# RKPM with Augmented Corrected Collocation Method for Treatment of Material Discontinuities 

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#### Abstract

An accurate numerical methodology for capturing the field quantities across the interfaces between material discontinuities, in the context of reproducing kernel particle method (RKPM), is of particular interest. For this purpose the innovative numerical technique, so-called augmented corrected collocation method is introduced; this technique is an extension of the corrected collocation method used for imposing essential boundary conditions (EBCs). The robustness of this methodology is shown by utilizing it to solve two benchmark problems of material discontinuities, namely the problem of circular inhomogeneity with uniform radial eigenstrain, and the problem of interaction between a crack and a circular inhomogeneity. Moreover, an efficient algorithm for computing the area associated to each particle for performing nodal quadrature in 2D in the context of RKPM is proposed. The efficacy of this algorithm in determination of the elastic fields within a plate weakened by a hole under uniform far-field tension is demonstrated. This algorithm combined with augmented corrected collocation method provides a powerful tool for treating problems with material discontinuities.


Keywords: reproducing kernel particle method (RKPM), material discontinuity, augmented corrected collocation method, crack inhomogeneity interaction, nodal quadrature

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## 1 Introduction

In meshless methods such as element free Galerkin method (EFGM) [Belytschko, Lu , and Gu (1994)] and reproducing kernel particle method (RKPM) [Liu, Jun, and Zhang (1995)], the geometry of the problem is discretized merely by introducing particles rather than using elements. Therefore, the difficulties due to mesh generation and remeshing encountered in classical finite element method (FEM), for example when dealing with large deformations or crack propagation, can be circumvented. Another important feature of meshless methods is that, by choosing appropriate window function and basis functions, shape functions with any desired smoothness can be readily constructed. Although this property can be very useful in solving various type of boundary value problems (BVPs) of solid mechanics and it obviates the need to any extra post-processing after computing strains and stresses, it may cause some troubles at interfaces in problems in which there are material discontinuities. It should be noted that, at a perfectly bonded interface at which sliding and debonding are not allowed, the displacements and traction stresses are continuous, whereas the strain components are discontinuous. Consequently, it may lead one to try to come up with appropriate RKPM shape functions which possess the necessary discontinuities at the interface. In fact, employing meshless approximation for which the strains are approximated as continuous functions would result in unacceptable solutions.
So far, many endeavors and researches have been devoted to somehow treat this issue. Cordes and Moran (1996) discretized each phase separately by use of moving least squares (MLS) and then enforced the continuity of the displacements at the interface of the two phases by means of Lagrange multiplier method. This process not only requires integration along the interface and possesses the usual problems of Lagrange multiplier method, but also results in an approximation of the field quantity with high overshoots and undershoots in its derivatives in the vicinity of the interface. Overall this methodology has less accuracy than FEM. Masuda and Noguchi (2006), and Krongauz and Belytschko (1998) modeled the aforementioned discontinuity by presenting special shape functions for which the derivatives have jump discontinuity across the interface. Kavashima and Noguchi (2000) used EFGM and employed the idea of separate discretization of phases, which was proposed by Cordes and Moran (1996), except they enforced the continuity of the displacement at the interface by means of penalty method instead of Lagrange multipliers method. Later, Li, Shen, Han, and Atluri (2003), in the framework of MLPG, presented a promising method for solving these problems in multi-dimensional domains which was relatively simple and efficient. Of course utilizing this method requires paying special care to some small details for determining the domain of influence of the particles which are located in the vicinity of
the interface.
In the present work the concept of separate discretization of phases is used, but the continuity condition of the displacement at the interface is imposed by an augmentation of the corrected collocation method. The efficacy of the proposed generalization is studied by investigation of several fundamental examples such as Eshelby's problem [Eshelby (1957)], and the problem of crack-circular inclusion interaction considered by Erdogan, Gupta, and Ratwani (1974). It should be emphasized that in the current study the discretization is done by RKPM. To this end, it is needed to compute the nodal domain associated to each particle. It is important to note that a suitable determination of the nodal domain will directly affect the accuracy of nodal quadrature. In some of previous studies, for example in the works of Chen, Han, You, and Meng (2003) and Jin, Li, and Aluru (2001), in order to facilitate the computations, the nodal domain is considered to be equal to one for all particles. It is noteworthy to point out that this simplification is not compatible with the concept of discretization in RKPM. Khezri, Hashemian, and Shodja (2009) proposed an efficient algorithm to compute the nodal domains in the context of RKPM. They analyzed a benchmark problem in fracture mechanics (an edge-cracked plate under uniform tension) and showed their proposed algorithm leads to more accurate results in comparison with the case in which all the nodal domains are taken to be equal to one. In the current study both methods will be applied to the problem of a plate with a hole under uniform tension.
The framework of the present paper is as follows. The next section is devoted to a brief review of RKPM and the proposed algorithm for computing the area associated to each particle (nodal domain). The theory of the augmented corrected collocation method will be comprehensively presented in section 3 . Section 4, presents some descriptive numerical experiments. Isotropic plate with a hole under uniform tension is examined in section 4.1. The problem of circular inhomogeneity with uniform radial eigenstrain distribution is addressed in section 4.2. The problem of center crack in the vicinity of an inhomogeneity is studied in section 4.3. Finally, conclusions are stated in section 5.

## 2 Implementation of nodal domain in 2D RKPM

A given function, $u(\mathbf{x})$ defined over a two-dimensional domain, $\Omega$ can be reproduced by the following formula:

$$
\begin{equation*}
u^{R}(\mathbf{x})=\int_{\Omega} \bar{\varphi}_{a}(\mathbf{x}, \mathbf{y}) u(\mathbf{y}) d \mathbf{y} \tag{1}
\end{equation*}
$$

where $u^{R}(\mathbf{x})$ is the reproduced function, $\bar{\varphi}_{a}(\mathbf{x}, \mathbf{y})$ is the corrected kernel function in which the subscript $a$ is pertinent to the dilation parameter, $a(\mathbf{y})$. The corrected
kernel function may be expressed as

$$
\begin{equation*}
\bar{\varphi}_{a}(\mathbf{x}, \mathbf{y})=\frac{C(\mathbf{x}, \mathbf{x}-\mathbf{y})}{a(\mathbf{y})} \varphi\left(\frac{\|\mathbf{x}-\mathbf{y}\|}{a(\mathbf{y})}\right) \tag{2}
\end{equation*}
$$

where $\varphi$ is the kernel function, $\|$.$\| represents Euclidian norm and C(\mathbf{x} ; \mathbf{x}-\mathbf{y})$ is the correction function to be determined by the completeness conditions [Liu, Jun, and Zhang (1995)].


Figure 1: Determination of the area associated with $J^{\text {th }}$ particle according to the algorithm given in section 2

Discretization of the integral in equation (1) via trapezoidal rule leads to
$u^{R}(\mathbf{x})=\sum_{J=1}^{N P} N_{J}(\mathbf{x}) d_{J}$,
where $N P$ is the number of particles used to discretize $\Omega$ and $d_{J}$ represents nodal values of the function $u(\mathbf{x})$ at the $J^{t h}$ particle. $N_{J}(\mathbf{x})$ is the shape function associated with the $J^{\text {th }}$ particle and is given by
$N_{J}(\mathbf{x})=\bar{\phi}_{a}(\mathbf{x}, \mathbf{y}) \Delta S_{J}$,
where $\Delta S_{J}$ is the area pertinent to the $J^{t h}$ particle.

For the computational purposes an algorithm for determination of the particle area is employed [Khezri, Hashemian, and Shodja (2009)]. As it will be seen in the results and discussion section, for the problem studied in this paper the proposed technique has a better overall performance as compared with the other commonly used techniques for particle area. In some studies, for simplicity of the computations $\Delta S_{J}=1$ is employed; see for example, Jin, Li, and Aluru (2001), and Chen, Han, You, and Meng (2003). It is noteworthy to mention that for $\Delta S_{J}=1$ and non-shifted basis the reproducing kernel approximation would be identical to the EFGM [Aluru and Li (2001)].

A brief discussion of the method proposed by Khezri, Hashemian and Shodja (2009) is given in the remainder of this section. Consider an arbitrary representative two-dimensional domain, $\Omega$ with boundary $\partial \Omega$. This domain is discretized into a large number of cells using a very fine grid (background mesh) as shown in Fig. 1. In Fig. 1, the circular points are the particles used to discretize $\Omega$.
As a direct result of application of the trapezoidal rule, a given cell within $\Omega$ belongs to the associated area of the particle which is nearest to that cell. In this manner $\Delta S_{J}$, which is the area associated with a typical particle $J$, can be calculated systematically. The accuracy of this procedure is increased by utilization of finer grids. The algorithm for the proposed methodology is given below

## Loop over the constructed cells

1) Loop over the particles
a) Calculate the distances between the particles and the considered cell
b) Determine the nearest particle to the considered cell
2) End particles loop
3) Assign the subjected cell to the area of its nearest particle

## End cells loop

## 3 Treatment of material discontinuities via an augmented corrected collocation method

This section is aimed to develop a straightforward technique for treatment of material discontinuity in an arbitrary domain in the context of RKPM. The proposed method takes advantage of the concept of corrected collocation method, which was previously developed for imposing the essential boundary conditions (EBCs) in the context of meshless methods [Wagner and Liu (2000)].

In collocation methods the EBCs, which in second order linear elliptic BVP of linear elasticity are associated with displacements, are satisfied exactly at the boundary particles. Wagner and Liu (2000) came up with the corrected collocation method which can conveniently enforce the EBCs at the boundary particles in the context of usual RKPM. This methodology is not applicable to the derivative type EBCs. Recently, Shodja and Hashemian (2007a, 2007b, 2008) and Hashemian and Shodja (2008a, 2008b) generalized the corrected collocation method to satisfy the EBCs which involve not only the unknown function but also the first derivatives of the unknown function. The mathematical analysis of generalized corrected collocation method for imposing EBCs, which involve the unknown function and its derivatives of any order, is provided by Behzadan, Shodja and Khezri (2010). The present work chooses to satisfy not only the EBCs at the boundary particles, but also satisfy the continuity of the displacements at the interface particles exactly. This treatment which leads to enhancement of solution in the vicinity of interfaces will be referred to as "augmented corrected collocation method".
For convenience the formulation will be presented for two adjacent phases $\Omega^{\alpha}$ and $\Omega^{\beta}$ with interface $\Gamma$ (Fig. 2). The extension to the case of multi-phase follows in a straightforward manner.


