# Dispersion of One Dimensional Stochastic Waves in Continuous Random Media

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**Abstract:**　　Second, or higher, order harmonics have great potential in fatigue life prediction. In this study, the dispersion properties of waves propagating in the nonlinear random media are investigated. An one dimensional nonlinear model based on the nonlinear Hikata stress-strain relation is used. We applied perturbation method, the Liouville transformation and the smoothing approximation method to solve the one dimensional nonlinear stochastic wave equation. We show easily that the dispersion equations for all higher order terms will be the same with the corresponding linear random medium by perturbation method. The linear stochastic equation with two random coefficients is greatly simplified to an equatin with just one random coefficients by the Liouville transformation. And without using any more approximations, the Green function for the first term and the closed form dispersion equation are obtained explicitly. The numerical solution of the dispersion equation shows that the phase velocity for the same wave number will decrease when damage factor—a measure of the total damage/inhomogeneties—increases and will increase to the velocity of the undamaged material for a given damage factor when the circular frequency increases. And very excitingly, we find that there is cutoff wave number which is rarely found before. Like phonon which has forbidden bands for frequency, a medium with particular randomly distributed damages/inhomogeneities will have forbidden bands for wave length. This may have special applications in industry. The simplification method of stochastic equations with multiple coefficients can also be used to other stochastic problems. And the dispersion equation and its properties obtained in this study may give theoretical support to the nonlinear NDE community for predicting fatigue life.

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## 1   Introduction

For many materials such as metals, more than 80% of the total fatigue life of materials will be consumed before a macro-crack appears. Thus, it is critically important to evaluate qualitatively the fatigue life of engineering structures in service. Many works (Morris, Buck, and Inman, 1979; Yost and Cantrell, 1990; Deng, Scharf, and Barnard, 1997; Cantrell and Yost, 2000; Kim, Qu, Jacobs, Littles, and Savage, 2006; Pruell, Kim, Qu, and Jacobs, 2009) have demonstrated experimentally the strong correlation between fatigue life and the second order harmonic. Thus the nonlinear ultrasonic test offers a great potential to characterize and quantify the fatigue life.

Dislocations have long been recognized as the main source of higher order harmonics (Granato and Lucke, 1956; Hikata, Chick, and Elbaum, 1965). However, systematic studies on establishing a quantitative relationship between dislocations and higher order harmonics were not conducted until the 1980s. In a series of papers (Cantrell and Yost, 2000; Cantrell, 2004, 2006), Cantrell and his colleagues formulated a theory that establishes the relationship between second order harmonics and dislocation densities, dislocation dipoles, precipitates, micro-cracks, etc. Although dislocation dynamics (initiation, annihilation, motion, accumulation, and localization) is responsible for fatigue damage, the precise information about their types, distribution, or even the density, is not clear to the metal fatigue community, primarily because the large number of dislocations and the fact that the characteristic length of dislocations is typically in the nanometer range, so that experimental measurements/observations of every individual dislocation is very difficult (more information on dislocation dynamics could be found in Ghoniem' review paper (Ghoniem and Cho, 2002)).

Although the precise information about dislocations in the fatigue process is hard to get, the stochastic information, e.g. correlation length and density distribution, can be obtained experimentally (Kozlowski, Paszkiewicz, Korbutowicz, and Tlaczala, 2001; Beigmohamadi, Niyamakom, Farahzadi, Kremers, Michely, and Wuttig, 2008; Jacques, Le Bolloc'h, and Ravy, 2009). Therefore, a nonlinear model built upon stochastic process will have the potential to practically predict the fatigue life. Some useful numerical methods have been developed to tackle the linear/nonlinear stochastic problems (Radhika, Panda, Manohar, and Source, 2008; Tian, Yang, and Source, 2008; Manjuprasad, Manohar, and Source, 2007; Stroud, Krishnamurthy, and Smith, 2002). Kim and Qu et al. (Kim, Qu, Jacobs, Littles, and

Savage, 2006) have formulated a delicate nonlinear model which considers various dislocations as initial plastic deformation randomly distributed. However, a procedure which can analytically analyze the properties of nonlinear stochastic wave propagation problems is still unavailable today.

To understand the basic properties of the nonlinear wave in random media, e.g. the dispersion curves, we propose a simplified nonlinear stochastic model based on the Hikata nonlinear strain-stress relation (Hikata, Chick, and Elbaum, 1965), and to consider the random properties of the damaged material, the coefficients of the Hikata relation is changed to random variables.

Once we build and solve the stochastic equation which relates the damage factor—a measure of the total fatigue damage—and the nonlinearity parameter, which could be got from the first and the second terms of the linearized stochastic equation, and get the relationship between the damage factor and the cumulative plastic strain, then we can use many existing models (Mediratta, Ramaswamy, and Rao, 1986; Singh, Sundararaman, Chen, and Wahi, 1991; Laird and Feltner, 1967; Santner and Fine, 1977) in fatigue that relate the cumulative plastic strain in a material to its fatigue life.

Figure 1 is a scheme of how to predict the fatigue life using our model. Our model is on the third stage, i.e. the stage from the nonlinear parameter to the damage factor .
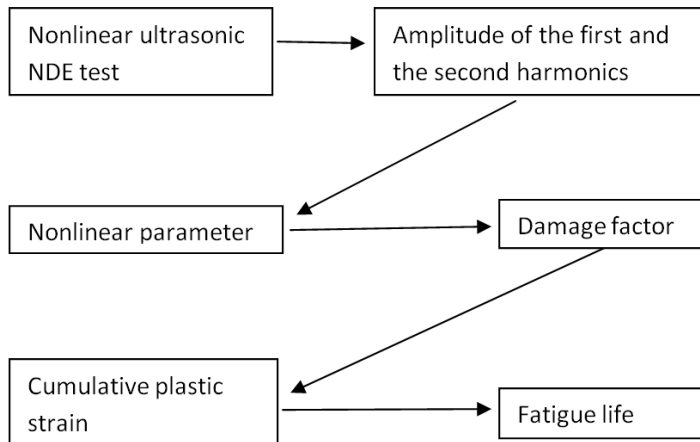
Figure 1: scheme of predicting fatigue life from nonlinear ultrasonic test

Although the model we proposed is greatly simplified, the one dimensional nonlinear stochastic model is still unlikely to be solved directly. Thus the perturbation

method is used to linearize the model. Linear elastic or electromagnetic stochastic wave problems have long been studied. Karal and Keller (Karal Jr and Keller, 1964) analyzed propagation of any type of wave in a linear random medium by small perturbation method. Frisch (Frisch, 1970) investigated thoroughly the mathematical foundation of the diagram method and the smoothing method and their application to multiple scattering problems. Sobczyk (Sobczyk, 1985) gave a detailed description on the linear stochastic differential equation theory, including modeling of continuous and discrete random media, and also reviewed various methods to analyze the wave propagation, e.g. the Born approximation, geometical optics, Rytov method, parabolic approximation, smoothing method, functional approach etc. After comparing various methods, Pao-Liu Chow (Chow, 1975) have concluded that the smoothing method is the most useful analytical method to tackle the random wave propagation problem. Turner and his colleagues used the first order smoothing approximation (FOSA) method to investigated propagation and scattering of elastic waves in heterogeneous media with local isotropy (Turner and Anugonda, 2001) and in solids with uniaxially aligned cracks (Yang and Turner, 2003a). However the smoothing method available today is very complex for problems with 2 random coefficients or more, even for a non-dissipative medium, because it needs all the auto- and cross-correlations of the random variables. This has greatly limited the practical application of the smoothing method.

