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Reduced Polynomials and Their Generation in Adomian Decomposition Methods

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Abstract: Adomian polynomials are constituted of reduced polynomials and derivatives of nonlinear operator. The reduced polynomials are independent of the form of the nonlinear operator. A recursive algorithm of the reduced polynomials is discovered and its symbolic implementation by the software Mathematica is given. As a result, a new and convenient algorithm for the Adomian polynomials is obtained.

Keywords: Adomian polynomials, Adomian decomposition method, Reduced polynomials, Mathematica, Nonlinear operator.

1 Introduction

The Adomian decomposition method and its modifications [Adomian (1986, 1989, 1994); Lai, Chen, and Hsu (2008); Soliman and Abdou (2008); Wazwaz (1999, 2009); Wazwaz and El-Sayed (2001)] provide an effective procedure for analytical solution of many kinds of, linear or nonlinear, functional equations in science and engineering. The advantage of the decomposition method is that it is straightforward, without restrictive assumptions, and does not change the problem into a convenient one for the use of linear theory.

Let us recall the basic principles of the Adomian decomposition methods. Consider an equation in the form

$$Lu + Ru + Nu = g, \tag{1}$$

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where L is an easily invertible linear operator, R is the remaining linear part, N represents an analytical nonlinear operator and g is a given function.

For an initial value problem, for example, we assume that $L^{-1}Lu = u - \phi$. Applying operator L^{-1} on both sides of (1) gives

$$u = \phi + L^{-1}g - L^{-1}Ru - L^{-1}Nu. \tag{2}$$

The tactic of the method is to look for a solution in the series form $u = \sum_{n=0}^{\infty} u_n$ and to decompose the nonlinear term Nu into a series

$$Nu = \sum_{n=0}^{\infty} A_n, \tag{3}$$

where A_n depends on $u_0, u_1, \cdots, u_n$, called the Adomian polynomials that are obtained for the analytical nonlinearity $Nu = f(u)$ by the formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left(\sum_{n=0}^{\infty} u_n \lambda^n\right)\right]_{\lambda=0}, \ n = 0, 1, 2, \cdots. \tag{4}$$

The first few Adomian polynomials are

$$\begin{aligned} A_0 &= f(u_0), \\ A_1 &= f'(u_0)u_1, \\ A_2 &= f'(u_0)u_2 + f''(u_0)\tfrac{u_1^2}{2!}, \\ A_3 &= f'(u_0)u_3 + f''(u_0)u_1u_2 + f^{(3)}(u_0)\tfrac{u_1^3}{3!}. \end{aligned}$$

The decomposition method consists in identifying the u_n's by means of the formulae

$$u_0 = \phi + L^{-1}g, \tag{5}$$

$$u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n, n = 0, 1, 2, \cdots. \tag{6}$$

Convergence of this method was studied in, e.g., [Abbaoui and Cherruault (1994); Cherruault (1989); Gabet (1994); Rach (2008)].

The calculation of the Adomian polynomials is a key issue and different algorithms were proposed [Abdelwahid (2003); Azreg-Aïnou (2009); Babolian and Javadi (2004); Biazar and Shafiof (2007); Rach (1984, 2008); Wazwaz (2000); Zhu, Chang, and Wu (2005)].

Symbolic implementation of the algorithms by using software Mathematica or Maple was considered in, e.g, [Azreg-Aïnou (2009); Chen and Lu (2004); Choi and Shin (2003); Pourdarvish (2006)].

Most of the algorithms involve with parametrization, derivatives about the parameter, expanding and regrouping, etc. Recursive methods for A_n should be more efficient. The algorithms in [Babolian and Javadi (2004); Biazar and Shafiof (2007)] are recursive, used self-defined operator and derivatives about parameter, respectively.

In this article we give a new recursive algorithm for A_n in terms of the reduced polynomials.