Figure 2: Medium $\Omega$ consisting of two phases

The classical form of the governing BVP on the whole domain, which is the CauchyNavier's equations of elasticity, can be written as follows

$$
\begin{align*}
& -\frac{\partial}{\partial x_{j}}\left(C_{i j k l}(\mathbf{x}) \frac{\partial u_{k}}{\partial x_{l}}\right)=b_{i}(\mathbf{x}) \quad \text { in } \Omega \quad i=1,2, \\
& \text { B.C. }\left\{\begin{array}{lll}
\mathrm{EBC}: u_{i}(\mathbf{x})=g_{i}(\mathbf{x}) & \forall \mathbf{x} \in\left(\partial \Omega_{i}\right)_{e} & i=1,2, \\
C_{i j k l}(\mathbf{x}) \frac{\partial u_{k}}{\partial x_{l}} n_{j}=t_{i}(\mathbf{x}) & \forall \mathbf{x} \in\left(\partial \Omega_{i}\right)_{t} & i=1,2,
\end{array}\right. \tag{5}
\end{align*}
$$

where $u_{i}(\mathbf{x})$ denotes the components of displacement vector, $n$ is the unit outward normal vector, and $C_{i j k l}(\mathbf{x})$ is the elastic constant. Also $b_{i}(\mathbf{x}), g_{i}(\mathbf{x})$, and $t_{i}(\mathbf{x}), i=$ 1,2 , are given functions corresponding to body forces, prescribed displacements and prescribed tractions, respectively. $\partial \Omega$ denotes the boundary of $\Omega$ and $\left(\partial \Omega_{i}\right)_{t}$ and $\left(\partial \Omega_{i}\right)_{e} i=1,2$, are subsets of $\partial \Omega$ such that $\left(\partial \Omega_{i}\right)_{t} \cap\left(\partial \Omega_{i}\right)_{e}=\emptyset$.
It is well-known that the pertinent weak form is exactly the same as principle of virtual work in mechanics which can be written as follows:

Find $\mathbf{u} \in\left(H^{1}(\Omega)\right)^{2}$ such that

$$
\begin{align*}
& \text { 1) } a(\mathbf{u}, \mathbf{v})=l(\mathbf{v}) \quad \forall \mathbf{v} \in W  \tag{6}\\
& \text { 2) } u_{i}(\mathbf{x})=g_{i}(\mathbf{x}) \quad \forall \mathbf{x} \in\left(\partial \Omega_{i}\right)_{e} i=1,2
\end{align*}
$$

where

$$
\begin{gather*}
a(\mathbf{u}, \mathbf{v})=\int_{\Omega} C_{i j k l}(\mathbf{x}) \frac{\partial u_{k}}{\partial x_{l}} \frac{\partial v_{i}}{\partial x_{j}} d \mathbf{x} \\
=\int_{\Omega} \varepsilon_{i j}(\mathbf{v}(\mathbf{x})) \sigma_{i j}^{\mathbf{u}}(\mathbf{x}) d \mathbf{x}  \tag{7}\\
=\int_{\Omega}(\varepsilon(\mathbf{v}(\mathbf{x})))^{T} \sigma^{\mathbf{u}}(\mathbf{x}) d \mathbf{x} \\
l(\mathbf{v})=\int_{\Omega} b_{i}(\mathbf{x}) v_{i}(\mathbf{x}) d \mathbf{x}+\int_{\left(\partial \Omega_{1}\right)_{t}} t_{1}(\mathbf{x}) v_{1}(\mathbf{x}) d s+\int_{\left(\partial \Omega_{2}\right)_{t}} t_{2}(\mathbf{x}) v_{2}(\mathbf{x}) d s  \tag{8}\\
W=\left\{\mathbf{w} \in\left(H^{1}(\Omega)\right)^{2} \mid w_{i}(\mathbf{x})=0 \quad \forall \mathbf{x} \in\left(\partial \Omega_{i}\right)_{e} i=1,2\right\} \tag{9}
\end{gather*}
$$

In equation (7), $\sigma^{\mathbf{u}}(\mathbf{x})$ is the vector of stress components corresponding to the actual displacement vector $\mathbf{u}(\mathbf{x})$, and $\varepsilon(\mathbf{v}(\mathbf{x}))$ is the vector of strain components due to virtual displacement vector $\mathbf{v}(\mathbf{x})$.
From now on we employ the following convention:
For all functions $f(\mathbf{x})$ defined on $\Omega$

$$
\begin{equation*}
f^{\alpha}:=\left.f\right|_{\Omega^{\alpha}} \quad f^{\beta}:=\left.f\right|_{\Omega^{\beta}} \tag{10}
\end{equation*}
$$

With regard to the above convention, equation (7) may be written as:

$$
\begin{align*}
a(\mathbf{u}, \mathbf{v}) & =\int_{\Omega^{\alpha}} \varepsilon_{i j}\left(\mathbf{v}^{\alpha}(\mathbf{x})\right) \sigma_{i j}^{\mathbf{u}^{\alpha}}(\mathbf{x}) d \mathbf{x}+\int_{\Omega^{\beta}} \varepsilon_{i j}\left(\mathbf{v}^{\beta}(\mathbf{x})\right) \sigma_{i j}^{\mathbf{u}^{\beta}}(\mathbf{x}) d \mathbf{x}  \tag{11}\\
& :=a^{\alpha}\left(\mathbf{u}^{\alpha}, \mathbf{v}^{\alpha}\right)+a^{\beta}\left(\mathbf{u}^{\beta}, \mathbf{v}^{\beta}\right) .
\end{align*}
$$

Similarly (8) becomes:

$$
\begin{array}{r}
l(\mathbf{v})=\int_{\Omega^{\alpha}} b_{i}^{\alpha}(\mathbf{x}) v_{i}^{\alpha}(\mathbf{x}) d \mathbf{x}+\int_{\left(\partial \Omega_{1}^{\alpha}\right)_{t}} t_{1}^{\alpha}(\mathbf{x}) v_{1}^{\alpha}(\mathbf{x}) d s+\int_{\left(\partial \Omega_{2}^{\alpha}\right)_{t}} t_{2}^{\alpha}(\mathbf{x}) v_{2}^{\alpha}(\mathbf{x}) d s \\
+\int_{\Omega^{\beta}} b_{i}^{\beta}(\mathbf{x}) v_{i}^{\beta}(\mathbf{x}) d \mathbf{x}+\int_{\left(\partial \Omega_{1}^{\beta}\right)_{t}} t_{1}^{\beta}(\mathbf{x}) v_{1}^{\beta}(\mathbf{x}) d s+\int_{\left(\partial \Omega_{2}^{\beta}\right)_{t}} t_{2}^{\beta}(\mathbf{x}) v_{2}^{\beta}(\mathbf{x}) d s \\
=l^{\alpha}\left(\mathbf{v}^{\alpha}\right)+l^{\beta}\left(\mathbf{v}^{\beta}\right) \tag{12}
\end{array}
$$

Thus, with regard to the notations used in (11) and (12), the weak form of the BVP under consideration can be restated in the following manner:

Find $\quad \mathbf{u}^{\alpha} \in\left(H^{1}\left(\Omega^{\alpha}\right)\right)^{2}$,

$$
\begin{equation*}
\mathbf{u}^{\beta} \in\left(H^{1}\left(\Omega^{\beta}\right)\right)^{2} \tag{13}
\end{equation*}
$$

such that

$$
\begin{array}{lll}
a^{\alpha}\left(\mathbf{u}^{\alpha}, \mathbf{v}^{\alpha}\right)+a^{\beta}\left(\mathbf{u}^{\beta}, \mathbf{v}^{\beta}\right)=l^{\alpha}\left(\mathbf{v}^{\alpha}\right)+l^{\beta}\left(\mathbf{v}^{\beta}\right) & \forall\left(\mathbf{v}^{\alpha}, \mathbf{v}^{\beta}\right) \in T, \\
u_{i}^{\alpha}(\mathbf{x})=g_{i}^{\alpha}(\mathbf{x}) & \forall \mathbf{x} \in\left(\partial \Omega_{i}^{\alpha}\right)_{e} \quad i=1,2, \\
u_{i}^{\beta}(\mathbf{x})=g_{i}^{\beta}(\mathbf{x}) & \forall \mathbf{x} \in\left(\partial \Omega_{i}^{\beta}\right)_{e} \quad i=1,2 . \tag{15}
\end{array}
$$

Moreover, in view of convention (10), the following conditions hold:

$$
\begin{align*}
& \mathbf{u}^{\alpha}(\mathbf{x})=\mathbf{u}^{\beta}(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma,  \tag{16}\\
& T=\left\{\left(\mathbf{w}^{\alpha}, \mathbf{w}^{\beta}\right) \in\left(H^{1}\left(\Omega^{\alpha}\right)\right)^{2} \times\left(H^{1}\left(\Omega^{\beta}\right)\right)^{2}\right. \\
& \left.\left\lvert\,\left\{\begin{array}{cc}
w_{i}^{\alpha}(\mathbf{x})=0 & \text { on }\left(\partial \Omega_{i}^{\alpha}\right)_{e} \quad i=1,2, \\
w_{i}^{\beta}(\mathbf{x})=0 & \text { on }\left(\partial \Omega_{i}^{\beta}\right)_{e} \quad i=1,2, \\
\mathbf{w}^{\alpha}(\mathbf{x})=\mathbf{w}^{\beta}(\mathbf{x}) & \text { on } \Gamma
\end{array}\right\}\right.\right\} \tag{17}
\end{align*}
$$

In order to discretize the obtained weak form, we employ the RKPM shape functions which are obtained according to certain discretization of the phases. More strictly speaking each medium is considered as a detached phase and is surveyed separately in the sense that for each phase a distinct set of particles is considered and the corresponding shape functions are derived as if the other domain did not exist. Of course, to enforce the displacement continuity across the interface we require that the particles of $\bar{\Omega}^{\alpha}$ and $\bar{\Omega}^{\beta}$ share the same coordinates on $\Gamma$.

Therefore, the discretized form of the problem will be as follows:
Find $\quad u_{i}^{\alpha, h} \in \operatorname{Span}\left\{N_{I}^{\alpha}\right\}_{1 \leq I \leq N P^{\alpha}} \quad i=1,2$,

$$
\begin{equation*}
u_{i}^{\beta, h} \in \operatorname{Span}\left\{N_{I}^{\beta}\right\}_{1 \leq I \leq N P^{\beta}} \quad i=1,2, \tag{18}
\end{equation*}
$$

such that

$$
\begin{align*}
& \text { 1) } a^{\alpha}\left(\mathbf{u}^{\alpha, h}, \mathbf{v}^{\alpha, h}\right)+a^{\beta}\left(\mathbf{u}^{\beta, h}, \mathbf{v}^{\beta, h}\right)=l^{\alpha}\left(\mathbf{v}^{\alpha, h}\right)+l^{\beta}\left(\mathbf{v}^{\beta, h}\right)  \tag{19}\\
& \forall\left(\mathbf{v}^{\alpha, h}, \mathbf{v}^{\beta, h}\right) \in T^{h}, \\
& T^{h}=T \cap\left[\left(\operatorname{Span}\left\{N_{I}^{\alpha}\right\}_{1 \leq I \leq N P^{\alpha}}\right)^{2} \times\left(\operatorname{Span}\left\{N_{I}^{\beta}\right\}_{1 \leq I \leq N P^{\beta}}\right)^{2}\right],  \tag{20}\\
& \text { 2) } u_{i}^{\alpha, h}(\mathbf{x})=g_{i}^{\alpha}(\mathbf{x}) \quad \forall \mathbf{x} \in\left(\partial \Omega_{i}^{\alpha}\right)_{e} \quad i=1,2, \\
& u_{i}^{\beta, h}(\mathbf{x})=g_{i}^{\beta}(\mathbf{x}) \quad \forall \mathbf{x} \in\left(\partial \Omega_{i}^{\beta}\right)_{e} \quad i=1,2,  \tag{21}\\
& \mathbf{u}^{\alpha, h}(\mathbf{x})=\mathbf{u}^{\beta, h}(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma . \tag{22}
\end{align*}
$$