In this study, we utilize the Liouville substitution to transform the equation with 2 random coefficients into an equation with 1 random coefficient. By this way, we only need to know one auto-correlation function, thus the problem is greatly simplified. Then the transformed equation is solved analytically by the FOSA method. At last, the Green function and the dispersion equation are obtained explicitly. To analyze the properties of the dispersion equation, several kinds of dispersion curves of a thin aluminium (also for silicon crystal, titanium alloy) rod are plotted numerically. Analysis and discussion of these plots are then followed.

## 2    Mathematical modeling and analysis

We use the nonlinear stress-strain relation introduced by Hikata (Hikata, Chick, and Elbaum, 1965),

$$\sigma(x,t,\gamma) = A(x,\gamma)\frac{\partial u(x,t,\gamma)}{\partial x} + \frac{1}{2}B(x,\gamma)\left(\frac{\partial u(x,t,\gamma)}{\partial x}\right)^2 \tag{1}$$

Where $\sigma$ is the stress, $A$ is the second-order elastic constant, and $B$ is the nonlinear coefficient, which is the combination of the second and the third-order elastic constants. In Hikata's paper (Hikata, Chick, and Elbaum, 1965), $A, B$ were taken as constants. However, $A, B$ should not only vary with space, but also be random

in materials with randomly distributed dislocations and micro-cracks. So we introduce a random variable $\gamma$ here.

It should be pointed out that the $u$ in equation (1) is the displacement from initial to current state rather than from natural to current state, thus it can still be viewed as infinitesimal.

Substituting equation (1) into the wave equation

$$\rho(x,\gamma)\frac{\partial^2 u(x,t,\gamma)}{\partial t^2} = \frac{\partial \sigma(x,t,\gamma)}{\partial x} \tag{2}$$

yields,

$$\rho(x,\gamma)\frac{\partial^2 u(x,t,\gamma)}{\partial t^2} = \frac{\partial A(x,\gamma)}{\partial x}\frac{\partial u(x,t,\gamma)}{\partial x} + A(x,\gamma)\frac{\partial^2 u(x,t,\gamma)}{\partial x^2} +$$
$$\frac{1}{2}\frac{\partial B(x,\gamma)}{\partial x}\left(\frac{\partial u(x,t,\gamma)}{\partial x}\right)^2 +$$
$$B(x,\gamma)\frac{\partial u(x,t,\gamma)}{\partial x}\frac{\partial^2 u(x,t,\gamma)}{\partial x^2} \tag{3}$$

It should be noticed that in equation (3), $\rho$ is a also function of space and random variable because of the randomly distributed dislocations and micro-cracks. And from now on, we will omit some variables in the functions for convenience if there is no confusion.

It is very hard to solve the nonlinear stochastic equation (3) directly, as a consequence, we have to linearize it. The perturbation method is used to linearize equation (3). The amplitude $\varsigma$ of an input ultrasonic harmonic wave is taken as the small parameter for the perturbation. The displacement due to the input ultrasonic waves can be expressed by this series:

$$u = \varsigma u_1 + \varsigma^2 u_2 + \dots \tag{4}$$

Substituting equation (4) into equation (3), we could get the first two terms of the series,

$$O(\varsigma): \qquad \rho\frac{\partial^2 u_1}{\partial t^2} - \frac{\partial}{\partial x}\left(A\frac{\partial u_1}{\partial x}\right) = 0 \tag{5}$$

$$O(\varsigma^2): \qquad \rho\frac{\partial^2 u_2}{\partial t^2} - \frac{\partial}{\partial x}\left(A\frac{\partial u_2}{\partial x}\right) = \frac{1}{2}\frac{\partial B}{\partial x}\left(\frac{\partial u_1}{\partial x}\right)^2 + B\frac{\partial u_1}{\partial x}\frac{\partial^2 u_1}{\partial x^2} \tag{6}$$

It could be seen that the operators for the first and second terms are the same with the corresponding linear random medium , therefore, their dispersion equations

should be the same. And we could also show further by perturbation method that all the dispersion equations for the higher terms should be the same with the first order equation.

Applying Fourier transformation to equation (5) and (6), we get,

$$O(\varsigma): \qquad \rho\omega^2 U_1 + \frac{\partial}{\partial x}\left(A\frac{\partial U_1}{\partial x}\right) = 0 \tag{7}$$

$$O(\varsigma^2): \qquad \rho\omega^2 U_2 + \frac{\partial}{\partial x}\left(A\frac{\partial U_2}{\partial x}\right) = -\frac{1}{2\sqrt{2\pi}}\frac{\partial B}{\partial x}\left(\frac{\partial U_1}{\partial x} * \frac{\partial U_1}{\partial x}\right)(\omega) -$$

$$\frac{1}{\sqrt{2\pi}}B\left(\frac{\partial U_1}{\partial x} * \frac{\partial^2 U_1}{\partial x^2}\right)(\omega) \tag{8}$$

In which,

$$U_{1,2}(x,\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_{1,2}(x,t)e^{-i\omega t}\,\mathrm{d}t \tag{9}$$

And,

$$\left(\frac{\partial U_1}{\partial x} * \frac{\partial U_1}{\partial x}\right)(\omega) = \int_{-\infty}^{\infty}\frac{\partial}{\partial x}U_1(x,\varphi)\frac{\partial}{\partial x}U_1(x,\omega-\varphi)\,\mathrm{d}\varphi$$

$$= \int_{-\infty}^{\infty}\left(\frac{\partial}{\partial x}u_1(x,t)\right)^2 e^{-i\omega t}\,\mathrm{d}t \tag{10}$$

$$\left(\frac{\partial U_1}{\partial x} * \frac{\partial^2 U_1}{\partial x^2}\right)(\omega) = \int_{-\infty}^{\infty}\frac{\partial}{\partial x}U_1(x,\varphi)\frac{\partial^2}{\partial x^2}U_1(x,\omega-\varphi)\,\mathrm{d}\varphi$$

$$= \int_{-\infty}^{\infty}\frac{\partial}{\partial x}u_1(x,t)\frac{\partial^2}{\partial x^2}u_1(x,t)e^{-i\omega t}\,\mathrm{d}t \tag{11}$$

## 3　The Green function and the dispersion equation

Since the operators for $U_1, U_2$ are the same, we choose to get the dispersion equation from equation (7).

There are two random coefficients in equation (7), so we apply the Liouville substitution (Heading, 1962) to transform it into an equation with just one random coefficient. Define two new coefficients : $a, z$

$$a = \sqrt{\frac{A}{\rho}} \qquad\qquad\qquad z = \sqrt{A\rho} \tag{12}$$

Since $A$ is the second order elastic constant, $a, z$ can be viewed as the velocity of sound and the characteristic impedance respectively.

