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Approximate Solution of an Inverse Problem for a Non-Stationary General Kinetic Equation

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Abstract: We investigate the solvability of an inverse problem for the non-stationary general kinetic equation. We also obtained the approximate solution of this problem by using symbolic computation. A comparison between the approximate solution and the exact solution of the problem is presented.

Keywords: Overdetermined Inverse Problem, Non-stationary Kinetic Equation, Galerkin Method, Symbolic Computation.

1 Introduction

The problem of determining coefficients, the right-hand side, initial conditions or boundary conditions of a differential equation from some additional information about a solution of the equation is called inverse problem, [Lavrent'ev (1967), Tikhonov and Arsenin (1979), Anikonov (2001), Amirov (2001)]. Such problems arise in many applications such as medical imaging, exploration geophysics and non-destructive evaluation where measurements made in the exterior of a body are used to deduce properties of the hidden interior [Ling and Atluri (2006), Huang and Shih (2007), Ling and Takeuchi (2008), Beilina and Klibanov (2008)]. Kinetic equations describe the continuity of motion of substance and are the basic equations of mathematical physics and naturel science. They can be used for quantitative and qualitative description of physical, chemical, biological, social and other processes. In this study, we consider an inverse problem for a non-stationary general kinetic equation. The physical interpretation of such problems consists in finding forces of particle interaction, scattering indicatrices, radiation sources and other physical parameters.

Let Ω be a domain in the Euclidean space $\mathbb{R}^{2n+1}$, $(n \geq 1)$. For the variables

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$(x,v,t) \in \Omega$, it is assumed that $x \in D$, $v \in G$, $t \in (0,T)$, where D and G are domains in $\mathbb{R}^n$ with boundaries of class C^3.

We deal with the following kinetic equation in Ω:

$$Lu \equiv \frac{\partial u}{\partial t} + \sum_{i=1}^{n} \left(v_i \frac{\partial u}{\partial x_i} + f_i \frac{\partial u}{\partial v_i} \right) + \int_G K\left(x,v,v'\right) u\left(x,v',t\right) dv' = \rho\left(x,t\right), \tag{1}$$

which typically models the density of particles $u(x,v,t)$ in the space of positions x and velocities v as a function of time t. In (1), $\rho(x,t)$ is an unknown source function, $K(x,v,v')$ is called scattering kernel which indicates the amount of particles scattering from a direction v into a direction v' at position x, $f = (f_1,...,f_n)$ is the force acting on a particle.

Problem 1 *Determine the functions $u(x,v,t)$ and $\rho(x,t)$ defined in Ω from equation (1), provided that $K(x,v,v')$, $f = (f_1,...,f_n)$ are given and the trace of $u(x,v,t)$ is known on the boundary.*

In this paper the solvability of Problem 1 is investigated. For this aim, uniqueness, existence and stability conditions for the solution of the problem are formulated. Approximate solution of the problem is computed by using a symbolic computation technique based on the Galerkin method.

Problem 1 is an overdetermined problem. In the theory of inverse problems, if the number of free variables in the additional data exceeds the number of free variables in the unknown coefficient or right hand side ($\rho(x,t)$) of the equation, then the problem is called overdetermined. The initial data for these problems can not be arbitrary; they should satisfy some "solvability conditions" which are difficult to establish [Amirov (1987), Amirov (2001)]. So one of our purposes is to demonstrate how to investigate the solvability of an overdetermined problem.

2 Solvability of the Problem

On using some extension of the class of unknown functions ρ, overdetermined problem in question replaced by a determined one. This is achieved by assuming the unknown function ρ depends not only upon the space variable x and time variable t, but also upon the direction v in some special manner, i.e. consider $\rho(x,v,t)$. The dependence upon v of ρ is impossible to be arbitrary, for in the opposite case the problem would be underdetermined and it is easy to construct the nonuniqueness examples of a solution. Herein the special dependence of $\rho(x,v,t)$ upon the direction v means that $\rho(x,v,t)$ satisfies a certain differential equation $\left(\hat{L}\rho = 0\right)$

such that the new problem which we call Problem 2 with the function $\rho(x,v,t)$ becomes a determined one and the sufficiently smooth functions $\rho(x,t)$ satisfy this equation.