Where $\left\{N_{I}^{\alpha}\right\}_{1 \leq I \leq N P^{\alpha}}$ and $\left\{N_{I}^{\beta}\right\}_{1 \leq I \leq N P^{\beta}}$ are shape functions pertinent to particle distributions in $\Omega^{\alpha}$ and $\Omega^{\beta}$, respectively. Also the index $h$ is used to emphasize that we are seeking for an approximate solution. Now, it should be noticed that in the context of the collocation methods, since the ultimate goal is to satisfy the EBCs merely at the boundary degrees of freedom (DOFs), instead of the above problem, the following problem should be solved:
Find $\quad u_{i}^{\alpha, h} \in \operatorname{Span}\left\{N_{I}^{\alpha}\right\}_{1 \leq I \leq N P^{\alpha}} \quad i=1,2$,

$$
\begin{equation*}
u_{i}^{\beta, h} \in \operatorname{Span}\left\{N_{I}^{\beta}\right\}_{1 \leq I \leq N P^{\beta}}^{1 \leq I \leq N P^{\beta}} \quad i=1,2, \tag{23}
\end{equation*}
$$

such that

$$
\begin{align*}
& \text { 1) } a^{\alpha}\left(\mathbf{u}^{\alpha, h}, \mathbf{v}^{\alpha, h}\right)+a^{\beta}\left(\mathbf{u}^{\beta, h}, \mathbf{v}^{\beta, h}\right)=l^{\alpha}\left(\mathbf{v}^{\alpha, h}\right)+l^{\beta}\left(\mathbf{v}^{\beta, h}\right)  \tag{25}\\
& \forall\left(\mathbf{v}^{\alpha, h}, \mathbf{v}^{\beta, h}\right) \in S^{h}, \\
& \text { 2) } u_{i}^{\alpha, h}\left(\mathbf{x}_{I}^{\alpha}\right)=g_{i}^{\alpha}\left(\mathbf{x}_{I}^{\alpha}\right) \quad \forall \mathbf{x}_{I}^{\alpha} \in\left(\partial \Omega_{i}^{\alpha}\right)_{e} \quad i=1,2, \\
& u_{i}^{\beta, h}\left(\mathbf{x}_{I}^{\beta}\right)=g_{i}^{\beta}\left(\mathbf{x}_{I}^{\beta}\right) \quad \forall \mathbf{x}_{I}^{\beta} \in\left(\partial \Omega_{i}^{\beta}\right)_{e} \quad i=1,2,  \tag{26}\\
& \mathbf{u}^{\alpha, h}\left(\mathbf{x}_{I}\right)=\mathbf{u}^{\beta, h}\left(\mathbf{x}_{I}\right) \quad \forall \mathbf{x}_{I} \in \Gamma, \tag{27}
\end{align*}
$$

$$
\begin{align*}
S^{h}=\left\{\left(\mathbf{w}^{\alpha}, \mathbf{w}^{\beta}\right) \in\left(\left[\begin{array}{c}
\left(\operatorname{Span}\left\{N_{I}^{\alpha}\right\}_{1 \leq I \leq N P^{\alpha}}\right)^{2} \\
\times \\
\left(\operatorname{Span}\left\{N_{I}^{\beta}\right\}_{1 \leq I \leq N P^{\beta}}\right)^{2}
\end{array}\right]\right)\right. \\
\left.\left\lvert\,\left[\begin{array}{c}
w_{i}^{\alpha}\left(\mathbf{x}_{I}^{\alpha}\right)=0 \forall \mathbf{x}_{I}^{\alpha} \in\left(\partial \Omega_{i}^{\alpha}\right)_{e} \quad i=1,2, \\
w_{i}^{\beta}\left(\mathbf{x}_{I}^{\beta}\right)=0 \forall \mathbf{x}_{I}^{\beta} \in\left(\partial \Omega_{i}^{\beta}\right)_{e} \quad i=1,2, \\
\mathbf{w}^{\alpha}\left(\mathbf{x}_{I}\right)=\mathbf{w}^{\beta}\left(\mathbf{x}_{I}\right) \quad \forall \mathbf{x}_{I} \in \Gamma
\end{array}\right]\right.\right\} . \tag{28}
\end{align*}
$$

Where $\mathbf{x}_{I}^{\alpha}$ and $\mathbf{x}_{I}^{\beta}$ denote the particles belonging to $\Omega^{\alpha}$ and $\Omega^{\beta}$, respectively.
The DOFs in each phase are numbered separately in the manner described below:

1. First the DOFs to which the EBCs have been associated are numbered. The total number of these DOFs corresponding to $\Omega^{\alpha}$ and $\Omega^{\beta}$ are denoted by $M_{e}^{\alpha}$ and $M_{e}^{\beta}$, respectively. Also we denote by $\mathbf{g}^{\alpha, \dagger}$ and $\mathbf{g}^{\beta, \dagger}$ the column vectors with $M_{e}^{\alpha}$ and $M_{e}^{\beta}$ components, respectively such that their $i$ th component is the prescribed displacement of $i$ th DOF.
2. Second, the DOFs which are located at the interface are numbered. The total number of these DOFs are denoted by $M_{i}^{\alpha}$ and $M_{i}^{\beta}$, respectively. Obviously, with regard to what was said earlier about the interface particles on $\Gamma$, we have $M_{i}^{\alpha}=M_{i}^{\beta}$.
3. Finally, the remaining DOFs are numbered. The total number of these DOFs are denoted by $M_{r}^{\alpha}$ and $M_{r}^{\beta}$, respectively. Also we set

$$
\begin{array}{ll}
M^{\alpha}:=M_{e}^{\alpha}+M_{i}^{\alpha}+M_{r}^{\alpha} & \left(M^{\alpha}=2 N P^{\alpha}\right), \\
M^{\beta}:=M_{e}^{\beta}+M_{i}^{\beta}+M_{r}^{\beta} & \left(M^{\beta}=2 N P^{\beta}\right) . \tag{29}
\end{array}
$$

Equation (22), may be written as:

$$
\begin{align*}
& \mathbf{u}^{\alpha, h}(\mathbf{x})=\left[\begin{array}{l}
u_{1}^{\alpha, h}(\mathbf{x}) \\
u_{2}^{\alpha, h}(\mathbf{x})
\end{array}\right] \\
& \quad=\left[\begin{array}{ccccccc}
N_{1}^{\alpha}(\mathbf{x}) & 0 & N_{2}^{\alpha}(\mathbf{x}) & 0 & \cdots & N_{N P^{\alpha}}^{\alpha}(\mathbf{x}) & 0 \\
0 & N_{1}^{\alpha}(\mathbf{x}) & 0 & N_{2}^{\alpha}(\mathbf{x}) & \cdots & 0 & N_{N P^{\alpha}}^{\alpha}(\mathbf{x})
\end{array}\right]\left[\begin{array}{c}
d_{11}^{\alpha} \\
d_{21}^{\alpha} \\
\vdots \\
d_{1\left(N P^{\alpha}\right)}^{\alpha} \\
d_{2\left(N P^{\alpha}\right)}^{\alpha}
\end{array}\right] . \tag{30}
\end{align*}
$$

According to the DOFs numbering that was described previously, one can rewrite the equality (27) as follows

$$
\mathbf{u}^{\alpha, h}(\mathbf{x})=\left[\begin{array}{lll}
\mathbf{N}_{e}^{\alpha}(\mathbf{x}) & \mathbf{N}_{i}^{\alpha}(\mathbf{x}) & \mathbf{N}_{r}^{\alpha}(\mathbf{x})
\end{array}\right]\left[\begin{array}{l}
\mathbf{d}_{e}^{\alpha}  \tag{31}\\
\mathbf{d}_{i}^{\alpha} \\
\mathbf{d}_{r}^{\alpha}
\end{array}\right]:=\mathbf{N}^{\alpha}(\mathbf{x}) \mathbf{d}^{\alpha}:=\left[\begin{array}{l}
\mathbf{N}^{\alpha, 1}(\mathbf{x}) \\
\mathbf{N}^{\alpha, 2}(\mathbf{x})
\end{array}\right] \mathbf{d}^{\alpha},
$$

where $\mathbf{d}_{e}^{\alpha}, \mathbf{d}_{i}^{\alpha}$ and $\mathbf{d}_{r}^{\alpha}$ are the vectors of nodal DOFs associated with the EBCs of $\Omega^{\alpha}$, DOFs at interface, and the remaining DOFs in $\Omega^{\alpha}$, respectively.
Similarly for $\mathbf{u}^{\beta}$ one can write
$\mathbf{u}^{\beta, h}(\mathbf{x})=\mathbf{N}^{\beta}(\mathbf{x}) \mathbf{d}^{\beta}$,
also, obviously

$$
\begin{align*}
\mathbf{v}^{\alpha, h}(\mathbf{x}) & =\mathbf{N}^{\alpha}(\mathbf{x}) \mathbf{c}^{\alpha} \\
\mathbf{v}^{\beta, h}(\mathbf{x}) & =\mathbf{N}^{\beta}(\mathbf{x}) \mathbf{c}^{\beta} \tag{33}
\end{align*}
$$

Consequently it can be written that

$$
\begin{align*}
& \boldsymbol{\varepsilon}\left(\mathbf{u}^{\alpha, h}(\mathbf{x})\right)=\mathbf{B}^{\alpha} \mathbf{d}^{\alpha}, \quad\left(\sigma^{\mathbf{u}^{\alpha, h}}=\mathbf{D}^{\alpha} \boldsymbol{\varepsilon}\left(\mathbf{u}^{\alpha, h}(\mathbf{x})\right)\right) \\
& \boldsymbol{\varepsilon}\left(\mathbf{v}^{\alpha, h}(\mathbf{x})\right)=\mathbf{B}^{\alpha} \mathbf{c}^{\alpha} \tag{34}
\end{align*}
$$

where $\mathbf{D}^{\alpha}$ is the elastic coefficient matrix for the phase $\Omega^{\alpha}$, and $\mathbf{B}^{\alpha}$ is a $3 \times M^{\alpha}$ matrix which is obtained by changing the order of the columns of the following matrix according to the previously mentioned numbering of DOFs.

$$
\left[\begin{array}{ccccc}
\frac{\partial N_{1}^{\alpha}}{\partial x_{1}} & 0 & \cdots & \frac{\partial N_{N p^{\alpha}}^{\alpha}}{\partial x_{1}} & 0  \tag{35}\\
0 & \frac{\partial N_{1}^{\alpha}}{\partial x_{2}} & \cdots & 0 & \frac{\partial N_{N p^{\alpha} \alpha}^{\partial x_{1}}}{\partial x_{2}^{\alpha}} \\
\frac{\partial N_{1}^{\alpha}}{\partial x_{2}} & \frac{\partial N_{1}^{\alpha}}{\partial x_{1}} & \cdots & \frac{\partial N_{N p^{\alpha}}^{\alpha}}{\partial x_{2}} & \frac{\partial N_{N P^{\alpha}}}{\partial x_{1}}
\end{array}\right] .
$$