Define two transformations:

$$y = <a> \int_0^x \frac{1}{a(\xi)} \, d\xi \qquad\qquad V_1(y) = \sqrt{z} U_1(x, \omega) \tag{13}$$

With these transformations, the heterogeneous elastic medium has been modeled with two new random parameters: the velocity of sound $a$ and the impedance $z$. We assume the medium to be statistically homogeneous, therefore the mean value of $a$ does not depend on $x$.

Applying the transformations to equation (7), we get (see appendix A for the calculation)

$$\frac{\partial^2 V_1}{\partial y^2} + \left( \frac{\omega^2}{<a>^2} + e(y) \right) V_1 = 0 \tag{14}$$

In which

$$e(y) = -\frac{1}{\sqrt{z}} \frac{\partial^2 \sqrt{z}}{\partial y^2} \tag{15}$$

Introduce two new functions, $\zeta(y), k_0^2$

$$\zeta(y) = e(y) - <e(y)> \qquad \text{i.e.} \qquad <\zeta(y)> = 0 \tag{16}$$

$$k_0^2 = \frac{\omega^2}{<a>^2} + <e(y)> \tag{17}$$

Here, we can see that $<e(y)>$ is a measure of the impact of the random damage on the squared wave number ,so we name it damage factor. Normally, the random disturbances is small, i.e. $|<e(y)>| < \frac{\omega^2}{<a>^2}$. Hence, the right side of equation (17) should be positive.

Applying equation (16) and (17), equation (14) can be rewritten as,

$$\frac{\partial^2 V_1}{\partial y^2} + \left( k_0^2 + \zeta(y) \right) V_1 = 0 \tag{18}$$

We use the first order smoothing approximation(FOSA)(Karal Jr and Keller, 1964; Kröner, 1967) to solve this stochastic ODE. The FOSA (see appendix B for the derivation) is considered as the simplest and, meanwhile, the most useful method available to tackle stochastic wave problems (Chow, 1975).

The FOSA equation is,

$$\left( L_0(y) - <L_1(y, \gamma) L_0(y)^{-1} L_1(y, \gamma)> \right) <u(y, \gamma)> = g(y) \tag{19}$$

If $G_0(y, y_1)$ is the Green function of the operator $L_0$, then

$$L_0^{-1} f(y) = \int G_0(y, y_1) f(y_1) \, dy_1 \tag{20}$$

and equation (19) takes the form

$$L_0(y) < u(y, \gamma) > - \left\langle L_1(y, \gamma) \int G_0(y, y_1) L_1(y_1, \gamma) < u(y_1, \gamma) > dy_1 \right\rangle = g(y) \tag{21}$$

Or

$$L_0(y) < u(y, \gamma) > - \int G_0(y, y_1) R(y, y_1) < u(y_1, \gamma) > dy_1 = g(y) \tag{22}$$

In which, $R(y, y_1)$ is the correlation function of $L_1(y, \gamma)$

$$R(y, y_1) = \int L_1(y, \gamma) L_1(y_1, \gamma) \, d\gamma \tag{23}$$

Equation (22) can also be rewritten as

$$< u(y, \gamma) >= L_0^{-1} g(y) + L_0^{-1} \int G_0(y, y_1) R(y, y_1) < u(y_1, \gamma) > dy_1 \Rightarrow \tag{24}$$

$$< u(y, \gamma) >= \int G_0(y, y_1) g(y_1) \, dy_1 + \iint G_0(y, y_1) R(y_1, y_2) \times$$
$$G_0(y_1, y_2) < u(y_2, \gamma) > dy_1 \, dy_2 \tag{25}$$

Taking $g(y)$ in equation (25) as the Dirac delta function gives the Green function for $< u(y, \gamma) >$:

$$g(y_1) = \delta(y_1 - y_0)$$
$$< G(y, y_0, \gamma) >= G_0(y, y_0) + \iint G_0(y, y_1) R(y_1, y_2) \times$$
$$G_0(y_1, y_2) < G(y_2, y_0, \gamma) > dy_1 \, dy_2 \tag{26}$$

The $L_0$, and $L_1$ of equation (18) are

$$L_0 = \frac{\partial^2}{\partial y^2} + k_0^2 \qquad\qquad L_1 = \zeta(y) \tag{27}$$

The Green function of $L_0$ is,

$$G_0(y - y_0) = \frac{e^{-ik_0|y - y_0|}}{2ik_0} \tag{28}$$

And we assume that the random field $L_1(x, \gamma)$ is statistically homogeneous, thus its covariance function depends only on the difference of the arguments, i.e.

$$R(y, y_1) = \int \zeta(y, \gamma) \zeta(y_1, \gamma) \, d\gamma = R(y - y_1) \tag{29}$$

According to equation (28) and (29), equation (26) can be rewritten as

$$< G(y - y_0) > = G_0(y - y_0) + \iint G_0(y - y_1) R(y_1 - y_2) \times$$
$$G_0(y_1 - y_2) < G(y_2 - y_0, \gamma) > dy_1 \, dy_2 \tag{30}$$

Since equation (30) is of the convolution type, it can be solved by using the Fourier transform (Frisch, 1970). Applying the non-unitary Fourier transform to equation (30) yields

$$< G(y - y_0) > = \frac{1}{2\pi} \int < \overline{G}(k) > e^{ik(y - y_0)} \, dk \tag{31}$$

$$G_0(y - y_0) = \frac{1}{2\pi} \int < \overline{G}_0(k) > e^{ik(y - y_0)} \, dk \tag{32}$$

And

$$\iint G_0(y - y_1) R(y_1 - y_2) G_0(y_1 - y_2) < G(y_2 - y_0, \gamma) > dy_1 \, dy_2$$
$$= \frac{1}{(2\pi)^3} \iiiint \overline{G}_0(k) \overline{M}(k_1) < \overline{G}(k_2) > \times \tag{33}$$
$$e^{i(k(y - y_1) + k_1(y_1 - y_2) + k_2(y_2 - y_0))} \, dy_1 \, dy_2 \, dk \, dk_1 \, dk_2$$
$$= \frac{1}{2\pi} \int \overline{G}_0(k) \overline{M}(k) < \overline{G}(k) > e^{ik(y - y_0)} \, dk \tag{34}$$

In which, $M$ is the mass or self-energy operator (Frisch, 1970),

$$M(y_1 - y_2) = R(y_1 - y_2) G_0(y_1 - y_2) \tag{35}$$

$$\overline{M}(k_1) = \int M(y_1 - y_2) e^{-ik_1(y_1 - y_2)} \, dk_1 \tag{36}$$

With equations (31), (32) and (34), equation (30) becomes

$$< \overline{G}(k) > = \overline{G}_0(k) + \overline{G}_0(k) \overline{M}(k) < \overline{G}(k) > \tag{37}$$

Then $< \overline{G}(k) >$ could be got from the above equation

$$< \overline{G}(k) > = \left[ \overline{G}_0^{-1}(k) - \overline{M}(k) \right]^{-1} \tag{38}$$

