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Unconditionally Stable Convergence with Power Principle-based Time-Integration Schemes

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Abstract: This manuscript introduces a novel sufficient condition for the unconditionally stable convergence of the general class of trapezoidal integrators. Contrary to standard energy-based approaches, this convergence criterion is derived from the power principles, in terms of both balance and dissipation. This result improves the well-known condition of stable convergence based on the energy method, extending its applicative spectrum to a variety of physical problems, whose constitutive prescriptions may be more appropriately characterized by means of evolving field equations.
Our treatment, tailored for generalized trapezoidal integrators, addresses both linear and nonlinear problems, extending its applicability to contexts where standard energy-based schemes present loss of stable convergence, as well as an uncontrolled increase in terms of energy. To appreciate such novel result, the Newmark-based numerical solution scheme is applied to a simplified nonlinear problem, with hardening plasticity and finite deformations within a one-dimensional description. The proposed test shows how the power-based method attains a stable convergence and overcomes the requirement for additional conservative invariants, such as energy and angular momentum.

Keywords: Numerical algorithms, Stable convergence, Constitutive behaviour

1 Introduction

Over the last several years, a fair amount of work has been devoted to the numerical solution of linear and nonlinear problems with time integration; the areas of application span diverse contexts, all accompanied by the use of energy-like quantities in order to stabilize the numerical convergence. Trapezoidal rules are often employed to solve these problems numerically; for a general overview, we refer the reader to Hughes (2000).

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A well-known proof of stable convergence for energy-based algorithms was only given for linear contexts, except for some specific nonlinear cases, *e.g.*, in structural dynamics with finite deformations by Belytschko and Schoeberle Belytschko and Schoeberle (1975). In more general nonlinear contexts, conserving energy-like conditions are often a fortiori introduced into the modeling process, in order to achieve more stable numerical properties. In fact, various nonlinear systems are particularly deficient in conservative properties, resulting in the loss of stable convergence yielding an unbounded growth of physical quantities expected to be conserved. Just to cite few cases, we mention the works in structural dynamics, where angular momentum conservation is prescribed a posteriori Gams, Planinc, and Saje (2008); Armero (2006), or in fluid dynamics, where similar constraints, derived from known solutions, are added to the model equations Wallstedt and Guilkey (2008). All these approaches are specifically tailored for individual physical problems. Nevertheless, an initial effort addressing nonlinear systems in a variational formulation was given in Carini and Genna (2000); Bardella and Genna (2005), investigating and extending the Newmark family of time integration schemes. However, a general framework regarding stable convergence still lacks, at the best of our knowledge.

We propose a fundamental shift of perspective: the use of a stable numerical criterion derived from a thermodynamically consistent framework, employed in the generation of a model. Such a framework has been introduced in the past by several scholars in various fields of continuum physics. In fact, as emphasized by Truesdell and Noll Truesdell and Noll (2004), a weak formulation of the governing equations—the *principle of virtual power* or the *null working* principle—does not give enough conditions to attain solutions of nonlinear problems: further assumptions concerning the preservation of the second law of thermodynamics allow the physical behavior to be properly restricted in terms of constitutive relations.

Specifically, to serve our purposes, we refer to the framework by DiCarlo and Quiligotti DiCarlo and Quiligotti (2002), allowing the definition of a problem in terms of balance (or of null working) and dissipation *power principles*, and widely employed in heterogeneous physical problems Cherubini, Filippi, Nardinocchi, and Teresi (2009); Olsson and Klarbring (2008). Hence, we discard the above-mentioned limitations of recovering suitable energy stabilization criteria, and we provide a stable convergence criterion proving a sufficient condition entirely based on the power balance and dissipation principles. We mention in passing that Simo and Tarnai Simo and Tarnow (1992) first attempted to suggest power dissipation principles in the context of numerical time discretization; however, no discussion has been purported aimed at proving stable convergence conditions, with the treatment being restricted to the one-parameter family of generalized mid-point rule

algorithms.

Besides overcoming the weakness imposed by energy-based approaches, our method not only possesses the advantage of scrapping the constraint of positive definiteness of the system response, but also applies to general nonlinear problems. In fact, convergence is obtained by means of the very same principles involved in the generation of the model, *i.e.*, in the constitutive prescription; moreover, as emphasized in our convergence proof, we widely make use of the non-negative definition of the *dissipated power*, without any additional restriction on the definition of other operators, *e.g.*, the structural response in structural dynamics Hughes (2000), a common requirement for stability.

Our treatment focuses on generic trapezoidal rules, with the special case of the Newmark family of time integrators, and may be applied to both first and second order differential problems.

In order to provide an operational application, we pose our focus on a one-dimensional mechanical problem, consisting in a mass constrained by a spring exhibiting a nonlinear response in terms of strain and stress Armero (2006). Such example is a simplified test, yet not simple, employed to stress the limitations of energy-based algorithms, which require the additional constraint of the angular momentum conservation. We will show how such a constraint is implicitly satisfied by our simulation, framed within the power paradigm as well as our modeling.

Overview

This manuscript is organized as follows. The ensuing Section 2 will provide the reader with the adequate knowledge of the general power framework, focused on the derivation of thermodynamically consistent equations from the power dissipation principle. Section 3 will then propose the novel stable convergence criterion, and therefore on the conditions of convergence, originating from the aforementioned principle. Finally, in order to bring forth a clear applicative perspective on our results, Section 4 will be focused on the comparison of a classical Newmark scheme with our power-dissipation based algorithm.

2 Background

This section is devoted to the introduction of basic notions needed to describe the general framework, exhaustively detailed in DiCarlo and Quiligotti (2002), based on the power balance (or *null working*) and dissipation principles, and giving rise to thermodynamically consistent models. Let us describe a process by means of the kinematic descriptor q; the evolutive equations governing the process are derived

from such physical principles, respectively:

$$\mathscr{P}^{\mathrm{e}} - \mathscr{P}^{\mathrm{i}} = 0 \tag{1a}$$

$$\mathscr{P}^{\mathrm{i}} - \dot{\psi} \geq 0\,, \tag{1b}$$

where we denoted as $\mathscr{P}^{\mathrm{e}}$ and $\mathscr{P}^{\mathrm{i}}$ the external and the internal powers, respectively; ψ indicates a generic free energy.

We may then restate the principles expressed in (1) independently from their algebraic representation as follows:

$$\langle Q^{\mathrm{e}}, \dot{q}\rangle - \langle Q^{\mathrm{i}}, \dot{q}\rangle = 0 \tag{2a}$$

$$\langle Q^{\mathrm{i}}, \dot{q}\rangle - \langle s, \dot{q}\rangle \geq 0\,, \tag{2b}$$

where $\langle\cdot,\cdot\rangle$ denotes an inner product, while Q^{e} and Q^{i} represent the dynamic actions spending power on $\dot{q}$; s is the internal response of the system, *i.e.*, the action arising from the differentiation of ψ with respect to $\dot{q}$. Generally speaking, Q^{e} and s are assumed to be known, while Q^{i} results as a constitutive prescription, holding the inequality in (2).