Relations (31) correspond to $\Omega^{\alpha}$; clearly similar relations hold on $\Omega^{\beta}$.
By employing the aforementioned notations one can write

$$
\begin{align*}
& a^{\alpha}\left(\mathbf{u}^{\alpha, h}, \mathbf{v}^{\alpha, h}\right)=\int_{\Omega^{\alpha}}\left(\boldsymbol{\varepsilon}\left(\mathbf{v}^{\alpha, h}(\mathbf{x})\right)\right)^{T} \sigma^{\mathbf{u}^{\alpha, h}}(\mathbf{x}) d \mathbf{x} \\
& =\int_{\Omega^{\alpha}}\left(\mathbf{c}^{\alpha}\right)^{T}\left(\mathbf{B}^{\alpha}\right)^{T} \mathbf{D}^{\alpha} \mathbf{B}^{\alpha} \mathbf{d}^{\alpha} d \mathbf{x}  \tag{36}\\
& =\left(\mathbf{c}^{\alpha}\right)^{T}\left(\int_{\Omega^{\alpha}}\left(\mathbf{B}^{\alpha}\right)^{T} \mathbf{D}^{\alpha} \mathbf{B}^{\alpha} d \mathbf{x}\right) \mathbf{d}^{\alpha} \\
& :=\left(\mathbf{c}^{\alpha}\right)^{T} \mathbf{K}^{\alpha} \mathbf{d}^{\alpha}
\end{align*}
$$

$$
\begin{align*}
l^{\alpha}\left(\mathbf{v}^{\alpha, h}\right)= & \int_{\Omega^{\alpha}} b_{i}^{\alpha}(\mathbf{x}) v_{i}^{\alpha, h}(\mathbf{x}) d \mathbf{x}+\int_{\left(\partial \Omega_{1}^{\alpha}\right)_{t}} t_{1}^{\alpha}(\mathbf{x}) v_{1}^{\alpha, h}(\mathbf{x}) d s \\
& +\int_{\left(\partial \Omega_{2}^{\alpha}\right)_{t}} t_{2}^{\alpha}(\mathbf{x}) v_{2}^{\alpha, h}(\mathbf{x}) d s \\
= & \left(\mathbf{c}^{\alpha}\right)^{T}\left[\int_{\Omega^{\alpha}}\left(\mathbf{N}^{\alpha}(x)\right)^{T} \mathbf{b}(\mathbf{x}) d \mathbf{x}+\int_{\left(\partial \Omega_{1}^{\alpha}\right)_{t}}\left(\mathbf{N}^{\alpha, 1}(x)\right)^{T} t_{1}^{\alpha}(\mathbf{x}) d s\right.  \tag{37}\\
& \left.+\int_{\left(\partial \Omega_{2}^{\alpha}\right)_{t}}\left(\mathbf{N}^{\alpha, 2}(x)\right)^{T} t_{2}^{\alpha}(\mathbf{x}) d s\right] \\
:= & \left(\mathbf{c}^{\alpha}\right)^{T} \mathbf{f}^{\alpha} .
\end{align*}
$$

In the same manner $\mathbf{K}^{\beta}$ and $\mathbf{f}^{\beta}$ are defined, and so equation (23) can be rewritten as follows

$$
\begin{equation*}
\left(\mathbf{c}^{\alpha}\right)^{T}\left(\mathbf{K}^{\alpha} \mathbf{d}^{\alpha}-\mathbf{f}^{\alpha}\right)+\left(\mathbf{c}^{\beta}\right)^{T}\left(\mathbf{K}^{\beta} \mathbf{d}^{\beta}-\mathbf{f}^{\beta}\right)=0 \tag{38}
\end{equation*}
$$

At this point, we note that

$$
\left(\mathbf{v}^{\alpha, h}, \mathbf{v}^{\beta, h}\right) \in S^{h} \Leftrightarrow\left\{\begin{array}{l}
\left\{\begin{array}{l}
\mathbf{v}^{\alpha, h}(\mathbf{x})=\mathbf{N}^{\alpha}(\mathbf{x}) \mathbf{c}^{\alpha} \\
\forall \mathbf{x}_{I}^{\alpha} \in\left(\partial \Omega_{i}^{\alpha}\right)_{e} \quad v_{i}^{\alpha, h}\left(\mathbf{x}_{I}^{\alpha}\right)=0 \quad \\
\left\{\begin{array}{l}
\mathbf{v}^{\beta, h}(\mathbf{x})=\mathbf{N}^{\beta}(\mathbf{x}) \mathbf{c}^{\beta} \\
\forall \mathbf{x}_{I}^{\beta} \in\left(\partial \Omega_{i}^{\beta}\right)_{e} \quad v_{i}^{\beta, h}\left(\mathbf{x}_{I}^{\beta}\right)=0 \quad i=1,2
\end{array}\right. \\
\forall \mathbf{x}_{I} \in \Gamma \quad \mathbf{v}^{\beta, h}\left(\mathbf{x}_{I}\right)=\mathbf{v}^{\alpha, h}\left(\mathbf{x}_{I}\right)
\end{array}\right. \tag{a}
\end{array}\right.
$$

Define a bijective function $Q^{\alpha}$ as follows
$Q^{\alpha}:\left\{1,2, \ldots, M^{\alpha}\right\} \rightarrow\left\{(I, j) \mid 1 \leq I \leq N P^{\alpha}, j=1,2\right\}$
$Q^{\alpha}(i)=(I, j)$ if and only if the number of the DOF associated with particle $\mathbf{x}_{I}^{\alpha}$ in the direction $x_{j}$ is $i$. Define
$\forall 1 \leq I \leq N P^{\alpha} \quad j=1,2 \quad \mathbf{H}_{(I, j)}^{\alpha}:=\mathbf{N}^{\alpha, j}\left(\mathbf{x}_{I}^{\alpha}\right)$,
and
$\boldsymbol{\Psi}^{\alpha}:=\left[\begin{array}{c}\mathbf{H}_{Q^{\alpha}(1)}^{\alpha} \\ \vdots \\ \mathbf{H}_{Q^{\alpha}\left(M_{e}^{\alpha}\right)}^{\alpha} \\ \mathbf{H}_{Q^{\alpha}\left(M_{e}^{\alpha}+1\right)}^{\alpha} \\ \vdots \\ \mathbf{H}_{Q^{\alpha}\left(M_{e}^{\alpha}+M_{i}^{\alpha}\right)}^{\alpha} \\ \mathbf{H}_{Q^{\alpha}\left(M_{e}^{\alpha}+M_{i}^{\alpha}+1\right)}^{\alpha} \\ \vdots \\ \mathbf{H}_{Q^{\alpha}\left(M^{\alpha}\right)}^{\alpha}\end{array}\right]:=\left[\begin{array}{lll}\boldsymbol{\Psi}_{e, e}^{\alpha} & \boldsymbol{\Psi}_{e, i}^{\alpha} & \boldsymbol{\Psi}_{e, r}^{\alpha} \\ \boldsymbol{\psi}_{i, e}^{\alpha} & \boldsymbol{\psi}_{i, i}^{\alpha} & \boldsymbol{\Psi}_{i, r}^{\alpha} \\ \boldsymbol{\Psi}_{r, e}^{\alpha} & \boldsymbol{\Psi}_{r, i}^{\alpha} & \boldsymbol{\Psi}_{r, r}^{\alpha}\end{array}\right]$.
In the right hand side of the above equality, $\boldsymbol{\psi}_{\xi, \eta}^{\alpha}$ represents a matrix of size $M_{\xi}^{\alpha} \times$ $M_{\eta}^{\alpha}$. Similar definitions and relations hold for the phase $\beta$.
With regard to the above notations, condition (36) (a) is equivalent to the following

$$
\begin{align*}
& \forall(I, j) \in Q^{\alpha}\left(\left\{1, \ldots, M_{e}^{\alpha}\right\}\right) \quad v_{j}^{\alpha, h}\left(\mathbf{x}_{I}^{\alpha}\right)=0 \\
& \Leftrightarrow \forall(I, j) \in Q^{\alpha}\left(\left\{1, \ldots, M_{e}^{\alpha}\right\}\right) \quad \mathbf{N}^{\alpha, j}\left(\mathbf{x}_{I}^{\alpha}\right) \mathbf{c}^{\alpha}=0  \tag{43}\\
& \Leftrightarrow \forall i \in\left\{1, \ldots, M_{e}^{\alpha}\right\} \quad \mathbf{H}_{Q^{\alpha}(i)}^{\alpha} \mathbf{c}^{\alpha}=0 \\
& \Leftrightarrow\left[\begin{array}{lll}
\boldsymbol{\psi}_{e, e}^{\alpha} & \boldsymbol{\psi}_{e, i}^{\alpha} & \boldsymbol{\psi}_{e, r}^{\alpha}
\end{array}\right] \mathbf{c}^{\alpha}=\mathbf{0} .
\end{align*}
$$

Similarly the conditions (36) (b) and (c) are equivalent to the following conditions

$$
\left[\begin{array}{lll}
\boldsymbol{\psi}_{e, e}^{\beta} & \boldsymbol{\psi}_{e, i}^{\beta} & \boldsymbol{\psi}_{e, r}^{\beta}  \tag{44}\\
\boldsymbol{\psi}_{i, e}^{\beta} & \boldsymbol{\psi}_{i, i}^{\beta} & \boldsymbol{\psi}_{i, r}^{\beta}
\end{array}\right] \mathbf{c}^{\beta}=\mathbf{0},\left[\begin{array}{lll}
\boldsymbol{\psi}_{i, e}^{\alpha} & \boldsymbol{\psi}_{i, i}^{\alpha} & \boldsymbol{\psi}_{i, r}^{\alpha}
\end{array}\right] \mathbf{c}^{\alpha} .
$$

Moreover, condition (24) can be rewritten as follows

$$
\left[\begin{array}{cccccc}
\boldsymbol{\psi}_{e, e}^{\alpha} & \boldsymbol{\psi}_{e, i}^{\alpha} & \boldsymbol{\Psi}_{e, r}^{\alpha} & \mathbf{0} & \mathbf{0} & \mathbf{0}  \tag{45}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\psi}_{e, e}^{\beta} & \boldsymbol{\psi}_{e, i}^{\beta} & \boldsymbol{\psi}_{e, r}^{\beta} \\
\boldsymbol{\psi}_{i, e}^{\alpha} & \boldsymbol{\psi}_{i, i}^{\alpha} & \boldsymbol{\psi}_{i, r}^{\alpha} & -\boldsymbol{\psi}_{i, e}^{\beta} & -\boldsymbol{\psi}_{i, i}^{\beta} & -\boldsymbol{\psi}_{i, r}^{\beta}
\end{array}\right]\left[\begin{array}{c}
\mathbf{d}^{\alpha} \\
\mathbf{d}^{\beta}
\end{array}\right]=\left[\begin{array}{c}
\mathbf{g}^{\alpha, \dagger} \\
\mathbf{g}^{\beta, \dagger} \\
\mathbf{0}
\end{array}\right]
$$

Hence, with regard to what has been said so far, the discretized form of the equations (13)-(17) can be written as follows
Find $\quad \mathbf{d}^{\alpha} \in \mathscr{R}^{M^{\alpha}}$

$$
\begin{equation*}
\mathbf{d}^{\beta} \in \mathscr{R}^{M^{\beta}} \tag{46}
\end{equation*}
$$

such that

1) $\left(\mathbf{c}^{\alpha}\right)^{T}\left(\mathbf{K}^{\alpha} \mathbf{d}^{\alpha}-\mathbf{f}^{\alpha}\right)+\left(\mathbf{c}^{\beta}\right)^{T}\left(\mathbf{K}^{\beta} \mathbf{d}^{\beta}-\mathbf{f}^{\beta}\right)=0$,