The Rach's Rule [Adomian (1989, 1994)] for the Adomian polynomials reads

$$A_m = \sum_{k=1}^{m} f^{(k)}(u_0)C(k,m), \tag{7}$$

where $C(k,m)$ are the sums of all probably products of k components of u whose subscripts sum to m, divided by the factorial of the number of repeated subscripts. The explicit expression of $C(k,m)$ is

$$C(k,m) = \sum_{\sum_{j=1}^{m} \nu_j = k,\ \sum_{j=1}^{m} j\nu_j = m} \frac{u_1^{\nu_1}}{\nu_1!} \cdots \frac{u_m^{\nu_m}}{\nu_m!}. \tag{8}$$

2 Reduced polynomials and their generation

From the difference of the equations $\sum_{j=1}^{m} j\nu_j = m$ and $\sum_{j=1}^{m} \nu_j = k$ one deduces that

$$\nu_{m-k+2} = \cdots = \nu_m = 0, \tag{9}$$

so (8) is refined as [Azreg-Aïnou (2009)]

$$\begin{aligned} C(k,m) &= Z_{m,k}(u_1, u_2, \cdots, u_{m-k+1}) \\ &= \sum_{\sum_{j=1}^{m-k+1} \nu_j = k,\ \sum_{j=1}^{m-k+1} j\nu_j = m} \frac{u_1^{\nu_1}}{\nu_1!} \cdots \frac{u_{m-k+1}^{\nu_{m-k+1}}}{\nu_{m-k+1}!}. \end{aligned} \tag{10}$$

The function $Z_{m,k}$ is used to replace $C(k,m)$ for convenience. $Z_{m,k}$ $(u_1, u_2, \cdots, u_{m-k+1})$ is a function of $m-k+1$ variables, called *reduced polynomials*. This terminology was first introduced in [Azreg-Aïnou (2009)].

Therein the reduced polynomials are described through solving the indeterminate equations under the $\sum$ in Eq. (10), the recursive algorithm does not be given.

We give a recursive generation method for the reduced polynomials as follows:

Algorithm for reduced polynomials:

$\langle 1\rangle$ For $m \geq 1, k=1$,

$$Z_{m,1}(u_1,u_2,\cdots,u_m) = u_m. \tag{11}$$

$\langle 2\rangle$ For $m \geq 2$, if $2 \leq k \leq \left[\frac{m}{2}\right]$, then

$$Z_{m,k}(u_1,u_2,\cdots,u_{m-k+1}) = \sum_{l=0}^{k-1} \frac{{u_1}^l}{l!} Z_{m-k,k-l}(u_2,\cdots,u_{m-2k+l+2}), \tag{12}$$

if $\left[\frac{m}{2}\right] < k \leq m$, then

$$Z_{m,k}(u_1,u_2,\cdots,u_{m-k+1}) = \int_0^{u_1} Z_{m-1,k-1}(u_1,u_2,\cdots,u_{m-k+1})du_1. \tag{13}$$

Proof of the algorithm: Eq. (11) is immediate from (10). Let $2 \leq k \leq \left[\frac{m}{2}\right]$. Then in Eq. (10) ν_1 can take the values $0,1,\cdots,k-1$. If $\nu_1 = l$ then $\nu_2,\cdots,\nu_{m-k+1}$ satisfy

$$\sum_{j=2}^{m-k+1} \nu_j = k-l, \quad \sum_{j=2}^{m-k+1} j\nu_j = m-l, \quad 0 \leq l \leq k-1.$$

Hence Eq. (10) can be rewritten as

$$Z_{m,k}(u_1,u_2,\cdots,u_{m-k+1}) = \sum_{l=0}^{k-1} \frac{{u_1}^l}{l!} \sum_{\substack{\sum_{j=2}^{m-k+1} \nu_j = k-l \\ \sum_{j=2}^{m-k+1} (j-1)\nu_j = m-k}} \prod_{j=2}^{m-k+1} \frac{u_j^{\nu_j}}{\nu_j!}.$$

The system of equations under $\sum$ is equivalent to

$$\sum_{j=2}^{m-2k+l+2} \nu_j = k-l, \quad \sum_{j=2}^{m-2k+l+2} (j-1)\nu_j = m-k,$$

and

$v_{m-2k+l+3} = \cdots = v_{m-k+1} = 0.$

According to the definition of the reduced polynomials one derives

$$Z_{m,k}(u_1, u_2, \cdots, u_{m-k+1}) = \sum_{l=0}^{k-1} \frac{{u_1}^l}{l!} Z_{m-k,k-l}(u_2, \cdots, u_{m-2k+l+2}).$$

If $\left[\frac{m}{2}\right] < k \leq m$, from the equations $\sum_{j=1}^{m-k+1} jv_j = m$, $\sum_{j=1}^{m-k+1} v_j = k$ it follows that $v_1 \geq 1$. The reduced polynomials in Eq. (10) are rewritten as

$$Z_{m,k}(u_1, u_2, \cdots, u_{m-k+1}) = \int_0^{u_1} \sum_{\substack{\sum_{j=1}^{m-k+1} jv_j = m \\ \sum_{j=1}^{m-k+1} v_j = k}} \frac{u_1^{v_1-1}}{(v_1-1)!} \prod_{j=2}^{m-k+1} \frac{u_j^{v_j}}{v_j!} du_1.$$