Definition (Remodeling action) *We define the remodeling action of the process the quantity* $\overset{+}{Q}$*, prescribed by a positive-defined operator* $\mathbb{M}$*, as the following:*

$$\overset{+}{Q} := Q^{\mathrm{i}} - s \equiv \mathbb{M}^{-1}\dot{q}\,. \tag{3}$$

Therefore, by virtue of (3), we satisfy the power dissipation principle (2b) as

$$\langle Q^{\mathrm{i}} - s, \dot{q}\rangle \equiv \langle \overset{+}{Q}, \dot{q}\rangle = \langle \mathbb{M}^{-1}\dot{q}, \dot{q}\rangle \geq 0\,. \tag{4}$$

As a side note, we stress the fact that $\mathbb{M}^{-1}$ represents a dissipative modulus, ruling the evolution of q, in both linear and nonlinear cases.

Hence, by picking a kinematic descriptor and a suitable representation of the free energy, the associated dynamic actions are directly defined and the overall process evolution is constitutively correct. The evolutive equation, *i.e.*, the strong form of (2a), may be then restated from (2) and (4) as follows:

$$Q^{\mathrm{e}} - s - \mathbb{M}^{-1}\dot{q} = 0\,. \tag{5}$$

Note that in this formulation, the inertia forces may be framed in Q^{e}, *i.e.*, as an external power contribution.

Example of Formulation

For clarity's sake, we will refer to a classical dynamic problem, characterized by the kinematic descriptor x. The system, with inertia I, is subject to an external force F varying in time, and it is described by a linear response with an elastic and a damping factor, K and C, respectively. Hence, the power expressions of the above-mentioned process are formalized by the following equations:

$$\begin{aligned}\mathscr{P}^{\mathfrak{e}} &= \langle (F - I\ddot{x}) , \dot{x} \rangle , \\ \psi &= \langle K x, \dot{x} \rangle .\end{aligned}$$

Let us use C as the positively defined coefficient to satisfy the dissipation principle; the constitutive prescription is therefore obtained as the ensuing inequality:

$$\mathscr{P}^{\mathfrak{e}} - \dot{\psi} =: \langle C\dot{x}, \dot{x} \rangle \geq 0 .$$

According to the previous equations, we finally attain the standard equilibrium condition of the system:

$$I\ddot{x} + C\dot{x} + Kx = F .$$

3 Power Stability

This section is devoted to the definition of a novel stability criterion for the family of Newmark time-dependent algorithms. We will show how the power dissipation principle, as stated in Section 2, yields a sufficient condition for the stability of generalized Newmark algorithms.

An unconditional stability criterion for the Newmark β-method has been shown in Belytschko and Schoeberle (1975), and later, the generalized Newmark scheme has been proved to be conditionally stable (see *e.g.*, Hughes (2000)). In such cases, stability conditions are based on the definition of mechanical energy, and the linearized system response is assumed to be positively defined. In contrast with previous works, we depend solely on the definition of the dissipative operator, discarding the limitation of positive-definiteness of the system response. Moreover, our results are directly applicable to the broad category of nonlinear problems, and introduces no additional constraints on integrating quantities, *e.g.*, energy, angular or linear momentum.

First, let us introduce two supplementary operators, the *average* and the *gap* operators, defined as follows. For any field ϕ, sampled at a specific time-step $t_n \div t_{n+1}$, we define:

$$\{\phi_n\} := {}^1\!/_2 \left(\phi_{n+1} + \phi_n \right) , \quad [\phi_n] := \phi_{n+1} - \phi_n .$$

Since the power dissipation principle holds for any test velocity $\dot{\tilde{q}}$, we may restate (1) using the dissipative prescription (3) as follows:

$$\mathscr{P}^{\mathfrak{e}} - \dot{\psi} \geq \varepsilon \, \dot{\tilde{q}} \cdot \mathbb{M}^{-1} \dot{\tilde{q}}, \tag{6}$$

with ε being an arbitrarily small positive constant, and the *dot operator* being the shot-hand form of an inner product. Since $\mathscr{P}^{\mathfrak{e}} = \dot{\tilde{q}} \cdot Q^{\mathfrak{e}}$ and $\dot{\psi} = \dot{\tilde{q}} \cdot s$, equation (6), by applying the average operator, reads as:

$$\dot{\tilde{q}} \cdot \{Q_n^{\mathfrak{e}} - s_n\} \geq \varepsilon \, \dot{\tilde{q}} \mathbb{M}^{-1} \dot{\tilde{q}},$$

and therefore by virtue of (5),

$$\dot{\tilde{q}} \cdot \{r_n\} + \dot{\tilde{q}} \cdot \{\mathbb{M}^{-1} \dot{q}_n\} \geq \varepsilon \, \dot{\tilde{q}} \mathbb{M}^{-1} \dot{\tilde{q}}, \tag{7}$$

where r_n is a numerical residue originating from the field equation (5). We may then proceed in defining our stability criterion based on the power dissipation principle.

Definition (Power stability) *We define the power stability criterion as follows:*

$$\left| \dot{\tilde{q}} \cdot \{r_n\} \right| \leq \varepsilon \, \dot{\tilde{q}} \cdot \mathbb{M}^{-1} \dot{\tilde{q}}_n, \tag{8}$$

where $\dot{\tilde{q}}$ defines a test velocity, r_n is a numerical residue, $\mathbb{M}^{-1}$ is a positively-defined operator, and ε represents an arbitrary positive constant.

Granted that the previous inequality holds, *i.e.*, the above-defined power stability condition is satisfied, we deduce from the expression (7) the following:

$$\dot{\tilde{q}} \cdot \{\mathbb{M}^{-1} \dot{q}_n\} \geq 0, \tag{9}$$

being valid for any test velocity $\dot{\tilde{q}}$. Therefore, the inequality (9) expresses a condition for stable convergence, as defined by the above criterion (8).

Note that the condition expressed in equation (9) states the non-negative definition of the mobility operator $\mathbb{M}^{-1}$. We use this as key-point for getting sufficient conditions of stability. Finally, we mention in passing that $\mathbb{M}^{-1}$ may also depend on time, but we may omit the more respectable form $\mathbb{M}_n^{-1}$, due to the fact that $\mathbb{M}^{-1}$ shall be utilized as an operator applied to a velocity, therefore implicitly containing information regarding the time dependence.