The above equality must hold for all the vectors $\mathbf{c}^{\alpha} \in \mathscr{R}^{M^{\alpha}}$ and $\mathbf{c}^{\beta} \in \mathscr{R}^{M^{\beta}}$ which satisfy the following conditions

$$
\begin{align*}
& {\left[\begin{array}{lll}
\boldsymbol{\psi}_{e, e}^{\alpha} & \boldsymbol{\Psi}_{e, i}^{\alpha} & \left.\boldsymbol{\psi}_{e, r}^{\alpha}\right] \mathbf{c}^{\alpha}=\mathbf{0} \\
{\left[\boldsymbol{\psi}_{e, e}^{\beta}\right.} & \boldsymbol{\psi}_{e, i}^{\beta} & \boldsymbol{\psi}_{e, r}^{\beta}
\end{array}\right] \mathbf{c}^{\beta}=\mathbf{0},}  \tag{47}\\
& {\left[\begin{array}{llll}
\boldsymbol{\Psi}_{i, e}^{\alpha} & \boldsymbol{\psi}_{i, i}^{\alpha} & \boldsymbol{\Psi}_{i, r}^{\alpha}
\end{array}\right] \mathbf{c}^{\alpha}-\left[\begin{array}{lll}
\boldsymbol{\Psi}_{i, e}^{\beta} & \boldsymbol{\Psi}_{i, i}^{\beta} & \boldsymbol{\psi}_{i, r}^{\beta}
\end{array}\right] \mathbf{c}^{\beta}=\mathbf{0}}
\end{align*}
$$

and
2) $\left[\begin{array}{cccccc}\boldsymbol{\Psi}_{e, e}^{\alpha} & \boldsymbol{\Psi}_{e, i}^{\alpha} & \boldsymbol{\psi}_{e, r}^{\alpha} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\psi}_{e, e}^{\beta} & \boldsymbol{\psi}_{e, i}^{\beta} & \boldsymbol{\psi}_{e, r}^{\beta} \\ \boldsymbol{\psi}_{i, e}^{\alpha} & \boldsymbol{\Psi}_{i, i}^{\alpha} & \boldsymbol{\psi}_{i, r}^{\alpha} & -\boldsymbol{\psi}_{i, e}^{\beta} & -\boldsymbol{\psi}_{i, i}^{\beta} & -\boldsymbol{\psi}_{i, r}^{\beta}\end{array}\right]\left[\begin{array}{c}\mathbf{d}^{\alpha} \\ \mathbf{d}^{\beta}\end{array}\right]=\left[\begin{array}{c}\mathbf{g}^{\alpha, \dagger} \\ \mathbf{g}^{\beta, \dagger} \\ \mathbf{0}\end{array}\right]$.

Clearly,

$$
\begin{align*}
& \left\{\mathbf{c}^{\alpha}=\left[\begin{array}{l}
\mathbf{c}_{e}^{\alpha} \\
\mathbf{c}_{i}^{\alpha} \\
\mathbf{c}_{r}^{\alpha}
\end{array}\right] \in \mathscr{R}^{M^{\alpha}} \left\lvert\, \begin{array}{lll}
\boldsymbol{\psi}_{e, e}^{\alpha} & \boldsymbol{\Psi}_{e, i}^{\alpha} & \left.\left.\boldsymbol{\psi}_{e, r}^{\alpha}\right] \mathbf{c}^{\alpha}=\mathbf{0}\right\}
\end{array}\right.\right.  \tag{49}\\
& =\left\{\left[\begin{array}{l}
\mathbf{c}_{e}^{\alpha} \\
\mathbf{c}_{i}^{\alpha} \\
\mathbf{c}_{r}^{\alpha}
\end{array}\right] \in \mathscr{R}^{M^{\alpha}} \left\lvert\,\left[\begin{array}{c}
\mathbf{c}_{i}^{\alpha} \text { and } \mathbf{c}_{r}^{\alpha} \text { are arbitrary, } \\
\left.\left.\mathbf{c}_{e}^{\alpha}=-\left(\boldsymbol{\Psi}_{e, e}^{\alpha}\right)^{-1}\left[\boldsymbol{\Psi}_{e, i}^{\alpha} \mathbf{c}_{i}^{\alpha}+\boldsymbol{\psi}_{e, r}^{\alpha} \mathbf{c}_{r}^{\alpha}\right]\right]\right\}
\end{array}\right.\right.\right.
\end{align*}
$$

If we set

$$
\boldsymbol{\chi}^{\alpha}=\left[\begin{array}{ccc}
\boldsymbol{\psi}_{e, e}^{\alpha} & \boldsymbol{\psi}_{e, i}^{\alpha} & \boldsymbol{\psi}_{e, r}^{\alpha}  \tag{50}\\
\mathbf{0} & \mathbf{I} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{I}
\end{array}\right]
$$

then

$$
\begin{align*}
\boldsymbol{\chi}^{\alpha} \mathbf{c}^{\alpha}= & {\left[\begin{array}{c}
\mathbf{0} \\
\mathbf{c}_{i}^{\alpha} \\
\mathbf{c}_{r}^{\alpha}
\end{array}\right] \Rightarrow \mathbf{c}^{\alpha}=\left(\boldsymbol{\chi}^{\alpha}\right)^{-1}\left[\begin{array}{c}
\mathbf{0} \\
\mathbf{c}_{i}^{\alpha} \\
\mathbf{c}_{r}^{\alpha}
\end{array}\right]=}  \tag{51}\\
& {\left[\begin{array}{cccc}
\left(\boldsymbol{\Psi}_{e, e}^{\alpha}\right)^{-1} & -\left(\boldsymbol{\Psi}_{e, e}^{\alpha}\right)^{-1} & \boldsymbol{\Psi}_{e, i}^{\alpha} & -\left(\boldsymbol{\Psi}_{e, e}^{\alpha}\right)^{-1} \boldsymbol{\Psi}_{e, r}^{\alpha} \\
\mathbf{0} & \mathbf{I} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{I}
\end{array}\right]\left[\begin{array}{c}
\mathbf{0} \\
\mathbf{c}_{i}^{\alpha} \\
\mathbf{c}_{r}^{\alpha}
\end{array}\right] . }
\end{align*}
$$

Therefore, we will have

$$
\begin{aligned}
& \left(\mathbf{c}^{\alpha}\right)^{T}\left(\mathbf{K}^{\alpha} \mathbf{d}^{\alpha}-\mathbf{f}^{\alpha}\right) \\
= & {\left[\begin{array}{lll}
(\mathbf{0})^{T} & \left(\mathbf{c}_{i}^{\alpha}\right)^{T} & \left(\mathbf{c}_{r}^{\alpha}\right)^{T}
\end{array}\right] } \\
& {\left.\left[\begin{array}{ccc}
\left(\boldsymbol{\Psi}_{e, e}^{\alpha}\right)^{-T} & \mathbf{0} & \mathbf{0} \\
-\left(\boldsymbol{\psi}_{e, i}^{\alpha}\right)^{T}\left(\boldsymbol{\psi}_{e, e}^{\alpha}\right)^{-T} & \mathbf{I} & \mathbf{0} \\
-\left(\boldsymbol{\Psi}_{e, r}^{\alpha}\right)^{T}\left(\boldsymbol{\Psi}_{e, e}\right)^{-T} & \mathbf{0} & \mathbf{I}
\end{array}\right]\left(\begin{array}{lll}
\mathbf{K}_{e, e}^{\alpha} & \mathbf{K}_{e, i}^{\alpha} & \mathbf{K}_{e, r}^{\alpha} \\
\mathbf{K}_{i, e}^{\alpha} & \mathbf{K}_{i, i}^{\alpha} & \mathbf{K}_{i, r}^{\alpha} \\
\mathbf{K}_{r, e}^{\alpha} & \mathbf{K}_{r, i}^{\alpha} & \mathbf{K}_{r, r}^{\alpha}
\end{array}\right] \mathbf{d}^{\alpha}-\left[\begin{array}{c}
\mathbf{f}_{e}^{\alpha} \\
\mathbf{f}_{i}^{\alpha} \\
\mathbf{f}_{r}^{\alpha}
\end{array}\right]\right) } \\
:= & {\left[\begin{array}{lll}
(\mathbf{0})^{T} & \left(\mathbf{c}_{i}^{\alpha}\right)^{T} & \left(\mathbf{c}_{r}^{\alpha}\right)^{T}
\end{array}\right]\left(\left[\begin{array}{lll}
\mathbf{L}_{e, e}^{\alpha} & \mathbf{L}_{e, i}^{\alpha} & \mathbf{L}_{e, r}^{\alpha} \\
\mathbf{L}_{i, e}^{\alpha} & \mathbf{L}_{i, i}^{\alpha} & \mathbf{L}_{i, r}^{\alpha} \\
\mathbf{L}_{r, e}^{\alpha} & \mathbf{L}_{r, i}^{\alpha} & \mathbf{L}_{r, r}^{\alpha}
\end{array}\right] \mathbf{d}^{\alpha}-\left[\begin{array}{l}
\mathbf{p}_{e}^{\alpha} \\
\mathbf{p}_{i}^{\alpha} \\
\mathbf{p}_{r}^{\alpha}
\end{array}\right]\right) } \\
:= & {\left[\begin{array}{lll}
(\mathbf{0})^{T} & \left(\mathbf{c}_{i}^{\alpha}\right)^{T} & \left(\mathbf{c}_{r}^{\alpha}\right)^{T}
\end{array}\right]\left[\begin{array}{l}
\mathbf{A}_{1}^{\alpha} \\
\mathbf{A}_{2}^{\alpha} \\
\mathbf{A}_{3}^{\alpha}
\end{array}\right] . }
\end{aligned}
$$

Thus,

$$
\begin{equation*}
\left(\mathbf{c}^{\alpha}\right)^{T}\left(\mathbf{K}^{\alpha} \mathbf{d}^{\alpha}-\mathbf{f}^{\alpha}\right)=\left(\mathbf{c}_{i}^{\alpha}\right)^{T} \mathbf{A}_{2}^{\alpha}+\left(\mathbf{c}_{r}^{\alpha}\right)^{T} \mathbf{A}_{3}^{\alpha} \tag{53}
\end{equation*}
$$

Similarly for $\beta$ phase one can write

$$
\begin{equation*}
\left(\mathbf{c}^{\beta}\right)^{T}\left(\mathbf{K}^{\beta} \mathbf{d}^{\beta}-\mathbf{f}^{\beta}\right)=\left(\mathbf{c}_{i}^{\beta}\right)^{T} \mathbf{A}_{2}^{\beta}+\left(\mathbf{c}_{r}^{\beta}\right)^{T} \mathbf{A}_{3}^{\beta} . \tag{54}
\end{equation*}
$$