Suppose that, such a differential equation for $\rho(x,v,t)$ has been found and a priori the exact data u_0^e of Problem 2 related to a function $\rho(x,t)$ is known. Then, utilizing u_0^e, a solution $\tilde{\rho}$ to Problem 2 can be constructed. By uniqueness of the solution, $\tilde{\rho}$ and $\rho(x,t)$ coincide. At the same time, knowing the approximate data u_0^a with $\left\|u_0^e - u_0^a\right\|_{H^3(\Gamma_1)} \leq \varepsilon$, an approximate solution $\rho^a(x,v,t)$ can be constructed such that $\|\rho - \rho^a\|_{L_2(\Omega)} \leq \varepsilon C$. Recall that, if ρ depends only on x and t, and u_0^a does not satisfy the "solvability conditions", the solution ρ^a depending only x and t does not exist. In other words, a regularising procedure constructed for Problem 2.

Problem 2 *Determine the functions $u(x,v,t)$ and $\rho(x,v,t)$ defined in Ω from equation (1), provided that the functions $K(x,v,v')$, $f=(f_1,...,f_n)$ are given, the trace of $u(x,v,t)$ is known on the boundary, i.e.,*

$$u|_{\partial\Omega} = u_0 \tag{2}$$

and the function $\rho(x,v,t)$ satisfies

$$\left\langle \rho, \widehat{L}\eta \right\rangle = 0 \tag{3}$$

for any $\eta \in C_0^\infty(\Omega)$. Here

$$\widehat{L} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i \partial v_i}.$$

The notations to be used in the sequel are introduced below:

The standart function space $C^m(\Omega)$ is the Banach space of functions that are m times continuously differentiable in Ω; $C^\infty(\Omega)$ is the set of functions that belong to $C^m(\Omega)$ for all $m \geq 0$; $C_0^\infty(\Omega)$ is the set of finite functions in Ω that belong to $C^\infty(\Omega)$; $L_2(\Omega)$ is the space of measurable functions that are square integrable in Ω, $H^k(\Omega)$ is the Sobolov space and $\overset{\circ}{H}{}^k(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm of $H^k(\Omega)$ [Lions and Magenes (1972), Mikhailov (1978)].

We define $\widetilde{C}_0^3(\Omega) = \left\{\varphi : \varphi \in C^3(\Omega), \varphi = 0 \text{ on } \partial\Omega\right\}$ and select a subset $\{w_1, w_2, ...\}$ of $\widetilde{C}_0^3$ which is orthonormal and everywhere dense in $L_2(\Omega)$. Let P_n be the orthogonal projector of $L_2(\Omega)$ onto M_n, where M_n is the linear span of $\{w_1, w_2, ...\}$. By $\Gamma(A)$ the set of functions u is denoted with the following properties:

i. $u \in \Gamma(A)$, $Au \in L_2(\Omega)$ in the generalized sense, where $Au = \widehat{L}Lu$;

ii. There exists a sequence $\{u_k\} \subset \widetilde{C}_0^3$ such that $u_k \to u$ in $L_2(\Omega)$ and $\langle Au_k, u_k\rangle \to \langle Au, u\rangle$ as $k \to \infty$.

The condition that $Au \in L_2(\Omega)$ in the generalized sense means that there exists a function $f \in L_2(\Omega)$ such that $\langle u, A^*\varphi\rangle = -\langle f, \varphi\rangle$ and $Au = f$ for all $\varphi \in C_0^\infty(\Omega)$, where A^* is the differential operator conjugate to A in the sense of Lagrange.

Theorem 1 *Let $f \in C^1(\Omega)$, $K(x,v,v') \in C^1\left(\overline{D}\times\overline{G}\times\overline{G}\right)$ and assume that the following inequalities hold for all $\xi \in \mathbb{R}^n$:*

$$\xi \in \mathbb{R}^n : \sum_{i,j=1}^{n} \frac{\partial f_i}{\partial x_j}\xi^i\xi^j \geq \alpha_1 |\xi|^2,$$

$$\alpha_1 - \frac{L_0}{2} > \alpha_2,\ L_0 = l_0 C,\ l_0 = \max_{1\leq i\leq n}\left\{\max_{x\in\overline{D}} \int_G\int_G K_{v_i}^2(x,v,v')\,dv\,dv'\right\}, \tag{4}$$

where α_1, α_2 are positive numbers. Then Problem 2 has at most one solution (u,ρ) such that $u \in \Gamma(A)$ and $\rho \in L_2(\Omega)$.

