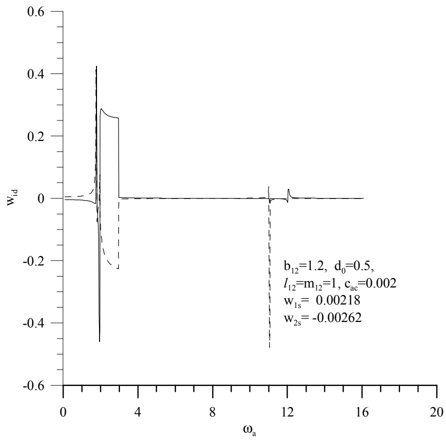
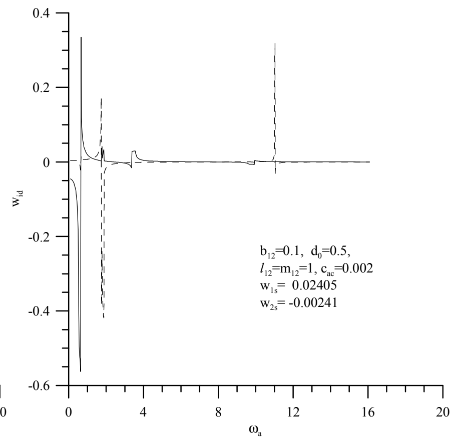


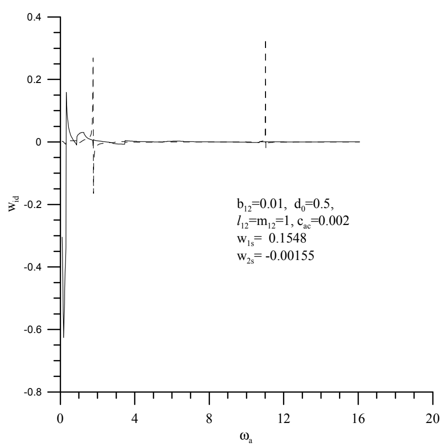
Figure 3: Influence of the a.c. frequency ω_{ac} and the distance d_0 between two beams on the dynamic tip amplitudes of the double- beams assembly with the same geometry and material properties [real line: for the first beam; dashed line: for the second beam].



(a)



(b)



(c)

Figure 4: Influence of the a.c. frequency ω_{ac} and the bending rigidity ratio b_{12} on the dynamic tip amplitudes of the double- beams assembly [real line: for the first beam; dashed line: for the second beam].

Figure 4 demonstrates the influence of the bending rigidity ratio b_{12} on the spectrum of the coupled beams. It is observed from Figure 3 and 4a that for the bending rigidity ratio b_{12} = 1.2 the first resonant domain change slightly and the second one becomes two resonant points. When bending rigidity ratio is changed to be b_{12} = 0.1, as shown in Figure 4b, the first resonant domain becomes several resonant points. When bending rigidity ratio is decreased to be b_{12} = 0.01, as shown in