Thus, the first part of the latter discretized weak form may be restated as follows:

$$
\begin{equation*}
\left(\mathbf{c}_{i}^{\alpha}\right)^{T} \mathbf{A}_{2}^{\alpha}+\left(\mathbf{c}_{r}^{\alpha}\right)^{T} \mathbf{A}_{3}^{\alpha}+\left(\mathbf{c}_{i}^{\beta}\right)^{T} \mathbf{A}_{2}^{\beta}+\left(\mathbf{c}_{r}^{\beta}\right)^{T} \mathbf{A}_{3}^{\beta}=0 \tag{55}
\end{equation*}
$$

the above equality must hold for all $\mathbf{c}_{i}^{\alpha} \in \mathscr{R}^{M_{i}^{\alpha}}, \mathbf{c}_{r}^{\alpha} \in \mathscr{R}^{M_{r}^{\alpha}}, \mathbf{c}_{i}^{\beta} \in \mathscr{R}^{M_{i}^{\beta}}$, and $\mathbf{c}_{r}^{\beta} \in$ $\mathscr{R}^{M_{r}^{\beta}}$ that satisfy the following condition (the third condition in (44))

$$
\left[\begin{array}{lll}
\boldsymbol{\psi}_{i, e}^{\beta} & \boldsymbol{\psi}_{i, i}^{\beta} & \boldsymbol{\psi}_{i, r}^{\beta}
\end{array}\right]\left(\boldsymbol{\chi}^{\beta}\right)^{-1}\left[\begin{array}{c}
\mathbf{0}  \tag{56}\\
\mathbf{c}_{i}^{\beta} \\
\mathbf{c}_{r}^{\beta}
\end{array}\right]=\left[\begin{array}{lll}
\boldsymbol{\psi}_{i, e}^{\alpha} & \boldsymbol{\Psi}_{i, i}^{\alpha} & \boldsymbol{\psi}_{i, r}^{\alpha}
\end{array}\right]\left(\boldsymbol{\chi}^{\alpha}\right)^{-1}\left[\begin{array}{c}
\mathbf{0} \\
\mathbf{c}_{i}^{\alpha} \\
\mathbf{c}_{r}^{\alpha}
\end{array}\right] .
$$

Now note that

$$
\begin{align*}
& {\left[\begin{array}{lll}
\boldsymbol{\psi}_{i, e}^{\alpha} & \boldsymbol{\psi}_{i, i}^{\alpha} & \boldsymbol{\psi}_{i, r}^{\alpha}
\end{array}\right]\left(\boldsymbol{\chi}^{\alpha}\right)^{-1}} \\
& =\left[\begin{array}{lll}
\boldsymbol{\psi}_{i, e}^{\alpha} & \boldsymbol{\Psi}_{i, i}^{\alpha} & \boldsymbol{\Psi}_{i, r}^{\alpha}
\end{array}\right]\left[\begin{array}{ccc}
\left(\boldsymbol{\Psi}_{e, e}^{\alpha}\right)^{-1} & -\left(\boldsymbol{\psi}_{e, e}^{\alpha}\right)^{-1} \boldsymbol{\psi}_{e, i}^{\alpha} & -\left(\boldsymbol{\psi}_{e, e}^{\alpha}\right)^{-1} \boldsymbol{\psi}_{e, r}^{\alpha} \\
\mathbf{0} & \mathbf{I} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{I}
\end{array}\right]  \tag{57}\\
& =[\underbrace{\boldsymbol{\psi}_{i, e}^{\alpha}\left(\boldsymbol{\psi}_{e, e}^{\alpha}\right)^{-1}}_{\mathbf{R}_{1}^{\alpha}} \underbrace{\boldsymbol{\psi}_{i, i}^{\alpha}-\boldsymbol{\psi}_{i, e}^{\alpha}\left(\boldsymbol{\psi}_{e, e}^{\alpha}\right)^{-1} \boldsymbol{\psi}_{e, i}^{\alpha}}_{\mathbf{R}_{2}^{\alpha}} \underbrace{\boldsymbol{\psi}_{i, r}^{\alpha}-\boldsymbol{\psi}_{i, e}^{\alpha}\left(\boldsymbol{\psi}_{e, e}^{\alpha}\right)^{-1} \boldsymbol{\psi}_{e, r}^{\alpha}}_{\mathbf{R}_{3}^{\alpha}}] .
\end{align*}
$$

With similar definitions for $\beta$ phase the condition (53) takes the following form
$\mathbf{R}_{2}^{\beta} \mathbf{c}_{i}^{\beta}+\mathbf{R}_{3}^{\beta} \mathbf{c}_{r}^{\beta}=\mathbf{R}_{2}^{\alpha} \mathbf{c}_{i}^{\alpha}+\mathbf{R}_{3}^{\alpha} \mathbf{c}_{r}^{\alpha}$,
and so,
$\mathbf{c}_{i}^{\beta}=\left(\mathbf{R}_{2}^{\beta}\right)^{-1} \mathbf{R}_{2}^{\alpha} \mathbf{c}_{i}^{\alpha}+\left(\mathbf{R}_{2}^{\beta}\right)^{-1} \mathbf{R}_{3}^{\alpha} \mathbf{c}_{r}^{\alpha}-\left(\mathbf{R}_{2}^{\beta}\right)^{-1} \mathbf{R}_{3}^{\beta} \mathbf{c}_{r}^{\beta}$.
Thus, with regard to (52) it can be concluded that

$$
\begin{align*}
& \forall \mathbf{c}_{i}^{\alpha} \in \mathscr{R}^{M_{i}^{\alpha}} \quad \forall \mathbf{c}_{r}^{\alpha} \in \mathscr{R}^{M_{r}^{\alpha}} \quad \forall \mathbf{c}_{r}^{\beta} \in \mathscr{R}^{M_{r}^{\beta}} \\
& \left(\mathbf{c}_{i}^{\alpha}\right)^{T}\left(\mathbf{A}_{2}^{\alpha}+\left(\mathbf{R}_{2}^{\alpha}\right)^{T}\left(\mathbf{R}_{2}^{\beta}\right)^{-T} \mathbf{A}_{2}^{\beta}\right)+\left(\mathbf{c}_{r}^{\alpha}\right)^{T}\left(\mathbf{A}_{3}^{\alpha}+\left(\mathbf{R}_{3}^{\alpha}\right)^{T}\left(\mathbf{R}_{2}^{\beta}\right)^{-T} \mathbf{A}_{2}^{\beta}\right)  \tag{60}\\
& +\left(\mathbf{c}_{r}^{\beta}\right)^{T}\left(\mathbf{A}_{3}^{\beta}+\left(\mathbf{R}_{3}^{\beta}\right)^{T}\left(\mathbf{R}_{2}^{\beta}\right)^{-T} \mathbf{A}_{2}^{\beta}\right)=0 .
\end{align*}
$$

Consequently

$$
\begin{align*}
& \mathbf{A}_{2}^{\alpha}+\left(\mathbf{R}_{2}^{\alpha}\right)^{T}\left(\mathbf{R}_{2}^{\beta}\right)^{-T} \mathbf{A}_{2}^{\beta}=\mathbf{0} \\
& \mathbf{A}_{3}^{\alpha}+\left(\mathbf{R}_{3}^{\alpha}\right)^{T}\left(\mathbf{R}_{2}^{\beta}\right)^{-T} \mathbf{A}_{2}^{\beta}=\mathbf{0}  \tag{61}\\
& \mathbf{A}_{3}^{\beta}+\left(\mathbf{R}_{3}^{\beta}\right)^{T}\left(\mathbf{R}_{2}^{\beta}\right)^{-T} \mathbf{A}_{2}^{\beta}=\mathbf{0}
\end{align*}
$$

Regarding to the way that $\mathbf{A}_{2}^{\alpha}, \mathbf{A}_{3}^{\alpha}, \mathbf{A}_{2}^{\beta}$ and $\mathbf{A}_{3}^{\beta}$ are defined in (49), the above equations can be rewritten as follows

$$
\begin{aligned}
& {\left[\begin{array}{lll}
\mathbf{L}_{i, e}^{\alpha} & \mathbf{L}_{i, i}^{\alpha} & \mathbf{L}_{i, r}^{\alpha}
\end{array}\right] \mathbf{d}^{\alpha}-\mathbf{p}_{i}^{\alpha}+\left(\mathbf{R}_{2}^{\alpha}\right)^{T}\left(\mathbf{R}_{2}^{\beta}\right)^{-T}\left(\left[\begin{array}{lll}
\mathbf{L}_{i, e}^{\beta} & \mathbf{L}_{i, i}^{\beta} & \mathbf{L}_{i, r}^{\beta}
\end{array}\right] \mathbf{d}^{\beta}-\mathbf{p}_{i}^{\beta}\right)=\mathbf{0},} \\
& {\left[\begin{array}{lll}
\mathbf{L}_{r, e}^{\alpha} & \mathbf{L}_{r, i}^{\alpha} & \mathbf{L}_{r, r}^{\alpha}
\end{array}\right] \mathbf{d}^{\alpha}-\mathbf{p}_{r}^{\alpha}+\left(\mathbf{R}_{3}^{\alpha}\right)^{T}\left(\mathbf{R}_{2}^{\beta}\right)^{-T}\left(\left[\begin{array}{lll}
\mathbf{L}_{i, e}^{\beta} & \mathbf{L}_{i, i}^{\beta} & \mathbf{L}_{i, r}^{\beta}
\end{array}\right] \mathbf{d}^{\beta}-\mathbf{p}_{i}^{\beta}\right)=\mathbf{0}} \\
& {\left[\begin{array}{lll}
\mathbf{L}_{r, e}^{\beta} & \mathbf{L}_{r, i}^{\beta} & \mathbf{L}_{r, r}^{\beta}
\end{array}\right] \mathbf{d}^{\beta}-\mathbf{p}_{r}^{\beta}-\left(\mathbf{R}_{3}^{\beta}\right)^{T}\left(\mathbf{R}_{2}^{\beta}\right)^{-T}\left(\left[\begin{array}{lll}
\mathbf{L}_{i, e}^{\beta} & \mathbf{L}_{i, i}^{\beta} & \mathbf{L}_{i, r}^{\beta}
\end{array}\right] \mathbf{d}^{\beta}-\mathbf{p}_{i}^{\beta}\right)=\mathbf{0} .}
\end{aligned}
$$

Finally, considering the above equations and equations (45) one can obtain the unknown coefficients by solving the following linear system of algebraic equations