Proof. Let (u,ρ) be a solution to Problem 2 such that $u = 0$ on $\partial\Omega$ and $u \in \Gamma(A)$. Equation (1) and condition (3) imply $Au = 0$. Since $u \in \Gamma(A)$, there exists a sequence $\{u_k\} \subset \widetilde{C}_0^3$ such that $u_k \to u$ in $L_2(\Omega)$ and $\langle Au_k, u_k\rangle \to 0$ as $k \to \infty$. Observing that $u_k = 0$ on $\partial\Omega$, we get

$$-2\langle Au_k, u_k\rangle = 2\sum_{i=1}^{n}\left\langle \frac{\partial}{\partial v_i}(Lu_k), u_{k_{x_i}}\right\rangle.$$

We have the following identity for the right-hand side of the last equality:

$$\sum_{i=1}^{n} 2\frac{\partial u_k}{\partial x_i}\frac{\partial}{\partial v_i}(Lu_k) = \sum_{i=1}^{n}\left(\frac{\partial u_k}{\partial x_i}\right)^2 + \sum_{i,j=1}^{n}\frac{\partial f_i}{\partial x_j}\frac{\partial u_k}{\partial v_i}\frac{\partial u_k}{\partial v_j}$$
$$+\sum_{i=1}^{n}\frac{\partial}{\partial v_i}\left[\frac{\partial u_k}{\partial t}\frac{\partial u_k}{\partial x_i}\right] + \sum_{i=1}^{n}\frac{\partial}{\partial t}\left[\frac{\partial u_k}{\partial v_i}\frac{\partial u_k}{\partial x_i}\right] - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[\frac{\partial u_k}{\partial t}\frac{\partial u_k}{\partial v_i}\right]$$
$$+\sum_{i,j=1}^{n}\frac{\partial}{\partial v_j}\left(v_i\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) + \sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(v_i\frac{\partial u_k}{\partial v_j}\frac{\partial u_k}{\partial x_j}\right) - \sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(v_i\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial v_j}\right)$$
$$+\sum_{i=1}^{n}\frac{\partial}{\partial v_i}\left[v_i\left(\frac{\partial u_k}{\partial x_i}\right)^2\right] + \sum_{i,j=1}^{n}\frac{\partial}{\partial v_j}\left(f_i\frac{\partial u_k}{\partial v_i}\frac{\partial u_k}{\partial x_j}\right) + \sum_{i,j=1}^{n}\frac{\partial}{\partial v_i}\left(f_i\frac{\partial u_k}{\partial v_j}\frac{\partial u_k}{\partial x_j}\right)$$
$$-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(f_i\frac{\partial u_k}{\partial v_i}\frac{\partial u_k}{\partial v_j}\right) + \int_G K_{v_i}(x,v,v')u_k\left(x,v',t\right)dv'\frac{\partial u_k}{\partial x_i}. \tag{5}$$

If the geometry of the domain Ω and the condition $u_k = 0$ on $\partial\Omega$ are taken into account, then from (5) we obtain

$$-\langle Au_k, u_k\rangle = J(u_k), \tag{6}$$

where

$$J(u_k) \equiv \frac{1}{2}\sum_{i=1}^{n}\int_{\Omega}\left(\left(\frac{\partial u_k}{\partial x_i}\right)^2 + \sum_{j=1}^{n}\frac{\partial f_i}{\partial x_j}\frac{\partial u_k}{\partial v_i}\frac{\partial u_k}{\partial v_j} + \int_{G} K_{v_i}(x,v,v')u_k(x,v',t)\,dv'\frac{\partial u_k}{\partial x_i}\right)d\Omega. \tag{7}$$

We can estimate the third term in (7) as follows:

$$\begin{aligned}
&2\sum_{i=1}^{n}\int_{\Omega}\int_{G} K_{v_i}(x,v,v')u_k(x,v',t)\,dv' u_{k_{x_i}} d\Omega \\
\geq\ & -\sum_{i=1}^{n}\int_{\Omega}\left((\int_{G} K_{v_i}(x,v,v')u_k(x,v',t)\,dv')^2 + u_{k_{x_i}}^2\right)d\Omega \\
\geq\ & -L_0\int_{\Omega}|\bigtriangledown_v u_k|^2 d\Omega - \int_{\Omega}|\bigtriangledown_x u_k|^2 d\Omega,
\end{aligned} \tag{8}$$

where $l_0 = \max\limits_{1\leq i\leq n}\left\{\max\limits_{x\in\overline{D}}\int\limits_{G}\int\limits_{G} K_{v_i}^2(x,v,v')dvdv'\right\}$, $L_0 = l_0 C$.