Preliminary definitions

By virtue of the integrating parameters, we may express the set of all test velocities $\dot{\tilde{q}}$ of interest by means of the initial and final velocities and displacements, with

three independent parameters for time discretization. Thus, we formally express the test velocities as the following three forms:

$$\dot{\tilde{q}}_1 = \dot{q}_{n+1} - [q_n]/h\,, \tag{10a}$$

$$\dot{\tilde{q}}_2 = [q_n]/h - \dot{q}_n\,, \tag{10b}$$

$$\dot{\tilde{q}}_3 = [q_n]/h\,, \tag{10c}$$

with $h := t_{n+1} - t_n$, *i.e.*, the time gap measure between two successive integrating steps.

Note that by combining $\dot{\tilde{q}}_1$ and $\dot{\tilde{q}}_2$ we obtain $[\dot{q}_n] = \dot{\tilde{q}}_1 + \dot{\tilde{q}}_2$ as a valid test velocity satisfying (9); analogously, $\{\dot{q}_n\}$ may be obtained as the linear combination $1/2(\dot{\tilde{q}}_1 - \dot{\tilde{q}}_2) + \dot{\tilde{q}}_3$, as a valid test velocity. Additionally to the above, other types of test velocities will be introduced as the number of integrating parameters increases, *e.g.*, in presence of acceleration terms.

First order time integration

We shall proceed then introducing a numerical integration scheme based on the Newmark method, *i.e.*, let us define the discrete kinematics with the following numerical integrating scheme:

$$q_{n+1} = q_n + h\,\dot{q}_{n+\alpha}\,,$$

where the numerical velocity $\dot{q}_{n+\alpha}$ is the velocity at a generic time step $n+\alpha$, *i.e.*:

$$\dot{q}_{n+\alpha} := \{\dot{q}_n\} + \Big(\alpha - 1/2\Big)[\dot{q}_n]\,,$$

$\alpha \in [0,1]$ being the integrating parameter ruling the variation of the kinematics.

Let us proceed then employing the aforementioned test velocities (10) in (9), obtaining the following inequalities that should hold in order to attain a stable convergence, as defined by the power stability criterion:

$$(1-\alpha)[\dot{q}_n] \cdot \{\mathbb{M}^{-1}\dot{q}_n\} \geq 0\,, \tag{11a}$$

$$\alpha[\dot{q}_n] \cdot \{\mathbb{M}^{-1}\dot{q}_n\} \geq 0\,, \tag{11b}$$

$$(\{\dot{q}_n\} + (\alpha - 1/2)[\dot{q}_n]) \cdot \{\mathbb{M}^{-1}\dot{q}_n\} \geq 0\,, \tag{11c}$$

Recalling the previously expressed relation between velocity and displacement, *i.e.*, $[q_n] := h(1-\alpha)\dot{q}_n + h\,\alpha\,\dot{q}_{n+1}$, we obtain three conditions to be satisfied by the integrating parameter α: $\alpha \leq 1$ from (11a), $\alpha \geq 0$ from (11b), and $\alpha \geq 1/2$ from (11c), since $\{\dot{q}_n\} \cdot \{\mathbb{M}^{-1}\dot{q}_n\} \geq 0$ and $[\dot{q}_n] \cdot \{\mathbb{M}^{-1}\dot{q}_n\} \geq 0$.

Therefore, a sufficient condition for the unconditionally stable convergence of the Newmark family of integrating schemes is $\alpha \in [1/2, 1]$.

Second order time integration

Let us consider now a general Newmark integrating scheme Hughes (2000), *i.e.*, accounting for both acceleration $\ddot{q}$ and velocity $\dot{q}$, numerically obtained as follows, given the integrating parameters β, γ:

$$q_{n+1} := q_n + h\dot{q}_n + \frac{h^2}{2}\Big(\{\ddot{q}_n\} + (2\beta - 1/2)\,[\ddot{q}_n]\Big)\,, \tag{12a}$$

$$\dot{q}_{n+1} := \dot{q}_n + h\Big(\{\ddot{q}_n\} + (\gamma - 1/2)\,[\ddot{q}_n]\Big)\,. \tag{12b}$$

We may therefore mimic the demonstration provided earlier, employing the aforementioned test velocities (10), and introducing an additional one, *i.e.*, $h\{\ddot{q}_n\}$. As a result, we obtain:

$$\dot{\tilde{q}}_1 = \dot{q}_{n+1} - [q_n]/h \equiv 1/2\,h\{\ddot{q}_n\} + h(\gamma - \beta - 1/4)[\ddot{q}_n]\,, \tag{13a}$$

$$\dot{\tilde{q}}_2 = [q_n]/h - \dot{q}_n \equiv 1/2\,h\{\ddot{q}_n\} + h(\beta - 1/4)[\ddot{q}_n]\,, \tag{13b}$$

$$\dot{\tilde{q}}_3 = [q_n]/h \equiv \{\dot{q}_n\} + h(\beta - 1/2\,\gamma)[\ddot{q}_n]\,, \tag{13c}$$

$$\dot{\tilde{q}}_4 = h\{\ddot{q}_n\}\,, \tag{13d}$$

and as explained previously for the first-order time integration, we obtain as valid test velocities $h[\ddot{q}_n]$ and $\{\dot{q}_n\}$, resulting as linear combination of $\dot{\tilde{q}}_1, \ldots, \dot{\tilde{q}}_4$. Hence, the following inequalities have to be satisfied:

$$(1/2\,h\{\ddot{q}_n\} + h(\gamma - \beta - 1/4)[\ddot{q}_n]) \cdot \{\mathbb{M}^{-1}\dot{q}_n\} \geq 0\,, \tag{14a}$$

$$(1/2\,h\{\ddot{q}_n\} + h(\beta - 1/4)[\ddot{q}_n]) \cdot \{\mathbb{M}^{-1}\dot{q}_n\} \geq 0\,, \tag{14b}$$

$$(h\{\dot{q}_n\} + h(\beta - 1/2\,\gamma)[\ddot{q}_n]) \cdot \{\mathbb{M}^{-1}\dot{q}_n\} \geq 0\,, \tag{14c}$$

$$(h\{\ddot{q}_n\} + h(\gamma - 1/2)[\ddot{q}_n]) \cdot \{\mathbb{M}^{-1}\dot{q}_n\} \geq 0\,. \tag{14d}$$

We therefore have that a sufficient condition for the unconditionally stable convergence of Newmark integrating schemes is the region delimited by the inequalities obtained from (14), *i.e.*, $\beta \leq \gamma - 1/4$, $\beta \geq 1/2\,\gamma$, and $\gamma \geq 1/2$. We mention in passing that the first condition imposed on the Newmark integration-parameters restrict the limitations known in literature, proved only for particular classes of physical problems.