$$
\begin{align*}
& {\left[\begin{array}{cccc}
\mathbf{L}_{i, e}^{\alpha} & \mathbf{L}_{i, i}^{\alpha} & \mathbf{L}_{i, r}^{\alpha} & \left(\mathbf{R}_{2}^{\alpha}\right)^{T}\left(\mathbf{R}_{2}^{\beta}\right)^{-T} \mathbf{L}_{i, e}^{\beta}
\end{array}\right.} \\
& \mathbf{L}_{r, e}^{\alpha} \quad \mathbf{L}_{r, i}^{\alpha} \quad \mathbf{L}_{r, r}^{\alpha} \quad\left(\mathbf{R}_{3}^{\alpha}\right)^{T}\left(\mathbf{R}_{2}^{\beta}\right)^{-T} \mathbf{L}_{i, e}^{\beta} \\
& \begin{array}{cccc}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{L}_{r, e}^{\beta}-\left(\mathbf{R}_{3}^{\beta}\right)^{T}\left(\mathbf{R}_{2}^{\beta}\right)^{-T} \mathbf{L}_{i, e}^{\beta} \\
\boldsymbol{\psi}^{\alpha} & \boldsymbol{\psi}^{\alpha} & \boldsymbol{\psi}^{\alpha} &
\end{array} \\
& \begin{array}{llll}
\boldsymbol{\psi}_{e, e}^{\alpha} & \boldsymbol{\Psi}_{e, i}^{\alpha} & \boldsymbol{\psi}_{e, r}^{\alpha} & \mathbf{0}
\end{array} \\
& \begin{array}{cccc}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\psi}_{e, e}^{\beta}
\end{array} \\
& \begin{array}{llll}
\boldsymbol{\psi}_{i, e}^{\alpha} & \boldsymbol{\Psi}_{i, i}^{\alpha} & \boldsymbol{\psi}_{i, r}^{\alpha} & -\boldsymbol{\psi}_{i, e}^{\beta}
\end{array} \\
& \left.\begin{array}{cc}
\left(\mathbf{R}_{2}^{\alpha}\right)^{T}\left(\mathbf{R}_{2}^{\beta}\right)^{-T} \mathbf{L}_{i, i}^{\beta} & \left(\mathbf{R}_{2}^{\alpha}\right)^{T}\left(\mathbf{R}_{2}^{\beta}\right)^{-T} \mathbf{L}_{i, r}^{\beta} \\
\left(\mathbf{R}_{3}^{\alpha}\right)^{T}\left(\mathbf{R}_{2}^{\beta}\right)^{-T} \mathbf{L}_{i, i}^{\beta} & \left(\mathbf{R}_{3}^{\alpha}\right)^{T}\left(\mathbf{R}_{2}^{\beta}\right)^{-T} \mathbf{L}_{i, r}^{\beta} \\
\mathbf{L}_{r, i}^{\beta}-\left(\mathbf{R}_{3}^{\beta}\right)^{T}\left(\mathbf{R}_{2}^{\beta}\right)^{-T} \mathbf{L}_{i, i}^{\beta} & \mathbf{L}_{r, r}^{\beta}-\left(\mathbf{R}_{3}^{\beta}\right)^{T}\left(\mathbf{R}_{2}^{\beta}\right)^{-T} \mathbf{L}_{i, r}^{\beta} \\
\mathbf{0} & \mathbf{0} \\
\boldsymbol{\Psi}_{e, i}^{\beta} & \boldsymbol{\psi}_{e, r}^{\beta} \\
-\boldsymbol{\psi}_{i, i}^{\beta} & -\boldsymbol{\psi}_{i, r}^{\beta}
\end{array}\right] \\
& {\left[\begin{array}{c}
\mathbf{d}^{\alpha} \\
\mathbf{d}^{\beta}
\end{array}\right]=\left[\begin{array}{c}
\mathbf{p}_{i}^{\alpha}+\left(\mathbf{R}_{2}^{\alpha}\right)^{T}\left(\mathbf{R}_{2}^{\beta}\right)^{-T} \mathbf{p}_{i}^{\beta} \\
\mathbf{p}_{r}^{\alpha}+\left(\mathbf{R}_{3}^{\alpha}\right)^{T}\left(\mathbf{R}_{2}^{\beta}\right)^{-T} \mathbf{p}_{i}^{\beta} \\
\mathbf{p}_{r}^{\beta}-\left(\mathbf{R}_{3}^{\beta}\right)^{T}\left(\mathbf{R}_{2}^{\beta}\right)^{-T} \mathbf{p}_{i}^{\beta} \\
\mathbf{g}^{\alpha, \dagger} \\
\mathbf{g}^{\beta, \dagger} \\
\mathbf{0}
\end{array}\right] .} \tag{63}
\end{align*}
$$

## 4 Numerical Experiments

### 4.1 Example 1: Infinite plate with a circular hole under uniform uniaxial tension

Consider an infinite isotropic plate weakened by a circular hole of radius $a$. The plate is subjected to far-field uniform tension $q$ in $x_{1}$-direction (Fig.3). According to the elasticity theory [Timoshenko and Goodier (1970)], the displacement field in terms of polar coordinate system $(r, \theta)$ with the origin located at the center of the
hole is as follows:

$$
\begin{align*}
& u_{r}=\frac{q}{4 G}\left\{r[(\kappa-1) / 2+\cos 2 \theta]+\frac{a^{2}}{r}[1+(1+\kappa) \cos 2 \theta]-\frac{a^{4}}{r^{3}} \cos 2 \theta\right\}, \\
& u_{\theta}=\frac{q}{4 G}\left[(1-\kappa) \frac{a^{2}}{r}-r-\frac{a^{4}}{r^{3}}\right] \sin 2 \theta, \tag{64}
\end{align*}
$$



Figure 3: A uniform tension field disturbed by a circular hole
where $u_{r}$ and $u_{\theta}$ are the radial and tangential displacements, respectively. In addition we have $G=E /(2(1+v))$ and $\kappa=(3-v) /(1+v)$ in which $E$ and $v$ are Young's modulus and Poisson's ratio, respectively. The corresponding stress field in Cartesian coordinate system is given by [Zhu and Atluri (1998)]:

$$
\begin{align*}
& \sigma_{11}=q\left[1-\frac{a^{2}}{r^{2}}(3 / 2 \cos 2 \theta+\cos 4 \theta)+\frac{3 a^{4}}{2 r^{4}} \cos 4 \theta\right], \\
& \sigma_{22}=-q\left[\frac{a^{2}}{r^{2}}(1 / 2 \cos 2 \theta-\cos 4 \theta)+\frac{3 a^{4}}{2 r^{4}} \cos 4 \theta\right],  \tag{65}\\
& \sigma_{12}=-q\left[\frac{a^{2}}{r^{2}}(1 / 2 \sin 2 \theta+\sin 4 \theta)-\frac{3 a^{4}}{2 r^{4}} \sin 4 \theta\right] .
\end{align*}
$$

For numerical demonstration, $E=1000, v=0.3, a=1$, and $q=1$ are selected. Due to symmetry, a quarter of the plate is considered for analysis. The corresponding

EBCs are

$$
\begin{align*}
& u_{2}\left(x_{1}, 0\right)=0 \quad \forall x_{1} \geq 1 \\
& u_{1}\left(0, x_{2}\right)=0 \quad \forall x_{2} \geq 1 \tag{66}
\end{align*}
$$

For numerical analysis the dimensions of the domain of the problem are considered to be five times the radius of circular hole in both directions. Using equation (62), tractions along the edges $x_{1}=5,0 \leq x_{2} \leq 5$ and $x_{2}=5,0 \leq x_{1} \leq 5$ are calculated and imposed as the natural boundary conditions of the problem. The problem has been solved for two different particle configurations. In the first configuration (Fig. 4) 340 particles and in the second one (Fig. 5) 675 particles are used. The numerical integration has been performed by employing Gauss quadrature with $6 \times 6$ Gaussian points in each virtually constructed cell which is formed by the particles.
The purpose of this example is to study the efficacy of the proposed algorithm for determining the area associated with each particle (nodal domain) in nodal quadrature. The results are compared with the case in which the nodal domains are taken to be equal to one. To this end, $\sigma_{11}\left(0, x_{2}\right), 1 \leq x_{2} \leq 5$ has been calculated by use of 340 particles and its diagram has been shown in Fig. 6. Magnification of the result in the vicinity of the hole is shown in Fig. 7. It can be seen that the result of the proposed method is in reasonable agreement with the exact solution and the error is reduced with distance from the hole. While the result produced using nodal domain, $\Delta S_{J}=1$ is not as accurate as that of the proposed methodology.
Note that, the stress concentration factors in the vicinity of the hole are defined as:

$$
\begin{equation*}
K_{11}=\lim _{r \rightarrow a^{+}}\left(\left.\sigma_{11}\right|_{\theta=\pi / 2}\right), K_{22}=\lim _{r \rightarrow a^{+}}\left(\left.\sigma_{22}\right|_{\theta=0}\right) . \tag{67}
\end{equation*}
$$

By substituting equation (62) into relations (64) one can easily arrive at the following results:

$$
\begin{equation*}
K_{11}=3, K_{22}=-1 \tag{68}
\end{equation*}
$$

Using RKPM, the stress concentration factors for each configuration of particles are calculated and presented in Tab. 1. It is evident that calculation of nodal domain via the present algorithm leads to more accurate results. Also from the displayed data in Tab. 1, it is observed that by nearly doubling the number of particles (from 340 to 675), the results of the proposed algorithm improve remarkably, while such a convergence rate is not observed when $\Delta S_{J}=1$ is used.

### 4.2 Example 2: Circular inhomogeneity with uniform radial eigenstrain

When the material properties of a subdomain of a body are different from those of its surrounding matrix, the subdomain is referred to as inhomogeneity. Moreover,


Figure 4: Particles configuration pertinent to example 1; 340 particles


Figure 5: Particles configuration pertinent to example 1; 675 particles


Figure 6: Stress component $\sigma_{11}$ along $x_{1}=0$ using 340 particles


Figure 7: Detailed behavior of $\sigma_{11}$ in the vicinity of the hole; magnification of figure 6

Table 1: Stress concentration factors corresponding to example 1

| $N P$ | $\Delta S_{J}$ | $K_{11}$ | Error | $K_{22}$ | Error |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 340 | Calculated | 2.9481 | $\mathbf{- 1 . 7 3 \%}$ | -0.9748 | $\mathbf{2 . 5 2 \%}$ |
|  | 1 | 2.8413 | $\mathbf{- 5 . 2 9 \%}$ | -0.9418 | $\mathbf{5 . 8 2 \%}$ |
| 675 | Calculated | 2.9808 | $\mathbf{- 0 . 6 4 \%}$ | -0.9894 | $\mathbf{1 . 0 6 \%}$ |
|  | 1 | 2.8397 | $\mathbf{- 5 . 3 4 \%}$ | -0.9436 | $\mathbf{5 . 6 4 \%}$ |

if an eigenstrain field is prescribed within the inhomogeneity, then the subdomain is referred to as inhomogeneous inclusion, [Mura (1987)].
The circular inhomogeneous inclusion with uniform eigenstrain field is one of the benchmark problems of material discontinuities; for example, fiber reinforced composites with misfit strains. This problem has been used extensively as a reference problem in a large number of numerical studies; among them one can mention [Cordes, Moran (1996)], [Krongauz, Belytschko (1998)], [Li, Shen, Atluri (2003)] and [Masuda, Noguchi (2006)].
In this example, the robustness of the augmented corrected collocation method has been examined. Fig. 8 presents a schematic description of the problem.
The exact solution for this problem in polar coordinates can be expressed as [Cordes, Moran (1996)]
$u_{r}=\left\{\begin{array}{ll}C_{1} r, & r \leq R, \\ C_{1} R^{2} / r, & r>R,\end{array} \quad C_{1}=\frac{\left(\mu^{(1)}+\lambda^{(1)}\right) \varepsilon^{*}}{\mu^{(1)}+\lambda^{(1)}+\mu^{(2)}}\right.$,
$u_{\theta}=0$,
where $R$ is the radius of the circular inhomogeneous inclusion with uniform eigenstrain, $\varepsilon_{r r}^{*}=\varepsilon^{*}$. The inhomogeneous inclusion and the surrounding matrix are referred to as phase 1 and 2 , respectively. Both phases are isotropic, and $\mu^{(i)}$, $\lambda^{(i)}, i=1,2$ are Lame constants for phase $i$. The corresponding strains are
$\varepsilon_{r r}=\left\{\begin{array}{ll}C_{1}, & r<R, \\ -C_{1} R^{2} / r^{2}, & r>R,\end{array} \quad \varepsilon_{r \theta}=0, \quad \varepsilon_{\theta \theta}= \begin{cases}C_{1}, & r \leq R, \\ C_{1} R^{2} / r^{2}, & r>R\end{cases}\right.$
In this example it is assumed that $R=1$, and
$\left\{\begin{array}{l}E^{(1)}=1000, \\ v^{(1)}=0.28,\end{array} \quad\left\{\begin{array}{l}E^{(2)}=900, \\ v^{(2)}=0.33 .\end{array}\right.\right.$
Due to the axisymmetry of the problem, only a quarter of the domain ( $0 \leq \theta \leq \pi / 2$ ) whose size is $5 \times 5$ has been selected for numerical modeling. Each phase is dis-


Figure 8: Circular inhomogeneous inclusion in an infinite body with uniform eigen$\operatorname{strain} \varepsilon^{*}$
cretized separately; so the shape functions of the particles located in each phase are only defined in the corresponding phase. As it is shown in Fig. 9, 302 particles have been used for the analysis within the first phase and 503 particles have been used in the second phase. At the interface of the two phases, 82 particles ( 41 particles for each phase) are located (Fig. 10). The area associated with each particle for nodal quadraure (nodal domain) is determined using the proposed algorithm.
Since the selected domain is finite, the displacements in the right and upper edges of the plate have been calculated using the theoretical solution (66) and have been considered as EBCs. Also due to the symmetry, we have the following boundary conditions at the left and lower edges of the plate
$u_{1}\left(0, x_{2}\right)=0, u_{2}\left(x_{1}, 0\right)=0$.
Note that, at the interface of the two phases the following condition holds
$\mathbf{u}^{(1)}=\mathbf{u}^{(2)}$.