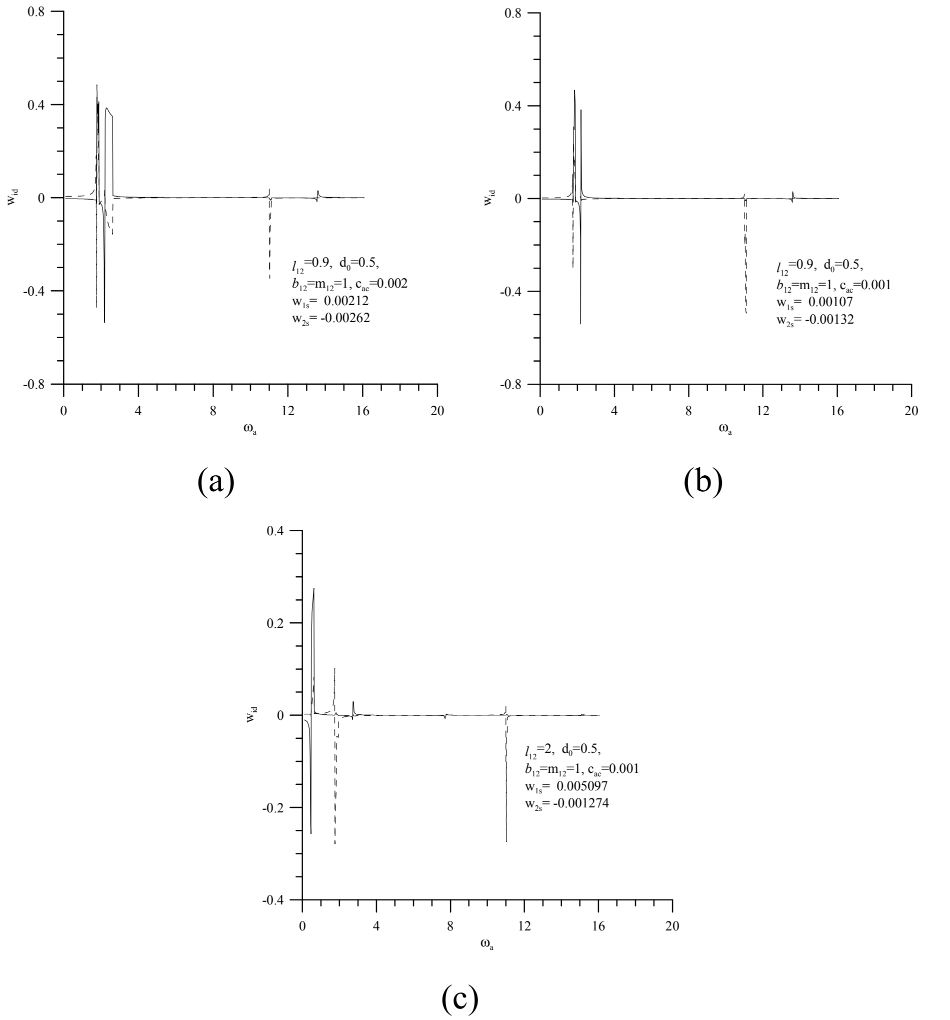


Figure 5: Influence of the a.c. frequency ω_{ac} and the beam length ratio l_{12} on the dynamic tip amplitudes of the double- beams assembly [real line: for the first beam; dashed line: for the second beam].

Figure 4c, the first and second resonant domains change further.

Figure 5a shows the influence of the length ratio l_{12} on the spectrum of the coupled beams. It is observed from Figures 3 and 5a that for the bending rigidity ratio l_{12} = 0.9 the first resonant domain separates obviously and the second one becomes two resonant points. Moreover, it is observed from Figures 5a and 5b that the resonant domain changes with the electrostatic coefficient. Further, if the length ratio is increased to be $l_{12} = 2$, as shown in Figure 5c, its spectrum is significantly different to that in Figure 5b.

5 Conclusion

In this study, the mathematical model of a double-beams assembly subjected to the a.c. nonlinear electrostatic force is established. The analytical solution for this system is presented. It is discovered that there is a large resonant domain for this system with two same beams. The spectrum of the system significantly changes with the bending rigidity ratio, the length ratio and the distance between two beams. Due to the variations of these parameters, the resonant domain will shift and even separate many resonant points. Moreover, it is well known that for a linear beam is excited by the harmonic force, the resonant point occurs at its natural frequency only. However, because the a.c. electrostatic force is nonlinear, there results in many resonant points in addition to these linear resonant phenomena.

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References

De Rosa, M.A.; Lippiello, M. (2007): Non-classical boundary conditions and DQM for double-beams. Mechanics Research Communications, Vol. 34, pp. 538–544.

Forke, R.; Dirk Scheibner, D.; Mehner, J.E.; Thomas Gessner, T.; Wolfram Dötzel, W. (2008): Electrostatic force coupling of MEMS oscillators for spectral vibration measurements. Sensors and Actuators A, Vol. 142, pp. 276–283.

Gao, J.X.; Cheng, L. (2005): Modeling and control of power flow in a double-beam system. Thin-Walled Structures, Vol. 43, pp. 751–771.

Hassanpoura, P.A.; Cleghorna, W.L.; Esmailzadehb, E.; Millsa, J.K. (2007): Vibration analysis of micro-machined beam-type resonators. Journal of Sound and Vibration, Vol. 308, pp. 287–301.

Huang, C.H.; Shih, C.C. (2007): An inverse problem in estimating simultaneously the time-dependent applied force and moment of an Euler-Bernoulli beam. *CMES: Computer Modeling in Engineering and Science*, vol.21, no.3, pp.239-254.

Kang, J.W.; Kwon, O.K; Lee, J.H.; Choi, Y.G.; Hwang, H.J. (2009): Frequency change by inter-walled length difference of double-wall carbon nanotube resonator. Solid State Communications, Vol. 149, pp. 1574-1577.

Lee, S.Y.; Lin, S.M. (1996): Dynamic analysis of non-uniform beam with time dependent elastic boundary conditions. *ASME Journal of Applied Mechanics*, vol. 63, no.2, pp. 474-478.

Lee, S.Y.; Lin, S.M. (1998): Nonuniform Timoshenko beams with time dependent elastic boundary conditions. *Journal of Sound and Vibration*, vol. 217, no. 2, pp. 223-238.

Lee, S.Y.; Lin, S.M.; Lee, C.S.; Lu, S.Y.; Liu, Y.T. (2008[a]): Exact Large Deflection of Beams with Nonlinear Boundary Conditions. *CMES: Computer Modeling in Engineering and Science*, vol.30, no.1, pp.27-36.

Lee, S.Y.; Lu, S.Y.; Liu, Y.T.; Huang, H.C. (2008[b]): Exact Large Deflection Solutions for Timoshenko Beams with Nonlinear Boundary Conditions. *CMES: Computer Modeling in Engineering and Science*, vol.33, no.3, pp.219-312.