Substituting $G_0$ in equation (28) into equation (32) yields

$$\overline{G}_0(k) = \int\limits_{-\infty}^{\infty} \frac{e^{-ik_0|y-y_0|}}{2ik_0} e^{-ik(y-y_0)} \, d(y-y_0) = \frac{k}{k_0(k_0{}^2 - k^2)} \tag{39}$$

Now, take the random operator $L_1$ as Uhlenbeck-Ornstein process (Frisch, 1970). It is known that this process is a centered, stationary, Gaussian and Markovian random function and its correlation function is

$$R(y, y_1) = \int L_1(y, \gamma) L_1(y_1, \gamma) \, d\gamma = \varepsilon^2 e^{-\frac{|y-y_1|}{R_c}} \tag{40}$$

In which, $\varepsilon = \sqrt{<L_1{}^2>}$ and it is the standard deviation of the random heterogeneity. And $R_c$ is the integral radius of the correlation (the correlation length), which physically means the scale of heterogeneity. Conventionally $R_c$ is defined as

$$R_c = \frac{1}{2R(0)} \int\limits_{-\infty}^{\infty} R(r) \, dr \tag{41}$$

The Fourier transform of the mass operator $M(y_1 - y_2)$ is

$$\begin{aligned}
\overline{M(k)} &= \int\limits_{-\infty}^{\infty} M(y_1 - y_2) e^{-ik(y_1-y_2)} \, d(y_1 - y_2) \\
&= \frac{R_c \varepsilon^2}{2ik_0} \left( \frac{1}{1 + iR_c(k_0 + k)} - \frac{1}{1 + iR_c(k_0 - k)} \right)
\end{aligned} \tag{42}$$

Substituting equation (42) and (39) into (38), we obtain

$$\begin{aligned}
<\overline{G}(k)> = \Bigg[ &\frac{k_0(k_0{}^2 - k^2)}{k} - \\
&\frac{R_c \varepsilon^2}{2ik_0} \left( \frac{1}{1 + iR_c(k_0 + k)} - \frac{1}{1 + iR_c(k_0 - k)} \right) \Bigg]^{-1}
\end{aligned} \tag{43}$$

According to the Residue Theorem in complex analysis, the inverse Fourier transform of $<\overline{G}(k)>$ will be determined by the zeros of the dominator of equation (43). Therefore, the dispersion equation for the nonlinear stochastic wave problem is

$$\frac{k_0(k_0{}^2 - k^2)}{k} - \frac{R_c \varepsilon^2}{2ik_0} \left( \frac{1}{1 + iR_c(k_0 + k)} - \frac{1}{1 + iR_c(k_0 - k)} \right) = 0 \tag{44}$$

The phase velocity is given by the real part of $k$ and the attenuation by the imaginary part (Turner and Anugonda, 2001). Thus to plot the dispersion curves, we only need to use the real part of $k$.

## 4    Numerical examples and discussions

As an example, we give several plots of the dispersion curves of a pure Aluminum thin rod with randomly distributed micro-damages or inhomogeneities. We take $5000 \, m/s$ as the initial longitudinal wave velocity $< a >$ of the undamaged Aluminum thin rod. To be simple, the correlation length $R_c$ and the standard deviation $\varepsilon$ are set on $0.00001 \, m$ and $1 \, m^{-2}$ respectively. In the following plots $c, k, \omega, < e >$ denote the phase velocity ($m * s^{-1}$), the wave number ($m^{-1}$), the circular frequency ($s^{-1}$), and the damage factor ($m^{-2}$) respectively. The numerical plots of the pure Aluminum thin rod clearly show that:

1. If there is no damage, i.e. $< e >= 0 \, m^{-2}$, the phase velocity is the initial longitudinal wave velocity $5000 \, m/s$ (figure 2, the dash dot line)
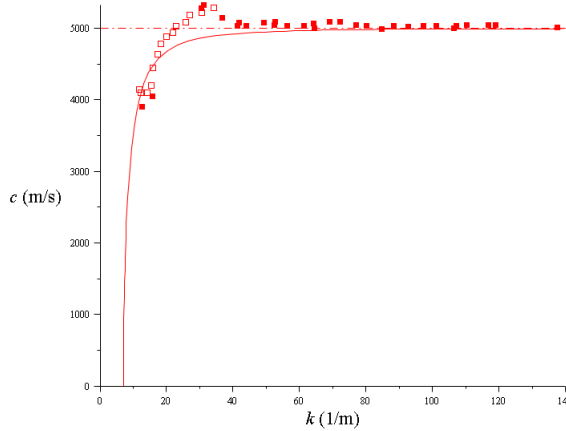


Figure 2: Dispersion curve of the thin Aluminium rod. Dash dot line: without random damage $< e >= 0 \, m^{-2}$; Solid line: with random damage $< e >= 50 \, m^{-2}$ ; Solid box points: experiment by Kinra et al (Kinra, Ker, and Datta, 1982) with volume fraction of random inclusions as 5%; Box points: experiment by Kinra et al (Kinra and Rousseau, 1987) with volume fraction of random inclusions as 5%.

2. The phase velocity $c$ for a wave number $k$ will decrease when the damage factor $< e >$ increases (figure 3).

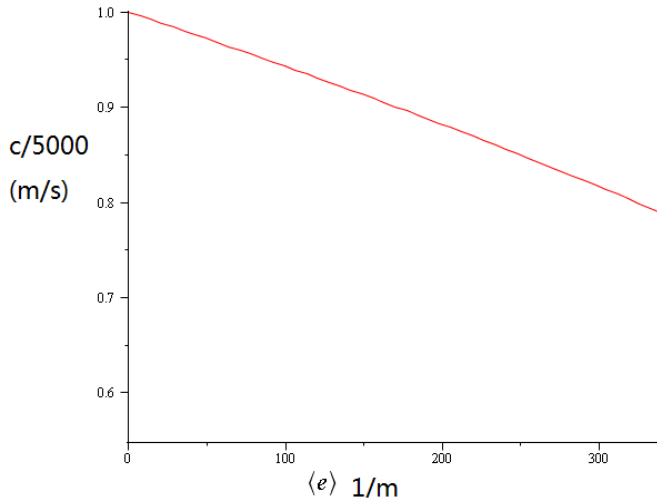Figure 3: Phase velocity $c$—damage factor $<e>$ curve when wave number $k$ is $30\,m^{-1}$

This trend agrees qualitatively well with Turner's theatrical result ((Yang and Turner, 2003b)).

3. The phase velocity $c$ will increase to the initial longitudinal wave velocity for a damage factor $<e>$ when the wave number $k$ increases (see figure 2, figure 4, and figure 5). As we can see from figure 2 and figure 5 that, this trend agrees qualitatively well with Kinral's experiments (the normalized data are from Kinral papers (Kinra, Ker, and Datta, 1982; Kinra and Rousseau, 1987) and Kim's paper (Kim, Ih, and Lee, 1995) ). Karal's experiments were done on various composites with inclusions, however, since materials with random damages/inhomogeneities can be viewed as composites with random inclusions, the well fitness of the theocratical prediction and the experiments proves partly the correctness of our theory.