On rewriting the system of equations under $\sum$ as

$$(v_1 - 1) + \sum_{j=2}^{m-k+1} jv_j = m - 1, (v_1 - 1) + \sum_{j=2}^{m-k+1} v_j = k - 1,$$

one obtains

$$Z_{m,k}(u_1, u_2, \cdots, u_{m-k+1}) = \int_0^{u_1} Z_{m-1,k-1}(u_1, u_2, \cdots, u_{m-k+1}) du_1.$$

The proof is completed. $\square$

Although Eq. (13) involves with integrals the calculation is very simple. One only needs to replace $\frac{u_1^{v_1}}{v_1!}$ in the expanding summation of $Z_{m-1,k-1}$ $(u_1, u_2, \cdots, u_{m-k+1})$ by $\frac{u_1^{v_1+1}}{(v_1+1)!}$. If u_1 does not appear in some summand we regard $\frac{u_1^0}{0!}$ is contained.

From $Z_{1,1} = u_1$ it follows that $Z_{2,2} = \frac{u_1^2}{2}$ by using (13). Further from $Z_{2,1}$ and $Z_{2,2}$ one obtains $Z_{3,2} = u_1u_2$ and $Z_{3,3} = \frac{u_1^3}{3!}$. $Z_{4,2}$ is given by using (12) from $Z_{2,1}$ and $Z_{2,2}$. We give the reduced polynomials $Z_{m,k}$ from $m = 1$ to $m = 6$ in Tab. 1.

Symbolic implementation using Mathematica for the reduced polynomials is as follows.

Table 1: Reduced polynomials $Z_{m,k}(u_1,u_2,\cdots,u_{m-k+1})$

	k=1	k=2	k=3	k=4	k=5	k=6
$m=1$	u_1					
$m=2$	u_2	$\frac{u_1^2}{2}$				
$m=3$	u_3	u_1u_2	$\frac{u_1^3}{3!}$			
$m=4$	u_4	$u_1u_3+\frac{u_2^2}{2}$	$\frac{u_1^2}{2}u_2,$	$\frac{u_1^4}{4!}$		
$m=5$	u_5	$u_1u_4+u_2u_3$	$\frac{u_1^2}{2}u_3+u_1\frac{u_2^2}{2}$	$\frac{u_1^3}{3!}u_2$	$\frac{u_1^5}{5!}$	
$m=6$	u_6	$u_1u_5+u_2u_4$ $+\frac{u_3^2}{2}$	$\frac{u_1^2}{2}u_4+u_1u_2u_3$ $+\frac{u_2^3}{3!}$	$\frac{u_1^3}{3!}u_3+\frac{u_1^2}{2}\frac{u_2^2}{2}$	$\frac{u_1^4}{4!}u_2$	$\frac{u_1^6}{6!}$

```
poly[n_]:=Module[{Z,U},Z=Table[0,{i,1,n},{j,1,i}];
 Z[[1,1]]=Subscript[u,1];U=Table[Subscript[u,1]^l/l!,{l,0,n}];
 For[m=2,m<=n,m++,Z[[m,1]]=Subscript[u,m];
  For[k=2,k<=Floor[m/2],k++,
   Z[[m,k]]=Expand[Take[U,k].(Table[Z[[m-k,k-l]],{l,0,k-1}]/.
     Table[Subscript[u,i]->Subscript[u,i+1],{i,1,n}])]];
  For[k=Floor[m/2]+1,k<=m,k++,
   Z[[m,k]]=Integrate[Z[[m-1,k-1]],Subscript[u,1]]]];
 Z];
```

Further the Adomian polynomials are given by the following Mathematica program.

```
Ado[n_]:=Module[{Z,dir},Z=poly[n];
   dir=Table[D[f[Subscript[u,0]],{Subscript[u,0],k}],{k,1,n}];
   For[m=1,m<=n,m++,Subscript[A,m]=Take[dir,m].Z[[m]]]];
```

We illustrate the calculation and use of Adomian polynomials by some examples.