Preliminary considerations

Contrary to a classical energy-driven approach, limiting the scope of its utilization to specific nonlinear problems Belytschko and Schoeberle (1975), our approach is directly applicable to a broader class of nonlinear systems. Moreover, in nonlinear

contexts, approaches based on the energy method require some efforts in order to stabilize the energy, usually introducing additional constraints in the numerical integration process: as the ensuing numerical example will show, a power-based approach embodies the supplementary conditions needed by the energy method—for more details, we refer the reader to Crisfield and Shi (1994); Armero (2006); Gonzalez (2000), and Section 4.5 of this manuscript.

4 A Derivation Example of Hardening Plasticity

In order to provide an elucidatory application, we adapt the modeling framework reported in Section 2 to a one-dimensional example of hardening plasticity, generally described by postulating stress yield criterions governing the nonlinear response. For an extensive treatise on classical elasto-plastic processes, we refer the reader to Simo and Hughes (2000).

Our approach follows a different perspective, found practical by other authors in recent scholarly works Gurtin (2000); Del Piero (2009); Gurtin and Reddy (2009), where isotropic hardening plasticity is formulated by means of variational inequalities.

We hence refer to one mass-spring system with an elasto-plastic behavior. Such a system, even in a simplified form, presents crucial hardships in convergence when time discretization techniques are employed, as we will show in Section 4.4, where a detailed numerical investigation is reported.

The following sections are devoted to introduce a classical approach to elasto-plasticity, and next we present the same laws derived from the power dissipation principle, as presented in DiCarlo and Quiligotti (2002). For simplicity's sake, we initially neglect to define the strain measure explicitly, namely ε, while focusing only on the constitutive assumptions modeling the inelastic part; we furthermore consider the absence of external loads.

4.1 The Classic Approach

Let the constitutive coefficients of the aforementioned spring be σ_Y, E, K and H, defining the limit plastic stress, the elastic coefficient, the isotropic and the kinematic hardening moduli, respectively. The elastic stress response in the spring is then equal to

$$\sigma = E\,(\varepsilon - p)\,,$$

where p is the plastic part of the total strain ε. In the classical perspective on elasto-plasticity, the equilibrium equation, in absence of external loads, is defined by the

following:

$$I\ddot{\varepsilon} + E(\varepsilon - p) = 0. \tag{15}$$

We may therefore proceed in characterizing the hardening plasticity, by postulating the yield criterion:

$$f(\sigma, b, a) := |\sigma - \tau| - (\sigma_Y + Ka) \leq 0, \tag{16}$$

where τ represents the back stress, and the internal isotropic hardening is denoted by the variable a, corresponding to the amount of the accumulated plastic strain. The overall hardening plastic process, given the rate $\lambda \geq 0$, is characterized by the following:

$$\dot{p} = \lambda \operatorname{sign}(\sigma - \tau), \tag{17a}$$

$$\dot{\tau} = \lambda H \operatorname{sign}(\sigma - \tau), \tag{17b}$$

$$\dot{a} = \lambda. \tag{17c}$$

The above-mentioned equations are the flow rule, the kinematic, and isotropic hardening law, respectively. Note that the equation (17b) represents the well-known Ziegler rule, *i.e.*,

$$\dot{\tau} := H\dot{p}. \tag{18}$$

Finally, the plastic parts τ and a are allowed to evolve as defined by (17), abiding the Kuhn-Tucker complementary conditions

$$\begin{aligned} &\lambda \geq 0 \\ &f(\sigma, \tau, a) \leq 0 \\ &\lambda f(\sigma, \tau, a) = 0, \end{aligned} \tag{19a}$$

and the consistency conditions:

$$\lambda \dot{f}(\sigma, \tau, a) = 0, \quad \text{if } f(\sigma, \tau, a) = 0. \tag{19b}$$

For sake of completeness, we mention the fact that classical finite element methods evaluate the plastic increments by means of the *return mapping* algorithm, *i.e.*, via an iterative scheme. This process is governed by the strain measure, with the basic independent variable set as $\dot{\varepsilon}$, and the outcomes being $\dot{\sigma}$ and the internal (p, a, τ) variables. In the case of a one dimensional problem, as we present in this section, the equations (17) and (19) are satisfied by

$$\lambda = \frac{\operatorname{sign}(\sigma - \tau)}{E + H + K} E\dot{\varepsilon}, \tag{20}$$

and hence the elasto-plastic tangent modulus is:

$$\begin{cases} E & \text{if } \lambda = 0 \\ E\dfrac{H+K}{E+H+K} & \text{if } \lambda > 0\,. \end{cases}$$

4.2 A Growth and Remodeling Approach

The power-based approach needs, in order to be properly defined, an internal energy ψ, whose time-derivative has to be compared with the external power.

From equation (1) we have that $\mathscr{P}^{\mathfrak{e}} - \dot{\psi} \geq 0$, and therefore, for our elasto-plastic example, we posit:

$$\psi := \frac{1}{2}E(\varepsilon - p)^2 + \frac{1}{2}H^{-1}\tau^2 + \frac{1}{2}Ka^2\,. \tag{21}$$

Note that different definitions of a free energy are admissible in order to produce similar models Gurtin and Reddy (2009); Del Piero (2009), as we will show in the present section.

Hence, since $\mathscr{P}^{\mathfrak{e}} = (-I\,\ddot{\varepsilon})\,\dot{\varepsilon}$, we have

$$\begin{aligned} \mathscr{P}^{\mathfrak{e}} - \dot{\psi} = {} & (-I\,\ddot{\varepsilon} - E\,(\varepsilon - p))\,\dot{\varepsilon} \\ & + \big(E(\varepsilon - p)\dot{p} - (H^{-1}\tau)\dot{\tau} - (Ka)\big)\,\dot{a} \geq 0\,. \end{aligned} \tag{22}$$

The first term represents the balance conditions, as expressed in equation (15): in other words, the dissipative term associated to $\dot{\varepsilon}$ is zero. The choices for the remaining dissipative quantities, *i.e.*, for the remaining velocities, shall define the overall plastic evolution and complete the governing equations of our problem as constitutive laws. We may in the following simplify the inequality (22) by applying Ziegler's rule (18), obtaining the subsequent inequality:

$$(E(\varepsilon - p) - \tau)\dot{p} - (Ka)\,\dot{a} \geq 0\,. \tag{23}$$

Finally, by considering the yield criterion (16), we may require that $\dot{p}$ evolves in such a way that the effective stress expending power with respect to $\dot{p}$, namely $(\sigma - \tau)$, tends to a limit condition: $(\sigma - \tau) \to \sigma_Y + Ka$. This limitation is consistent with (16). Moreover, under this perspective, we may require that the plastic rate tends to be proportional to the strain rate, *i.e.*, by virtue of equation (20):

$$\dot{p} \;\to\; |\,\dot{\varepsilon}\,|\,\frac{E}{E+H+K}\,.$$

Therefore, uncoupling the evolution of both p and a, and assuming $\dot{a} = |\dot{p}|$, hence neglecting the prescription regarding the dissipative term associated with $\dot{a}$, the previous considerations may be recast in the following evolving condition:

$$\dot{p} := \left(\frac{\sigma - \tau}{\sigma_Y + Ka} \right)^N |\dot{\varepsilon}| \frac{E}{E + H + K}, \tag{24}$$

where an odd $N \in \mathbb{N}$ satisfies the inequality (23). In fact, in equation (24), the only negative term can be the effective stress $\sigma - \tau$, the same term that multiplies $\dot{p}$ and giving rise to a non-negative dissipation.