It is also noticed that, to fit the experiment data, the damage factor $<e>$ goes from $50\,m^{-2}$ to $170\,m^{-2}$ when the volume fraction of random inclusions in Kinra's experiment goes from 5% to 15%. This indicates that the damage factor $<e>$ is also strongly correlated with the volume fraction of the random damages/inclusions in a material. A physical explanation for the damped wave velocity is that the damages or inhomogeneities distributed according to the correlation function will resonate, therefore, the energy carried by the wave will be partly transmitted to the randomly distributed damages
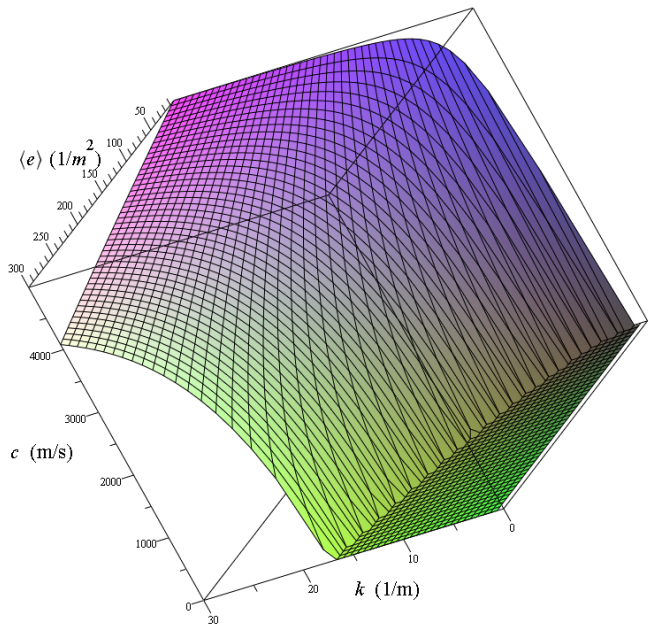
Figure 4: Phase velocity *c*—wave number *k*—damage factor $< e >$ surface
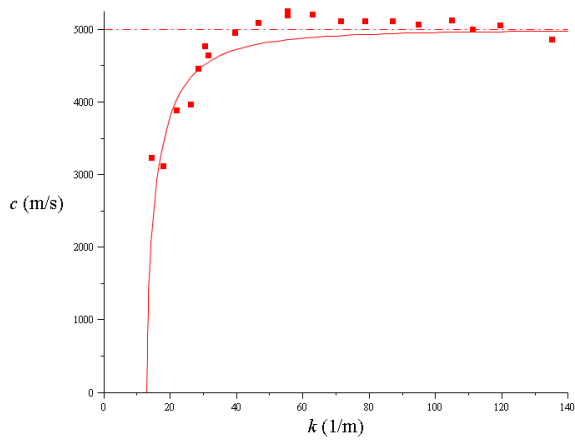


Figure 5: Dispersion curve of the thin Aluminium rod. Dash dot line: without random damage $< e >= 0\ m^{-2}$; Solid line: with random damage $< e >= 170\ m^{-2}$; Solid box points: experiment by Kinra et al (Kinra and Rousseau, 1987) with volume fraction of random inclusions as 15%.

or inhomogeneities. Thus the wave is damped.

4. It is interesting to note that, the phase velocity $c$ stays at 0 when the wave number $k$ is smaller than a particular value for a damage factor $< e >$. For example, the phase velocity $c$ would be 0 for waves with wave number $k$ smaller than $17\,m^{-1}$ when the randomness factor $< e >$ is $300\,m^{-2}$ (figures 4 and 6). Unlike the common waveguides we familiar with, e.g. strings on an
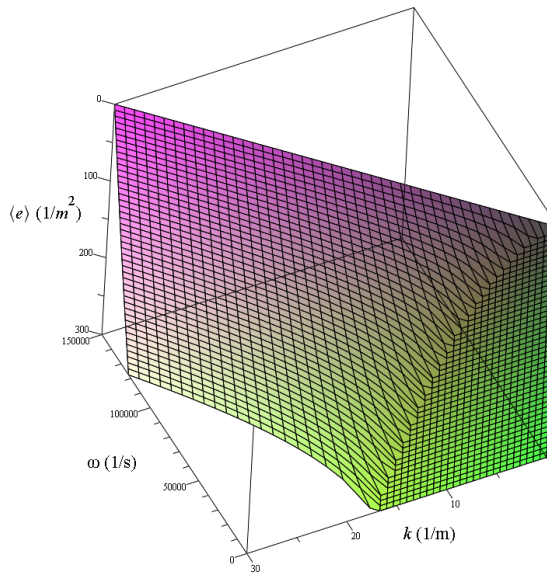


Figure 6: Circular frequency $\omega$—wave number $k$—damage factor $< e >$ surface

elastic foundation (Graff, 1991) and thin plates (Achenbach, 1973), which have cutoff frequency, the thin rod with randomly distributed damages has cutoff wave number (also see figure 2 and figure 5). And the upper limit of the cutoff wave number goes larger when the damage factor increases (figures 4 and 6). A mathematical explanation for this phenomenon is that the circular frequency $\omega$ becomes a negative pure imaginary number (see the imaginary part the solid line in figure 7 ) when the wave number $k$ is below $17\,m^{-1}$. If we describe the wave in the form of a plane wave, $A * exp(i(kx - \omega t))$, the wave becomes $A * exp(-\omega_I t) * exp(ikx)$ when the circular frequency $\omega$ becomes a negative pure imaginary number $-i\omega_I$ ($\omega_I$ is a real positive number here). Therefore, this wave decreases gradually to zero when time goes on, so this wave can not propagate further, as a consequence, it's phase velocity becomes zero. A physical explanation is similar to the explanation for
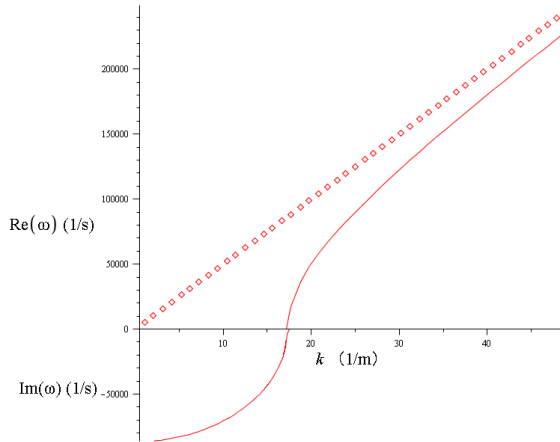
Figure 7: Frequency spectrum of the thin aluminium rod. Diamond point line: without random damage $< e >= 0\,m^{-2}$; Solid line: with random damage $< e >= 300\,m^{-2}$

the damped wave velocity, but the energy carried by the wave will be fully transmitted to the randomly distributed damages or inhomogeneities. Thus, waves with wave number below a value can not propagate further.