Li, J.; Hua, H. (2008): Dynamic stiffness vibration analysis of an elastically connected three-beam system. Applied Acoustics, Vol. 69, pp. 591–600.

Lin, S.M. (1998): Pretwisted nonuniform beams with time dependent elastic boundary conditions. *AIAA Journal*, vol.36, no.8, pp.1516-1523.

Lin, S.M.; Lee, S.Y. (2002): The forced vibration and boundary control of pretwisted Timoshenko beams with general time dependent boundary conditions. *Journal of Sound and Vibration*, vol. 254, no.27, pp. 69-90.

Lin, S.M. (2005): Exact Solution of the frequency shift in dynamic force microscopy. *Applied Surface Science*, Vol. 250, pp. 228-237.

Lin, S.M. (2007[a]): Analytical solutions of the frequency shifts of several modes in dynamic force microscopy Subjected to AC electrostatic force. *IEEE Transaction Nanotechnology*, July, Vol.6, Issue 4, pp. 404-412.

Lin, S.M. (2007[b]): Energy dissipation and dynamic response of an AM-AFM subjected to a tip-sample viscous force. *Ultramicroscopy*, Vol. 107, pp.245-253.

Lin, S.M. (2009[a]): Vibration suppression of a moving beam subjected to an active-control electrostatic force. *CMES: Computer Modeling in Engineering and Science*, Vol. 43, No. 1, pp. 73-90.

Lin, S.M. (2009[b]): Vibrations of in-plane non-constant inward and outward rotating beams. *CMES: Computer Modeling in Engineering and Science*, Vol. 52, No. 1, pp. 105-124.

Lin, S.M. and Lin, C.C. (2009): Phase shifts and energy dissipations of several modes of AFM: Minimizing topography and dissipation measurement errors. *Precision Engineering*, Vol. 33, pp. 371-377.

Oniszczuk, Z. (2003): Forced transverse vibrations of an elastically connected complex simply supported double-beam system. Journal of Sound and Vibration, Vol. 264, pp. 273–286.

Prabhakar, S.; Paidoussis, M.P.; Vengallatore, S. (2009): Analysis of frequency shifts due to thermoelastic coupling in flexural-mode micromechanical and nanomechanical resonators. Journal of Sound and Vibration, Vol. 323, pp. 385–396.

Sadek, I.; Abualrub, T.; Abukhaled, M. (2007): A computational method for solving optimal control of a system of parallel beams using Legendre wavelets. Mathematical and Computer Modelling, Vol. 45, pp. 1253–1264.

Zhang, Y.Q.; Lub, Y.; Wanga, S.L.; Liua, X. (2008): Vibration and buckling of a double - beam system under compressive axial loading. Journal of Sound and Vibration, Vol. 318, pp. 341–352.

Appendix

The relation between the dimensionless natural frequencies of the two beams

The free vibraton model of the first beam is expressed as

$$\frac{\partial^4 w_1}{\partial \xi_1^4} + b_{21} m_{12} l_{12}^4 \frac{\partial^2 w_1}{\partial \tau^2} = 0, \quad \xi_1 \in (0,1) \tag{A-1}$$

at $\xi_1 = 0$:

$$\frac{\partial \Psi}{\partial \xi} - \beta_1 \Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \tag{A-2}$$

$$\frac{\partial \Psi}{\partial \xi} - \beta_1 \Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \tag{A-3}$$

At $\xi_x = 1$:

$$\frac{\partial \Psi}{\partial \xi} - \beta_1 \Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \tag{A-4}$$

$$\frac{\partial^3 w_1(1,\tau)}{\partial \xi_1^3} = 0. \tag{A-5}$$

The free vibraton model of the second beam is expressed as

$$\frac{\partial^4 w_2}{\partial \xi_2^4} + \frac{\partial^2 w_2}{\partial \tau^2} = 0, \quad \xi_2 \in (0,1) \tag{A-6}$$

at $\xi_2 = 0$:

$$\frac{\partial \Psi}{\partial \xi} - \beta_1 \Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \tag{A-7}$$

$$\frac{\partial \Psi}{\partial \xi} - \beta_1 \Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \tag{A-8}$$

At $\xi_y = 1$:

$$\frac{\partial \Psi}{\partial \xi} - \beta_1 \Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \tag{A-9}$$

$$\frac{\partial^3 w_2(1,\tau)}{\partial \xi_2^3} = 0. \tag{A-10}$$

The jth dimensionless natural frequencies of the two beams are defined as $\{\omega_{1j}, \omega_{2j}\}$. Obviously, the relation between these natural frequencies is

$$\omega_{2j}^2 = \left(b_{21} m_{12} l_{12}^4\right) \omega_{1j}^2, \quad j = 1, 2, 3, \ldots \tag{A-11}$$

According to the method given by Lin (2005), the exact dimensionless natural frequencies $\{\omega_{1j}, \omega_{2j}\}$ of two independent beams are determined and listed in Table 1. Moreover, the relation (A-11) is verified here.