Equations (15) and (24) completely establish the elasto-plastic problem, once the assumptions $\dot{\tau} = H\dot{p}$ and $\dot{a} = |\dot{p}|$ have been accounted for.

4.3 Preliminary Numerical Comparisons

We may now test the classic formulation of an elasto-plastic process with the power-based methodology. In order to highlight how classical plastic constitutive prescriptions may be promptly recovered in our alternative model, we compare a β-Newmark scheme based on the power dissipation principle, with a reference numerical solution of the same problem following a standard approach.

Our tests are conducted imposing a cyclic and a monotonic strain, with the following conditions: $E = 50$ GPa, $\sigma_Y = 1.5$ GPa, $H = 0.25$ GPa, $K = 2.5$ GPa, and $N = 5$. The imposed strain functions employed in this example are

$$\varepsilon = v_0 kT \left(1 - e^{-t/50} \right),$$

$$\varepsilon = v_0 kT \sin\left(\frac{\pi}{2Tk} t \right),$$

with $v_0 = 0.001$ s^{-1} being the initial strain rate, the total time of the experiment being $T = 600$ seconds. The time-step employed in the numerical integration is $h = 1$ s, with $k = 0.25$. The algorithm is detailed in Alg. 1.

It is worth noticing as the results do not differ from a reference numerical solution obtained with a commercial software. The outcomes of computations, for both our algorithm and the reference, are pictured in Fig. 1 for both the cyclic and monotonic prescribed displacement.

function PLASTICITY(h, x_n, $\dot{x}_n$) ▷ Solves (24) at time step $n+1$
  $j \leftarrow 0$
  $x^0_{n+1} \leftarrow x_n + (1-\beta)h\dot{x}_n$ ▷ $x := (p,\, \tau,\, a)^\top$
  $c \leftarrow E/(E+H+K)$

  while $e \geq$ toll **do**
    $r^j_{n+1} \leftarrow \dot{p}^j_{n+1} - c\,\mathrm{abs}(\dot{p}_{n+1})\,((E(\varepsilon_{n+1} - p_{n+1}) - \tau^j_{n+1})/(\sigma_Y + H\alpha^j_{n+1}))^N$

    $\delta\dot{p}_{n+1} \leftarrow -(\mathbf{J}^j_{n+1})^{-1} r^j_{n+1}$ ▷ Update the current solution
    $\dot{p}^{j+1}_{n+1} \leftarrow \dot{p}^j_{n+1} + \delta\dot{p}_{n+1}$
    $\dot{\tau}^j_{n+1} \leftarrow H\dot{p}^{j+1}_{n+1}$
    $\dot{a}^j_{n+1} \leftarrow \mathrm{abs}(\dot{p}^j_{n+1})$
    $e \leftarrow \delta\dot{p}^2_{n+1}K$
    $j \leftarrow j+1$
  end while

  $x^{j*}_{n+1} \leftarrow x^j_{n+1}$ ▷ Solution at convergence
  $\dot{x}^{j*}_{n+1} \leftarrow \dot{x}^j_{n+1}$
  return x^{j*}_{n+1} **and** $\dot{x}^{j*}_{n+1}$
end function

Algorithm 1: Sketch of the algorithm employed to solve equation (24).

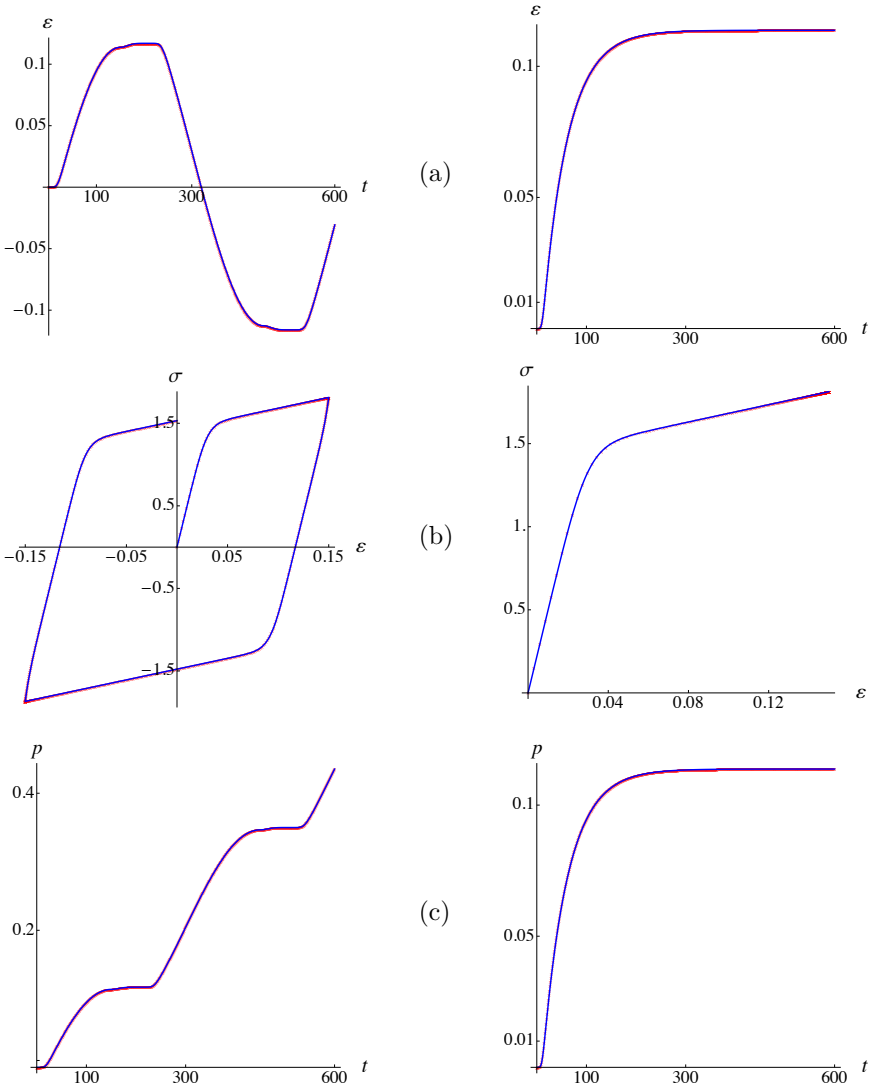

Figure 1: The results of cyclic (on the left) and monotonic (on the right) load with a Newmark time-integration algorithm (in red) compared to a reference solution (in blue): by rows, the time-dependent strain (a), the strain-stress cycle outcome (b), and plasticity (c) graphed over time.