It is known that there is phonon which has forbidden bands for frequency. According to our study here, a medium with particular randomly distributed damages/inhomogeneities will have forbidden bands for wave length, that is, a wave with wave length in the forbidden bands can not propagate through the medium.

5. When doing experiments, the frequency rather than the wave number of the ultrasonic wave is used as the controlling signal, so we give some plots with circular frequency $\omega$ as variable here (figures 8, 9, 10). And it could be seen from these plots that:

   (a) There is no cutoff frequency (figure 8,9 );

   (b) The phase velocity $c$ will increase to the initial longitudinal wave velocity for a damage factor $< e >$ when the circular frequency $\omega$ increases (figure 9);

   (c) The phase velocity $c$ for a circular frequency $\omega$ will decrease slowly to 0 when the damage factor $< e >$ increases (figure 10).

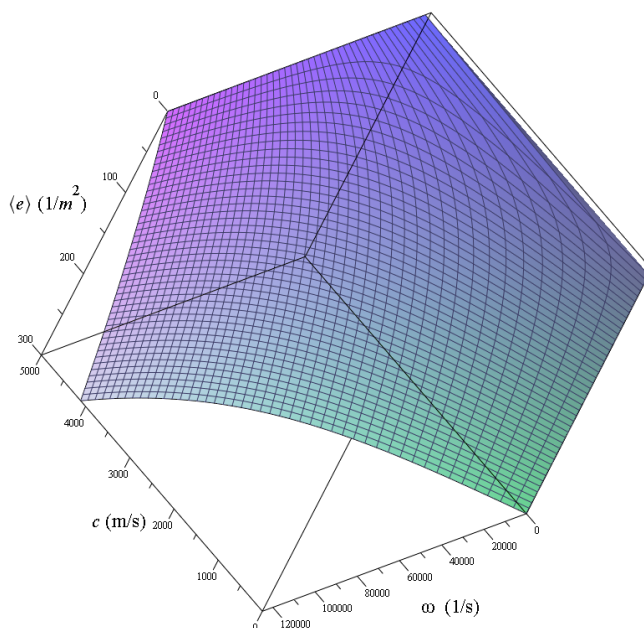Figure 8: Phase velocity $c$—circular frequency $\omega$—damage factor $< e >$ surface

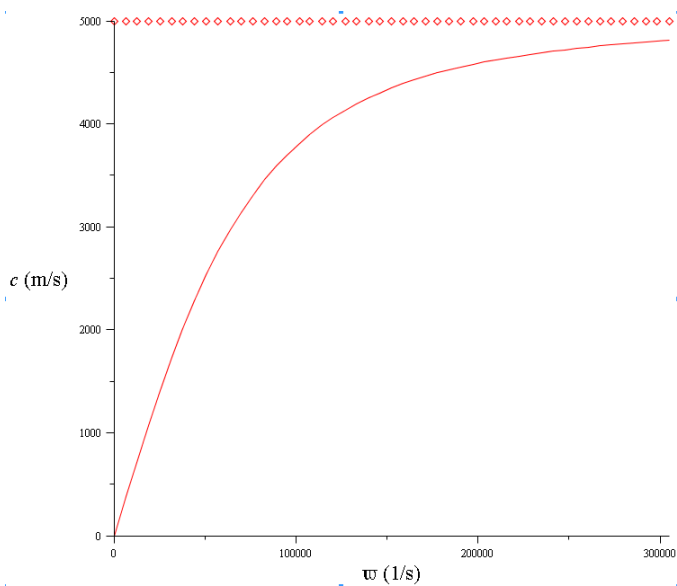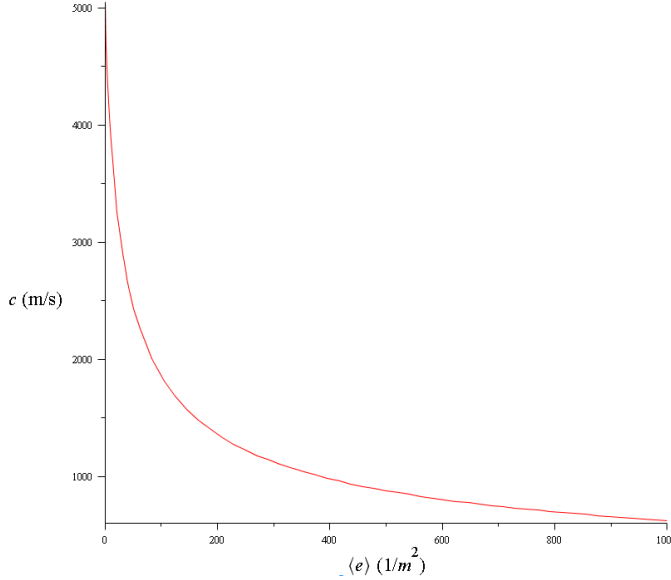Figure 9: Solid line: with random damage, $< e >$ is $300\ m^{-2}$; Diamond point line: without random damage, $< e >= 0\ m^{-2}$

Figure 10: phase velocity $c$—damage factor $< e >$ curve when circular frequency $\omega$ is 20000 $1/s$

The dispersion properties found for the pure aluminum can be applied to other materials as well. As examples, we present here the dispersion curves and frequency spectrums of a silicon crystal thin rod and a titanium alloy (Ti6Al4V Grade 5) thin rod. We take 8433 $m/s$ as the initial longitudinal wave velocity $< a >$ for the undamaged silicon crystal thin rod, and 5078 $m/s$ as that of the titanium alloy (Ti6Al4V Grade 5) thin rod. The correlation length $R_c$ and the standard deviation $\varepsilon$ for both materials are set on 0.00001 $m$ and 1 $m^{-2}$ respectively. The computed results are figures 11, 12, 13, and 14. From these plots, we can see that the dispersion properties for silicon crystals and titanium alloys are similar to that of pure aluminum.