Example 1. Consider the Riccati equation

$$u'(t)=u^2,\ 0<t<1,\ u(0)=1.$$

The exact solution of the equation is $u^*(t)=\frac{1}{1-t},0\le t<1$.

Integrating the equation yields

$$u = 1 + \int_0^t u^2 dt.$$

Let $u = \sum_{n=0}^{\infty} u_n$. The Adomian polynomials for u^2 are

$$A_0 = u_0^2,\ A_1 = 2u_0u_1,\ A_2 = u_1^2 + 2u_0u_2,\ A_3 = 2u_1u_2 + 2u_0u_3,$$
$$A_4 = u_2^2 + 2u_1u_3 + 2u_0u_4,\ \cdots.$$

By iteration

$$u_0 = 1,\ u_n = \int_0^t A_{n-1} dt, n = 1, 2, \cdots,$$

we obtain

$$u_1 = t,\ u_2 = t^2,\ u_3 = t^3,\ u_4 = t^4, \cdots.$$

The solution is derived

$$u(t) = 1 + t + t^2 + t^3 + \cdots = \frac{1}{1-t}, 0 \le t < 1.$$

Example 2. Consider the Riccati equation

$$u' = t^2 + u^2,\ u(0) = 0.$$

The exact solution of the equation is $u^*(t) = \frac{tJ_{3/4}(t^2/2)}{J_{-1/4}(t^2/2)}, 0 \le t < c$ [Edwards and Penney (2004)], where $J_p(z)$ is the Bessel function of the first kind, $c = 2.00315\cdots$ satisfies $u^*(t) \to +\infty$, as $t \to c^-$.
By integrations we get

$$u = \frac{t^3}{3} + \int_0^t u^2 dt.$$

Applying the iteration

$$u_0 = \frac{t^3}{3},\ u_n = \int_0^t A_{n-1} dt, n = 1, 2, \cdots,$$

Table 2: Error $|\phi_{22}(t) - u^*(t)|$

t	error
0.0	0.0
0.2	4.33681×10^{-19}
0.4	1.04083×10^{-17}
0.6	0.0
0.8	2.77556×10^{-17}
1.0	5.55112×10^{-17}
1.2	4.44089×10^{-16}
1.4	1.82077×10^{-14}
1.6	4.42956×10^{-9}
1.8	0.000340938

Table 3: Error $|\phi_5(t) - u^*(t)|$

t	error
0.5	1.28231×10^{-14}
1.0	1.18104×10^{-10}
1.5	1.77473×10^{-8}
2.0	4.6394×10^{-7}
2.5	3.51076×10^{-6}
3.0	5.13328×10^{-6}
3.5	0.000256459
4.0	0.00213891
4.5	0.0108976
5.0	0.0407117

the n-term approximation $\phi_n = \sum_{i=0}^{n-1} u_i$ can be obtained. Using the software Mathematica we calculate 22-term approximation ϕ_{22}, and the error $|\phi_{22}(t) - u^*(t)|$ in the interval [0, 1.8] is examined, see Tab. 2.

Example 3. Consider the pendulum equation

$$u'' + \frac{1}{4}\sin u = 0,\ u(0) = 0,\ u'(0) = \frac{1}{2}.$$

The solution can be expressed as $u^*(t) = 2\arcsin(\frac{1}{2}\mathrm{sn}(\frac{t}{2}, \frac{1}{4}))$, where sn($z$, m) is the Jacobi elliptic function.

Integrating the equation yields

$$u = \frac{t}{2} - \frac{1}{4}\int_0^t \int_0^t \sin u dt dt$$

The Adomian polynomials for $\sin u$ are

$$A_0 = \sin u_0,\ A_1 = u_1 \cos u_0,\ A_2 = u_2 \cos u_0 - \frac{u_1^2}{2}\sin u_0,$$
$$A_3 = -\frac{u_1^3}{6}\cos u_0 - u_1 u_2 \sin u_0 + u_3 \cos u_0,\ \cdots .$$