4.4 Nonlinear Numerical Results

We showed in the previous section as the power dissipation principle leads to a constraint of integrating parameters, a sufficient condition for satisfying the power stability criterion, as expressed in (8).

In the following, we will focus on a nonlinear problem, expressed within the power-based approach presented in Section 4.2, and solved numerically with a generalized Newmark scheme. This particular test, presented in Armero (2006), is an exemplification of the inability of classical Newmark schemes of solving nonlinear problems, *i.e.*, they exhibit an unbounded growth of physical energy. Nonlinearity is present in both the elasto-plastic material response, and the strain measure. As a side-note, we mention that nonlinear elasto-plasticity is still a open issue across several years of numerical researches in structural dynamics, and it is sometimes faced extending standard trapezoidal schemes by means of two additional integration parameters Casciaro (1975); Corigliano and Perego (1993).

As discussed before, we are proposing a radical change in perspective: employing power dissipation principles in lieu of energy conservation prescriptions. As we will show in the following, not only such variation benefits the overall stability, but also embodies prescriptions that are otherwise explicitly accounted for in the algorithmic development.

Let us consider the process underlying a rotating elasto-plastic spring-mass system fixed at one end, as pictured in Fig. 2. The system is exempt from external forces, evolving with the constraint that its associated angular momentum is constant in time.

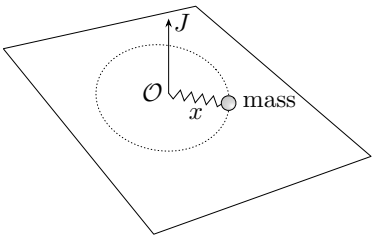

Figure 2: The mass-spring system detailed in Section 4.4.

The system, presenting a strong nonlinearity in both geometric and material properties, is modeled through equations expressed in terms of the mass position vector $x = (x_1, x_2)^\top$, rotating around the fixed end $\mathscr{O}$. The geometrical nonlinearity is

defined by the elastic and inelastic stretching, composed multiplicatively, *i.e.*,

$$\lambda := \frac{\|x\|}{L_0} = \lambda_p \lambda_i \,,$$

with $L_0 > 0$ being a reference length, and $\|x\| := \left(x_1^2 + x_2^2\right)^{1/2}$. The spring strain follows the multiplicative rule, *i.e.*, given the strain ε, it may be decomposed in its elastic and inelastic (plastic) parts, or in other words:

$$\varepsilon := \log(\lambda) = \varepsilon_e + \varepsilon_p =: \varepsilon_e + p \,,$$

given that $\varepsilon_e := \log(\lambda_e)$, $p := \log(\lambda_p)$, and $\varepsilon = \log(\|x\|) - \log(L_0)$. Therefore, proceeding along the same path we outlined in Section 4.2, and noting that the external power is now represented as $\mathscr{P}^{\mathfrak{e}} = (-I\ddot{x})^{\top}\dot{x}$, the combined inequality $\mathscr{P}^{\mathfrak{e}} - \dot{\psi} \geq 0$, with the free energy defined as in (21), yields the following:

$$\begin{aligned} \mathscr{P}^{\mathfrak{e}} - \dot{\psi} = &\left(-I\ddot{x} - E\,(\varepsilon - p)\,\|x\|^{-2} x\right)^{\top} \dot{x} \\ &+ \left(E(\varepsilon - p)\dot{p} - (H^{-1}\tau)\dot{\tau} - (K\,a)\right)\dot{a} \geq 0 \,. \end{aligned} \tag{25}$$

Similarly as before, we may then separate the two dynamic variables $\dot{x}$ and $\dot{a}$ obtaining the system of differential equations:

$$\begin{cases} I\ddot{x} + \|x\|^{-2}\sigma x = 0 \\ \sigma\dot{p} - (H^{-1}\tau)\dot{\tau} - (K\,a) = m\dot{a} \,, \end{cases} \tag{26}$$

where $\sigma = E(\varepsilon - p)$.

4.5 Results and Discussion

The algorithm employed to solve the system of differential equations reported in (26) is a standard Newmark time-integration scheme, and the algorithm is briefly sketched in Alg. 2. For a fair comparison, we made use of the same parameters as in Armero (2006), *i.e.*, with the discrete time-step $h \in \{0.3, 1.0, 3.0\}$ seconds, and Newmark integration parameters $\beta = 1/4$, and $\gamma = 1/2$.

The results are pictured in Fig. 3, where we outlined the mass trajectory, the angular momentum, the total energy, and the equivalent plastic strain; the results include the graph of the afore-mentioned quantities for all the three time-steps h. As we may notice from the algorithmic outline, no constraint was imposed explicitly in the integrating process: the system in itself, being exempt from external forces, exhibits a constant angular momentum, accounted for implicitly by the power-based approach described in previous sections.

function NONLINEARPLASTICITY(h, x_n, p_n, τ_n, a_n) ▷ Solves equations (26) at time step $n+1$
  $j \leftarrow 0$
  $x^0_{n+1} \leftarrow x_n + h\dot{x}_n + h^2/2\,(1-2\beta)\,\ddot{x}_n$ ▷ $x_n := (x_1, x_2)^\top$
  $[p,\tau,a]^0_{n+1} \leftarrow [p,\tau,a]_n + h\,[\dot{p},\dot{\tau},\dot{a}]_n(1-\alpha)$

  $\dot{x}^0_{n+1} \leftarrow \dot{x}_n + h\,(1-\gamma)\ddot{x}_n$
  $[\dot{p},\dot{\tau},\dot{a}]^0_{n+1} \leftarrow [\dot{p},\dot{\tau},\dot{a}]_n$
  $\ddot{x}^0_{n+1} \leftarrow 0$

  while $e \geq$ toll **do**
    $L \leftarrow \|x^j_{n+1}\|$
    $\varepsilon^j_{n+1} \leftarrow \log(L^j_{n+1}) - \log(L_0)$
    $\dot{\varepsilon}^j_{n+1} \leftarrow h\,\alpha^{-1}\,((\varepsilon^j_{n+1} - \varepsilon_n) - (1-\alpha)\dot{\varepsilon}_n)$
    $[p,\tau,a]^j_{n+1} \leftarrow$ PLASTICITY(h, $[p,\tau,a]_n$, $[\dot{p},\dot{\tau},\dot{a}]_n$) ▷ See Fig. 1