## 5 Conclusion

Second, or higher, order harmonics have great potential in fatigue life prediction. In this study, the dispersion properties of nonlinear waves in random media is studied. An one dimensional nonlinear model based on the nonlinear Hikata stress-strain relation (Hikata, Chick, and Elbaum, 1965) is used. And for practical course, the randomness of the damages/heterogeneities is considered by letting the coefficients of the dynamic equation to be stochastic functions. The perturbation method is
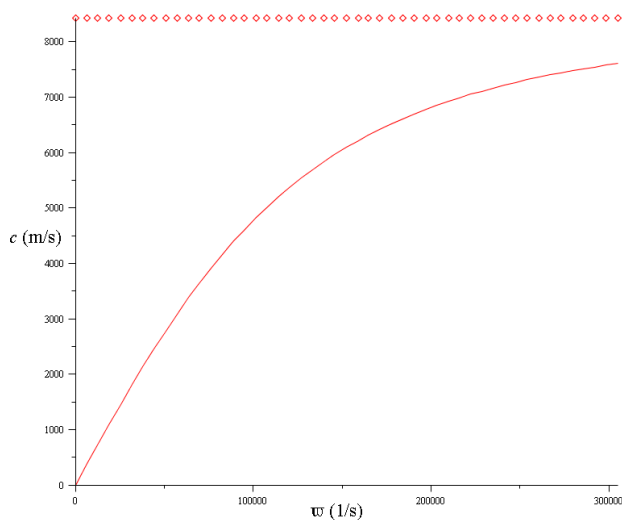
Figure 11: Dispersion curve of the thin silicon crystal rod. Solid line: with random damage, $< e >$ is $300\,m^{-2}$; Diamond point line: without random damage, $< e >= 0\,m^{-2}$.
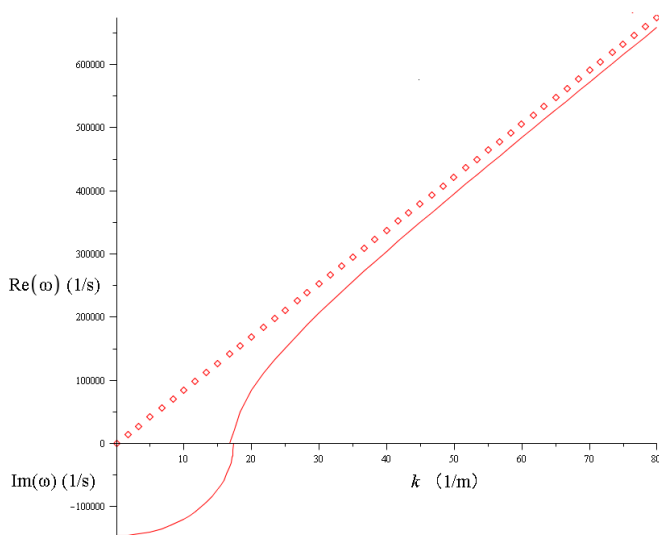


Figure 12: Frequency spectrum of the thin silicon crystal rod. Diamond point line: without random damage, $< e >= 0\,m^{-2}$; Solid line: with random damage, $< e >= 300\,m^{-2}$.
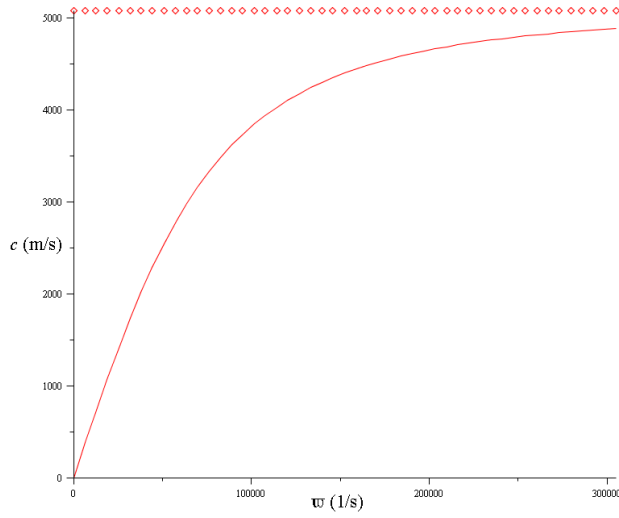
Figure 13: Dispersion curve of the thin titanium alloy rod. Solid line: with random damage, $< e >$ is $300\,m^{-2}$; Diamond point line: without random damage, $< e >= 0\,m^{-2}$.
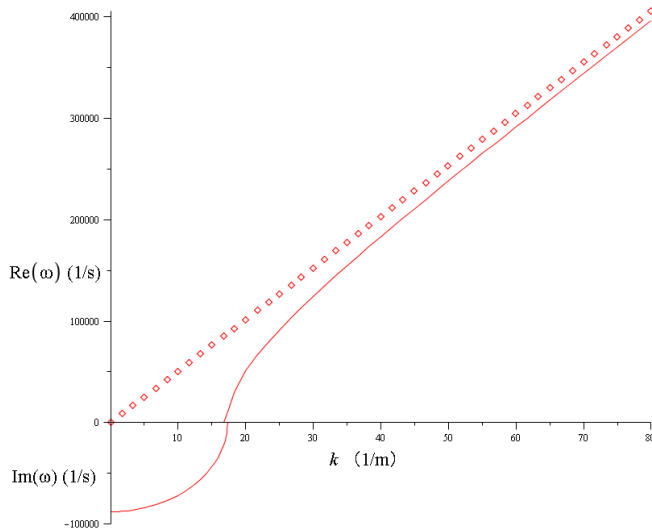


Figure 14: Frequency spectrum of the thin titanium alloy rod. Diamond point line: without random damage, $< e >= 0\,m^{-2}$; Solid line: with random damage, $< e >= 300\,m^{-2}$.

used to linearize the nonlinear stochastic equation. We show easily in this study that the dispersion equations for all higher order terms will be the same with the corresponding linear random medium by perturbation method. Then, the Liouville transformation is applied to transform the stochastic equation with two random coefficients into an equation with one random coefficient. Next, the widely used FOSA method is utilized to solve the stochastic equation. At last, without using any other approximations, the Green function and the closed form dispersion equation for all frequencies have been obtained explicitly.

As an example, several dispersion surfaces/curves of a thin aluminium rod are plotted numerically. These plots show that the phase velocity for the same wave number will decrease when the damage factor increases and will increase to the velocity of the undamaged thin rod for a given damage factor when the circular frequency increases. Comparison with Karal's experiment (Karal Jr and Keller, 1964) and Turner's theoretical result (Yang and Turner, 2003a) show that the damage factor $< e >$ is strongly correlated with the volume fraction and the density of the random damages/inclusions in a material. And very excitingly, we find that there is cutoff wave number which is rarely found before and the upper limit of which will increase with the damage factor. Just like phonon which has forbidden bands for frequency, a medium with particular randomly distributed damages/inhomogeneities will have forbidden bands for wave length. This may have special applications in civil or military industry. The dispersion properties found above can also be applied to other materials. The simplification method of stochastic equations with multiple coefficients into equations with just one coefficient can also be used to other stochastic problems. And since the nonlinear model in this study has considered the randomness of the heterogeneities, the dispersion equation and its properties obtained in this study may give theoretical support to the nonlinear NDE community for predicting fatigue life.

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## Appendix A: Calculation of the Liouville transformation

From equation (12), we know that

$$A = az \qquad\qquad \rho = \frac{z}{a} \qquad\qquad (45)$$

And, since $< a >$ does not depend on $x$, thus $\frac{d<a>}{dx} = 0$.