Also notice that both, the EBCs and the continuity condition at the interface are imposed via the augmented corrected collocation method.
Radial displacement along $\theta=\pi / 4$ has been plotted as a function of r in Fig. 11. The obtained results show a considerable accordance between the numerical and exact solutions. The graph is continuous but is not differentiable at the interface of the two phases which is in agreement with the fact that there is a jump discontinuity in the radial strain at the interface. To see this in more detail the radial strain along $\theta=\pi / 4$ has been plotted in Fig. 12. It is seen that, the mentioned discontinuity has been captured accurately without any undesired oscillation in the vicinity of the interface which verifies the efficacy of the augmented corrected collocation method.
The radial distributions of the hoop strain and radial stress along $\theta=\pi / 4$ are given in Figs. 13 and 14, respectively. Although, the variations of hoop strain and hoop stress are continuous, they are not classically differentiable at the interface. In both figures the numerical results are in accordance with the exact solution with noticeable accuracy.
Fig. 15 shows the radial variation of hoop stress along $\theta=\pi / 4$ which exhibits a jump discontinuity across the interface. The conformity of numerical results and exact solution, once again confirms the accuracy and the efficacy of the augmented corrected collocation method.

### 4.3 Example 3: Interaction between a crack and a circular inhomogeneity

The problem of crack-inhomogeneity interaction, due to its application in composites, is of particular interest. In order to show the ability of the proposed numerical technique in accurate computation of the elastic fields of such an interaction problem, a circular fiber in the vicinity of a horizontally oriented slit-like crack under far field a uniform uniaxial loading $\sigma_{22}(\mathbf{x})=\sigma^{0}$ as shown in Fig. 16 is considered. The analytical solution of this problem is available in the literature [Atkinson (1972)], [Erdogan, Gupta, and Ratwani (1974)], [Ojaghnezhad and Shodja (2009)]. In Fig. 16, $a_{f}$ is the radius of the inhomogeneity, $a_{c}$ is the halfcrack length, $C_{1}$ is the horizontal distance between the centers of the inhomogeneity and crack, and $C_{2}$ is the vertical distance between the center of the inhomogeneity and the crack plane. The crack-tip which is nearer to the inhomogeneity is labeled as $A$ and the crack-tip which is located farther away is labeled as $B$. The domain occupied by the inhomogeneity is denoted by $\Phi$ and its surrounding matrix is denoted by $\Omega$. The shear modulus and Poisson's ratio of the inhomogeneity are indicated as $\mu^{\phi}$ and $v^{\phi}$, respectively, whereas the shear modulus and Poisson's ratio of the matrix are denoted as $\mu^{\Omega}$ and $v^{\Omega}$, respectively. It is assumed that


Figure 9: Configuration of the particles used in example 2; one quarter of the domain is considered


Figure 10: The arrangement of particles in the first phase and in the interface boundary


Figure 11: Distribution of radial displacement in example 2


Figure 12: Distribution of radial strain in example 2


Figure 13: Distribution of hoop strain in example 2


Figure 14: Distribution of radial stress in example 2


Figure 15: Distribution of hoop stress in example 2
$a_{f}=2, \quad a_{c}=1, \quad C_{2}=1, v^{\Omega}=0.35$.
Computation of the stress intensity factors (SIFs) at the crack-tips $A$ and $B$ with reasonable accuracy is very important for predicting the crack behavior in the vicinity of the inhomogeneity. Define the normalized SIF, $\bar{K}_{\zeta}$ at crack-tip $\zeta=A$ or $B$ as:
$\bar{K}_{\zeta}=\frac{K_{\zeta}^{I}}{K_{\zeta}^{S}}=\lim _{r \rightarrow 0} \frac{\sigma_{\zeta}^{I}}{\sigma_{\zeta}^{S}}$,
where the superscripts $I$ and $S$ stand for "interacting" and "single", respectively. $K_{\zeta}^{I}$ is the SIF at the crack-tip $\zeta$ when the crack is interacting with the inhomogeneity, and $K_{\zeta}^{S}$ is the pertinent SIF in the absence of inhomogeneity. The corresponding stress component as the crack-tip $\zeta$ is approached $(r \rightarrow 0)$ in the presence and absence of the inhomogeneity are $\sigma_{\zeta}^{I}$ and $\sigma_{\zeta}^{S}$, respectively.
In order to solve the proposed problem by the present methodology, the dimensions of the plate are chosen reasonably large so that the crack and the inhomogeneity do not have any interactions with the remote boundaries. To this end, it is assumed that the plate's width is 28 units, while the plate's length depends on the value of $C_{1}$. For $C_{1}=4,6$ and 8 the plate's length are chosen to be 28,30 and 32 , respectively. For each value of $C_{1}$, a different configuration of particles has been considered. The domain size and particle distribution for cases, $C_{1}=4,6$ and 8 are displayed in Figs. 17-19, respectively. For demonstration, a magnified view of the particles
arrangement in the proximity of the inhomogeneity and the crack for the case of $C_{1}=4$ is shown in Fig. 20. Note that according to what was said in section 3, the inhomogeneity and surrounding medium have been discretized separately via two distinct sets of particles. The continuity condition between these two phases has been imposed using the augmented corrected collocation method.
The SIFs are computed by use of path-independent J-integral. Because of the asymmetric nature of the problem, the crack is under mixed mode conditions, but considering the type of loading, the first mode is the dominant one. For $\mu^{\phi} / \mu^{\Omega}=0$ and 23 (in the latter case $\nu^{\phi}$ is taken to be equal to 0.3 ), mode I SIFs, normalized according to (71), have been calculated and compared with the analytical results of [Ojaghnezhad and Shodja (2009)] and [Erdogan, Gupta, and Ratwani (1974)] in Tab. 2 and Tab. 3, respectively. Generally, the numerical results are within a reasonable range of the analytical results; the analytical results are not the exact solutions. It is noteworthy to mention that, in the case for which the inhomogeneity is a hole ( $\mu^{\phi} / \mu^{\Omega}=0$ ), the maximum values of SIFs which are displayed in Tab. 2 correspond to $C_{1}=4$ and decrease with increasing $C_{1}$. Both $\bar{K}_{A}$ and $\bar{K}_{B}$ are greater than 1 , which implies that the hole has anti-shielding effect. As the distance between the crack and the hole becomes larger the value of SIFs approaches 1, that is the value of the SIFs in the absence of the hole. In the case where the inhomogeneity has a larger shear modulus ( $\mu^{\phi} / \mu^{\Omega}=23$ ), among the displayed data in Tab. 3, the minimum values of SIFs, which are smaller than 1 , correspond to $C_{1}=4$ and become larger as $C_{1}$ becomes larger. This trend shows that the hard inhomogeneity shields the crack when the crack is located near the inhomogeneity. When $C_{1}=8$ the values of SIFs are both smaller than 1 , but very close to 1 , that is the effect of the inhomogeneity becomes nearly negligible.

## 5 Conclusion

New capabilities have been incorporated into RKPM to numerically solve the elastic equilibrium equations in two dimensions. By extending the corrected collocation method, a simple and efficient method has been proposed for numerical analysis of domains which consist of several phases with different material properties. In this treatment, each phase is discretized by a separate set of particles and the corresponding shape functions for each phase are derived separately. Then, by using the obtained shape functions the weak form of the governing BVP is discretized. Finally, the continuity condition at the interface and the EBCs are imposed by augmented corrected collocation method. Moreover, the area associated with each particle for nodal quadrature has been numerically computed and imposed by an efficient algorithm. The presented solution of the problem of interaction between a crack and a circular inhomogeneity is an indicator of the high potential of the


Figure 16: A horizontally oriented slit-like crack near a circular inhomogeneity under uniform far-field tension


Figure 17: The domain size and particle distribution pertinent to crackinhomogenety interaction problem with $C_{1}=4$


Figure 18: The domain size and particle distribution pertinent to crackinhomogenety interaction problem with $C_{1}=6$


Figure 19: The domain size and particle distribution pertinent to crackinhomogenety interaction problem with $C_{1}=8$


Figure 20: Particles configuration in the vicinity of the inhomogeneity and crack tips in the case of $C_{1}=4$
augmented corrected collocation method in solving complicated problems.

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Table 2: The normalized SIFs at crack-tips $A$ and $B$ in the case of $\mu^{\phi} / \mu^{\Omega}=0$

|  | Erdogan, Gupta, and Ratwani (1974) |  | Ojaghnezhad and Shodja (2009) |  | Present Study |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $C_{1}$ | $\overline{\mathbf{K}}_{\text {A }}$ | $\bar{K}_{B}$ | $\overline{\mathbf{K}}_{\mathrm{A}}$ | $\bar{K}_{B}$ | $\overline{\mathbf{K}}_{\text {A }}$ | $\bar{K}_{\text {B }}$ |
| 4 | 1.364 | 1.193 | 1.546 | 1.262 | 1.497 | 1.280 |
| 6 | 1.109 | 1.070 | 1.114 | 1.082 | 1.121 | 1.089 |
| 8 | 1.049 | 1.036 | 1.050 | 1.037 | 1.060 | 1.030 |

Table 3: The normalized SIFs at crack-tips $A$ and $B$ in the case of $\mu^{\phi} / \mu^{\Omega}=23$

|  | Erdogan, Gupta, and Ratwani (1974) |  | Ojaghnezhad and Shodja (2009) |  | Present Study |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $C_{1}$ | $\overline{\mathbf{K}}_{\mathbf{A}}$ | $\bar{K}_{B}$ | $\overline{\mathbf{K}}_{\mathbf{A}}$ | $\bar{K}_{B}$ | $\overline{\mathbf{K}}_{\text {A }}$ | $\bar{K}_{\text {B }}$ |
| 4 | 0.833 | 0.914 | 0.839 | 0.915 | 0.828 | 0.917 |
| 6 | 0.952 | 0.970 | 0.956 | 0.972 | 0.960 | 0.974 |
| 8 | 0.980 | 0.985 | 0.982 | 0.987 | 0.983 | 0.985 |

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