    $\sigma^j_{n+1} \leftarrow E(\varepsilon^j_{n+1} - p^j_{n+1})$
    $r^j_{n+1} \leftarrow m\ddot{x}^j_{n+1} + L_0/L^2\,\sigma^j_{n+1}\,x^j_{n+1}$

    $\delta x_{n+1} \leftarrow -(\mathbf{J}^j_{n+1})^{-1} r^j_{n+1}$ ▷ Update the current solution
    $x^{j+1}_{n+1} \leftarrow x^j_{n+1} + \delta x_{n+1}$
    $\dot{x}^{j+1}_{n+1} \leftarrow \gamma(h^{-1})\,\beta^{-1}(x^{j+1}_{n+1} - x_n) + (1-\gamma/\beta)\,\dot{x}_n + h\,(1-\gamma/2\beta)\,\ddot{x}_n$
    $\ddot{x}^{j+1}_{n+1} \leftarrow (h^{-2})\,\beta^{-1}(x^{j+1}_{n+1} - x_n) - (h\,\beta)^{-1}\,\dot{x}_n + (1-1/2\beta)\,\ddot{x}_n$
    $e \leftarrow \delta x_{n+1} \cdot r^j_{n+1}$
    $j \leftarrow j+1$
  end while

  $[x,\dot{x},\ddot{x}]^{j*}_{n+1} \leftarrow [x,\dot{x},\ddot{x}]^j_{n+1}$ ▷ Solution at convergence
  $[p,\tau,a]^{j*}_{n+1} \leftarrow [p,\tau,a]^j_{n+1}$
  return $[x,\dot{x},\ddot{x}]^{j*}_{n+1}$ **and** $[p,\tau,a]^{j*}_{n+1}$
end function

Algorithm 2: Sketch of the algorithm used to solve the problem detailed in Section 4.4. The notation $[\cdot]$ indicates an *ordered list*, for brevity's sake.

Comparing our results to the reference solutions provided in Armero (2006), we may observe as the energy, angular momentum, and the equivalent plastic strain do not diverge. In particular, the total energy ψ, and the plastic strain a display a divergent behavior for $h = 3.0$, whereas in our tests they all saturate to a constant solution. Evidently, the diverging process we may perceive in Armero (2006) is not found, attaining convergence with no additional constraint on the angular momentum.

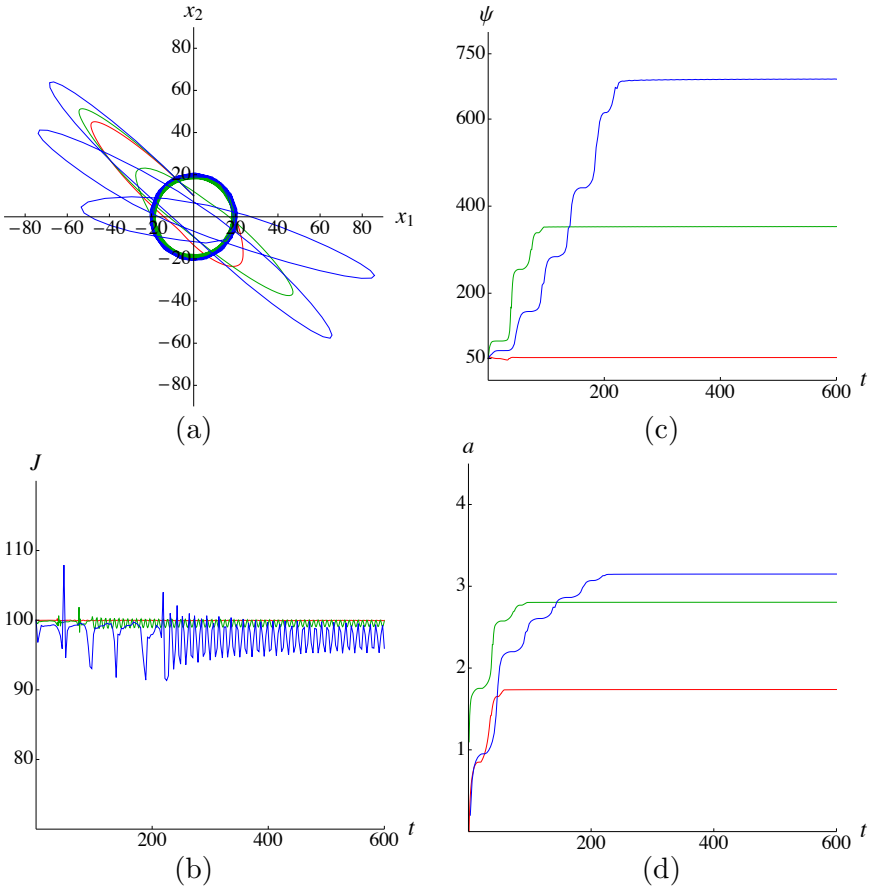

Figure 3: Solution for the mass-spring system detailed in Section 4.4 ($\beta = 1/4$ and $\gamma = 1/2$): the mass trajectory (a), the angular momentum (b), the total energy (b), and the equivalent plastic strain (d). The solution has been obtained with time lapses h of 0.3 (in red), 1.0 (in green), and 3.0 seconds (in blue).

Our results show clearly that with a power-based approach, the nonlinear problem detailed in Section 4.4 can be solved without explicitly imposing the constraint on the angular momentum, while converging for every time gap h, contrary to a standard energy-based Newmark integrating scheme. As a side-note, we highlight the fact that convergence is guaranteed by means of the same parameters employed in the construction of the (numerical) constitutive model.