From equation (13), we have

$$\frac{dy}{dx} = \frac{<a>}{a} \tag{46}$$

$$\frac{dU_1}{dx} = \frac{d(Vz^{-\frac{1}{2}})}{dx} = \frac{d(Vz^{-\frac{1}{2}})}{dy}\frac{dy}{dx} = \left(\frac{d(z^{-\frac{1}{2}})}{dy}V_1 + z^{-\frac{1}{2}}\frac{dV_1}{dy}\right)\frac{<a>}{a} \tag{47}$$

$$\frac{d^2U_1}{dx^2} = -\frac{<a>}{a^2}\frac{da}{dx}\left(\frac{d(z^{-\frac{1}{2}})}{dy}V_1 + z^{-\frac{1}{2}}\frac{dV_1}{dy}\right) +$$

$$\frac{<a>^2}{a^2}\left(\frac{d^2(z^{-\frac{1}{2}})}{dy^2}V_1 + 2\frac{d(z^{-\frac{1}{2}})}{dy}\frac{dV_1}{dy} + z^{-\frac{1}{2}}\frac{d^2V_1}{dy^2}\right) \tag{48}$$

Substituting equation (45) and the above relations in equation (7) gives

$$H_1\frac{dV_1}{dx} + H_2\frac{d^2V_1}{dx^2} + H_3V_1 = 0 \tag{49}$$

Where

$$H_1 = \frac{d(az)}{dx}\frac{<a>}{a}z^{-\frac{1}{2}} - \frac{z<a>}{a}\frac{da}{dx}z^{-\frac{1}{2}} + 2\frac{z<a>^2}{a}\frac{d}{dy}(z^{-\frac{1}{2}})$$

$$= \frac{<a>}{a}z^{-\frac{1}{2}}\left(\frac{d(az)}{dx} - z\frac{da}{dx} - a\frac{dz}{dx}\right)$$

$$= 0 \tag{50}$$

$$H_2 = \frac{\sqrt{z}}{a}<a>^2 \tag{51}$$

$$H_3 = \frac{<a>}{a}\frac{d(az)}{dx}\frac{d}{dy}(z^{-\frac{1}{2}}) - \frac{z<a>}{a}\frac{da}{dx}\frac{d}{dy}(z^{-\frac{1}{2}}) +$$

$$\frac{z<a>^2}{a}\frac{d^2}{dy^2}(z^{-\frac{1}{2}}) + \omega^2\frac{\sqrt{z}}{a}$$

$$= \frac{d(az)}{dx}\frac{d}{dx}(z^{-\frac{1}{2}}) - z\frac{da}{dx}\frac{d}{dx}(z^{-\frac{1}{2}}) - a\frac{dz}{dx}\frac{d}{dx}(z^{-\frac{1}{2}}) -$$

$$\frac{<a>^2}{a}\frac{d^2\sqrt{z}}{dy^2} + \omega^2\frac{\sqrt{z}}{a}$$

$$= \omega^2\frac{\sqrt{z}}{a} - \frac{<a>^2}{a}\frac{d^2\sqrt{z}}{dy^2} \tag{52}$$

In which, the following relations were used to get equation (51) and equation (52)

$$\frac{<a>}{a}\frac{\mathrm{d}(z^{-\frac{1}{2}})}{\mathrm{d}y} = \frac{\mathrm{d}(z^{-\frac{1}{2}})}{\mathrm{d}x} = -\frac{1}{2}z^{-\frac{3}{2}}\frac{\mathrm{d}z}{\mathrm{d}x} \tag{53}$$

$$\frac{\mathrm{d}(z^{-\frac{1}{2}})}{\mathrm{d}y} = -\frac{1}{z}\frac{\mathrm{d}\sqrt{z}}{\mathrm{d}y} \tag{54}$$

$$\frac{\mathrm{d}^2(z^{-\frac{1}{2}})}{\mathrm{d}y^2} = \frac{1}{z^2}\frac{\mathrm{d}z}{\mathrm{d}y}\frac{\mathrm{d}\sqrt{z}}{\mathrm{d}y} - \frac{1}{z}\frac{\mathrm{d}^2\sqrt{z}}{\mathrm{d}y^2} \tag{55}$$

Therefore, equation (49) can be written as

$$\frac{\sqrt{z}}{a}<a>^2\frac{\mathrm{d}^2V_1}{\mathrm{d}x^2} + \left(\omega^2\frac{\sqrt{z}}{a} - \frac{<a>^2}{a}\frac{\mathrm{d}^2\sqrt{z}}{\mathrm{d}y^2}\right)V_1 = 0 \tag{56}$$

And the above equation can be rewritten as,

$$\frac{\partial^2V_1}{\partial y^2} + \left(\frac{\omega^2}{<a>^2} + -\frac{1}{\sqrt{z}}\frac{\partial^2\sqrt{z}}{\partial y^2}\right)V_1 = 0 \tag{57}$$

## Appendix B:   To get the FOSA by CDAM

There are several ways to get the FOSA: the diagram method, the method of smoothing and the correlation discard approximation method (CDAM) (Frisch, 1970; Sobczyk, 1985).  The CDAM is easier to understand, thus we choose the CDAM to get the FOSA here. Consider this equation:

$$L_0(y)u(y,\gamma) + L_1(y,\gamma)u(y,\gamma) = g(y) \tag{58}$$

Where $L_0(y)$ is a given deterministic differential operator with respect to spatial coordinates and $L_1(y,\gamma)$ is a given centered random field, i.e.  $<L_1(y,\gamma)>= 0$. Averaging both sides of equation (58) gives

$$L_0(y)<u(y,\gamma)> + <L_1(y,\gamma)u(y,\gamma)>= g(y) \tag{59}$$

The equation includes not only $<u>$, but also the moment $<L_1u>$. To find the equation for $<L_1u>$, multiply both sides of equation (58) by $L_1(y_1,\gamma)$ and take the average. The resulting equation is

$$L_0<L_1(y_1,\gamma)u(y,\gamma)> + <L_1(y_1,\gamma)L_1(y,\gamma)u(y,\gamma)>=<L_1(y_1,\gamma)>g(y)$$
$$= 0 \tag{60}$$

Which, however, contains the third order moment, whence it is necessary to find another equation for it, etc. So, now we need to introduce truncations or closure assumptions. One of the closure assumptions is the correlation discard approximation method (CDAM), according to which,

$$< L_1(y_1,\gamma)L_1(y,\gamma)u(y,\gamma) >\approx< L_1(y_1,\gamma)L_1(y,\gamma) >< u(y,\gamma) > \qquad (61)$$

Note that this assumption is equivalent to

$$< L_1(y_1,\gamma)L_1(y,\gamma)\Delta u(y,\gamma) >= 0 \qquad (62)$$
$$\Delta u(y,\gamma) = u(y,\gamma)- < u(y,\gamma) >$$

which means that the correlation between the field fluctuation and the product of the values of a random coefficient (at different points) is neglected. Approximation in equation (61) is known as Bourret's local independence hypothesis.

From equation (60) and the CDAM assumption (61) , we could get

$$< L_1(y_1,\gamma)u(y,\gamma) >= - < L_1(y_1,\gamma)L_0(y)^{-1}L_1(y,\gamma) >< u(y,\gamma) > \qquad (63)$$

If $y,y_1$ in above equation are at the same point, then the above equation becomes

$$< L_1(y,\gamma)u(y,\gamma) >= - < L_1(y,\gamma)L_0(Y)^{-1}L_1(y,\gamma) >< u(y,\gamma) > \qquad (64)$$

Substituting equation (64) in equation (59) yields the first order smoothing approximation (FOSA)

$$\left(L_0(y)- < L_1(y,\gamma)L_0(y)^{-1}L_1(y,\gamma) >\right) < u(y,\gamma) >= g(y) \qquad (65)$$