Using the iteration

$$u_0 = \frac{t}{2}, u_n = -\frac{1}{4}\int_0^t \int_0^t A_{n-1} dt dt, n = 1, 2, \cdots,$$

the 5-term approximation $\phi_5(t)$ is obtained with the help of the software Mathematica

$$\phi_5 = \tfrac{t}{2} + \tfrac{1}{2}\left(2\sin\tfrac{t}{2} - t\right) + \tfrac{1}{4}\left(-\tfrac{5t}{2} + 8\sin\tfrac{t}{2} + \tfrac{\sin t}{2} - 2t\cos\tfrac{t}{2}\right) +$$
$$\tfrac{1}{48}\left(-3\left(2t^2 - 83\right)\sin\tfrac{t}{2} - 66t + 24\sin t + \sin\tfrac{3t}{2} - 78t\cos\tfrac{t}{2} - 6t\cos t\right) +$$
$$\tfrac{1}{384}\left(4t\left(2t^2 - 501\right)\cos\tfrac{t}{2} - 3\left(72t^2\sin\tfrac{t}{2} + 8t^2\sin t + 469t - 1936\sin\tfrac{t}{2} -\right.\right.$$
$$\left.\left.232\sin t - 16\sin\tfrac{3t}{2} + 4t\cos\tfrac{3t}{2} - (\sin t - 84t)\cos t\right)\right).$$

The graphs of the functions $u^*(t)$ and $\phi_5(t)$ on the interval [0,5] are plotted in Fig. 1.

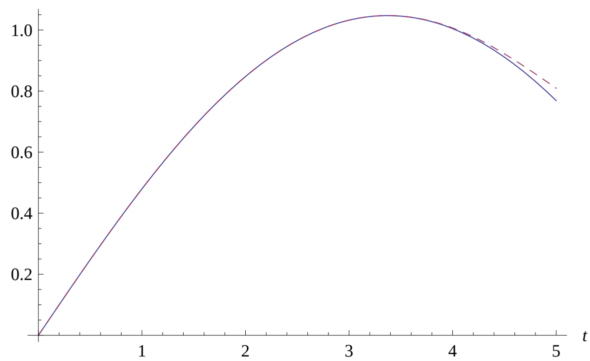

Figure 1: The exact solution $u^*(t)$ (solid line) and the approximate solution $\phi_5(t)$ (dashed line).

The error of the approximate solution $\phi_5(t)$ on the interval [0,5] is checked, see Tab. 3.

Example 4. Solve the inhomogeneous advection problem [Wazwaz (2009)]

$$u_t + \frac{1}{2}(u^2)_x = e^x + t^2 e^{2x},\ u(x,0) = 0.$$

Integrating with respect to t results in

$$u(x,t) = te^x + \frac{t^3}{3}e^{2x} - \frac{1}{2}\int_0^t \frac{\partial}{\partial x}u^2 dt.$$

Using Wazwaz's modification of the decomposition method [Wazwaz (1999)]

$$u_0 = te^x,\ u_1 = \frac{t^3}{3}e^{2x} - \frac{1}{2}\int_0^t \frac{\partial}{\partial x}A_0 dt, u_n = -\frac{1}{2}\int_0^t \frac{\partial}{\partial x}A_{n-1}dt, n = 2,3,\cdots,$$

yields $u_n = 0, n = 1,2,\cdots$. Thus $u(x,t) = te^x$, which is verified to be the solution.

In Examples 2 and 3 the programs generating the reduced polynomials and Adomian polynomials are carried out by Mathematica 7.

Adomian polynomials occur also in the power series method (modified decomposition method [Rach, Adomian, and Meyers (1992)]) for nonlinear problems. For instance, consider the differential equation

$$u'(t) + h(t)f(u) = g(t),\ u(0) = a, \tag{14}$$

where we suppose $h(t) = \sum_{n=0}^{\infty} h_n t^n,\ g(t) = \sum_{n=0}^{\infty} g_n t^n$.

Let $u(t) = \sum_{n=0}^{\infty} c_n t^n$. Then $f(u) = \sum_{n=0}^{\infty} A_n(c_0, c_1, \cdots, c_n)t^n$. Substituting into the differential equation and comparing the like power terms, and applying the initial value yield [Adomian (1994); Rach, Adomian, and Meyers (1992)]

$$c_0 = a, c_{n+1} = \frac{1}{n+1}\left(g_n - \sum_{k=0}^{n} h_{n-k}A_k(c_0, c_1, \cdots, c_k)\right), n = 0,1,\cdots. \tag{15}$$

3 Conclusion

The reduced polynomials constituting Adomian polynomials are studied and their recursive algorithms are given. Based on the algorithms the symbolic implementation by the software Mathematica for the reduced polynomials and Adomian polynomials is obtained. We illustrate by some nonlinear examples the Adomian decomposition method gives the exact analytical solutions or approximate analytical solutions.

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