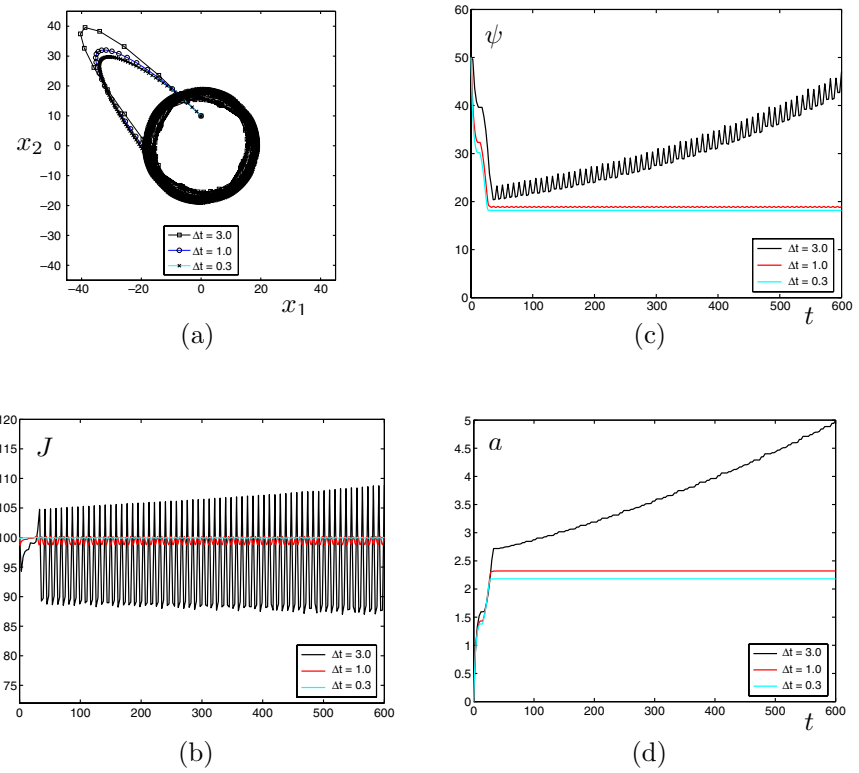

Figure 4: Solution for the mass-spring system detailed in Section 4.4, provided in Armero (2006) and obtained through a standard Newmark scheme ($\beta = 1/4$ and $\gamma = 1/2$): the mass trajectory (a), the angular momentum (b), the total energy (b), and the equivalent plastic strain (d). The solution has been obtained with time lapses $h = \Delta t$, as reported in the legends.

5 Concluding Remarks

This manuscript presents a novel stability criterion for the trapezoidal rule integrating schemes, proving a sufficient condition for the unconditionally stable convergence. Contrary to other standard energy-based approaches, relying on the power balance and dissipation principle allows us to generalize this result to evolutive nonlinear problems.

In fact, the power criterion does not require any particular property concerning the system response in order to prove stable convergence, such as positive-definiteness. Moreover, convergence criteria are originated from the (power) principles adapted to generate a thermodynamically consistent model, without introducing additional

energy-like conserving constraints, apt at limiting energy oscillations leading to convergence loss. The numerical stability was proved for both first and second order nonlinear problems, extending the applicability of such numerical schemes previously employed in linear contexts Hughes (2000), and very specific nonlinear ones Belytschko and Schoeberle (1975).

We numerically tested a simplified one-dimensional problem, consisting of a one mass-spring system, nonlinearly modeled in terms of both finite deformations and hardening plasticity. As detailed in the last section, such power-based approach benefits the numerical stability when compared to a standard trapezoidal scheme, and convergence has been attained, specifically without imposing the angular momentum conservation. In particular, the absence of such a constraint would lead a standard Newmark algorithm to loose convergence Armero (2006) with increasing integrating time steps.

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References

Armero, F. (2006): Energy-dissipative momentum-conserving time-stepping algorithms for finite strain multiplicative plasticity. *Computer Methods in Applied Mechanics and Engineering*, vol. 195, no. 37–40, pp. 4862–4889.

Bardella, L.; Genna, F. (2005): Newmark's time integration method from the discretization of extended functionals. *Trans. ASME Journal of Applied Mechanics*, vol. 72, no. 4, pp. 527–537.

Belytschko, T. B.; Schoeberle, D. F. (1975): On the unconditional stability of an implicit algorithm for nonlinear structural dynamics. *Journal of Applied Mechanics*, vol. 17, pp. 865–869.

Carini, A.; Genna, F. (2000): Saddle-point principles and numerical integration methods for second-order hyperbolic equations. *Computer Methods in Applied Mechanics and Engineering*, vol. 190, no. 13-14, pp. 1663–1678.

Casciaro, R. (1975): Time evolutional analysis of nonlinear structures. *Meccanica*, vol. 10, pp. 156–167.

Cherubini, C.; Filippi, S.; Nardinocchi, P.; Teresi, L. (2009): Electromechanical modelling of cardiac tissue. In Kamkin, A.; Kiseleva, I.(Eds): *Mechanosensitivity of the Heart*, volume 3, chapter 16. Springer.

Corigliano, A.; Perego, U. (1993): Generalized midpoint finite element dynamic analysis of elastoplastic systems. *International Journal for Numerical Methods in Engineering*, vol. 36, no. 3, pp. 361–383.

Crisfield, M. A.; Shi, J. (1994): A co-rotational element/time-integration strategy for non-linear dynamics. *International Journal for Numerical Methods in Engineering*, vol. 37, no. 11, pp. 1897–1913.

Del Piero, G. (2009): On the method of virtual power in continuum mechanics. *Journal of Mechanics of Materials and Structures*, vol. 4, no. 2, pp. 281–292.

DiCarlo, A.; Quiligotti, S. (2002): Growth and balance. *Mechanics Research Communications*, vol. 29, no. 6, pp. 449–456.

Gams, M.; Planinc, I.; Saje, M. (2008): Energy conserving time integration scheme for geometrically exact beam. *Computer Methods in Applied Mechanics and Engineering*, vol. 196, no. 17-20, pp. 2117–2129.

Gonzalez, O. (2000): Exact energy and momentum conserving algorithms for general models in nonlinear elasticity. *Computer Methods in Applied Mechanics and Engineering*, vol. 190, no. 13–14, pp. 1763–1783.

Gurtin, M. E. (2000): On the plasticity of single crystals: free energy, microforces, plastic-strain gradients. *Journal of the Mechanics and Physics of Solids*, vol. 48, pp. 989–1036.

Gurtin, M. E.; Reddy, B. D. (2009): Alternative formulations of isotropic hardening for Mises materials, and associated variational inequalities. *Continuum Mechanics and Thermodynamics*, vol. 21, no. 3, pp. 237–250.

Hughes, T. J. R. (2000): *The Finite Element Method: Linear Static and Dynamic Finite Element*. Dover.

Olsson, T.; Klarbring, A. (2008): Residual stresses in soft tissue as a consequence of growth and remodeling: application to an arterial geometry. *European Journal of Mechanics A/Solids*, vol. 27, no. 6, pp. 959–974.

Simo, J.; Hughes, T. J. R. (2000): *Computational Inelasticity*. Springer.

Simo, J.; Tarnow, N. (1992): The discrete energy-momentum method. conserving algorithms for nonlinear elastodynamics. *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)*, vol. 43, no. 5, pp. 757–792.

Truesdell, C.; Noll, W. (2004): *The non-linear field theories of mechanics*. Springer-Verlag, Berlin, third edition.

Wallstedt, P. C.; Guilkey, J. E. (2008): An evaluation of explicit time integration schemes for use with the generalized interpolation material point method. *Journal of Computational Physics*, vol. 227, no. 22, pp. 9628–9642.