Since Ω is bounded and $u_k = 0$ on $\partial\Omega$, from the assumptions of the theorem and (8) it follows that

$$\begin{aligned}
J(u_k)\ &>\ \frac{1}{2}\int_{\Omega}|\nabla_x u_k|^2 d\Omega + \alpha_1\int_{\Omega}|\bigtriangledown_v u_k|^2 d\Omega - \frac{L_0}{2}\int_{\Omega}|\bigtriangledown_v u_k|^2 d\Omega - \frac{1}{2}\int_{\Omega}|\bigtriangledown_x u_k|^2 d\Omega \\
&\geq\ c\int_{\Omega}|u_k|^2 d\Omega,\ c > 0,
\end{aligned}$$

where $\nabla_x u_k = \left(u_{k_{x_1}}, \ldots, u_{k_{x_n}}\right)$. Using definition of $\Gamma(A)$, we have $c\int\limits_{\Omega} u^2 d\Omega \leq 0$.

Then equation (1) implies $\rho(x,v,t) = 0$. Hence uniqueness of the solution of the problem is proven. ■

If $u_0 \in C^3(\partial\Omega)$ and $\partial D \in C^3$, $\partial G \in C^3$ then Problem 2 can be reduced to the following problem [Mikhailov (1978)].

Problem 3 *Determine the pair* (u,ρ) *from the equation*

$$Lu = \rho\,(x,v,t) + F \tag{9}$$

provided that $F \in H^2(\Omega)$, *and the trace of the solution u on the boundary* $\partial\Omega$ *is zero and* ρ *satisfies condition (3).*

Theorem 2 *Suppose that* $F \in H_2(\Omega)$. *Under the assumptions of Theorem 1, there exists a solution* (u,ρ) *of Problem 3 such that* $u \in \Gamma(A) \cap H^1(\Omega)$, $\rho \in L_2(\Omega)$.

Proof. The proof can be established using a similar way to that of Theorem 2.2.2 in [Amirov (2001)]. ■

3 Approximate Solution of the Problem

An approximate solution to Problem 3 which contains a non-stationary kinetic equation with a scattering term can be computed from the following relation

$$u_N = \sum_{i_1,\dots,i_n,j_1,\dots,j_n,k=0}^{N-1} \alpha_{N_{i_1,\dots,i_n,j_1,\dots,j_n,k}} w_{i_1,\dots,i_n,j_1,\dots,j_n,k}\,\eta\,(x)\,\mu\,(v)\,\zeta\,(t), \tag{10}$$

where

$$w_{i_1,\dots,i_n,j_1,\dots,j_n,k} = \left\{x_1^{i_1}\dots x_n^{i_n} v_1^{j_1}\dots v_n^{j_n} t^k\right\}_{i_1,\dots,i_n,j_1,\dots,j_n,k=0}^{\infty}$$

and

$$\left\{x_1^{i_1}\dots x_n^{i_n}\right\}_{i_1,\dots,i_n=0}^{\infty},\ \left\{v_1^{j_1}\dots v_n^{j_n}\right\}_{j_1,\dots,j_n=0}^{\infty},\ \left\{1,t,t^2,\dots\right\}$$

are the complete systems in $L_2(D)$, $L_2(G)$ and $L_2(0,T)$ respectively. In (10), the functions $\eta\,(x)$, $\mu\,(v)$, $\zeta\,(t)$ are selected such that they vanish on the boundary and outside of the corresponding domain. Unknown coefficients

$$\alpha_{N_{i_1,\dots,i_n,j_1,\dots,j_n,k}}, i_1,\dots,i_n, j_1,\dots,j_n,k = 0,1,\dots,N-1$$

in expression (10), are obtained from the following system of linear algebraic equations:

$$\begin{aligned}&\sum_{i_1,\dots,i_n,j_1,\dots,j_n,k=0}^{N-1} \left(A\left(\alpha_{N_{i_1,\dots,i_n,j_1,\dots,j_n,k}} w_{i_1,\dots,i_n,j_1,\dots,j_n,k}\right)\eta\mu\zeta, w_{i'_1,\dots,i'_n,j'_1,\dots,j'_n,k'}\right)_{L_2(\Omega)}\\ &= \left(\mathscr{F}, w_{i'_1,\dots,i'_n,j'_1,\dots,j'_n,k'}\right)_{L_2(\Omega)}, i'_1,\dots,i'_n, j'_1,\dots,j'_n,k' = 0,1,\dots,N-1.\end{aligned} \tag{11}$$

The computations were carried out using a computer program written in MAPLE on a PC with Intel Core 2, 2.00 GHz processor and 1 Gb RAM. We have presented two experiments below which show that obtained approximate solution is in reasonable agreement with the exact solution. Also, as it can be seen from the figures, we obtain more accurate results for the greater values of N.

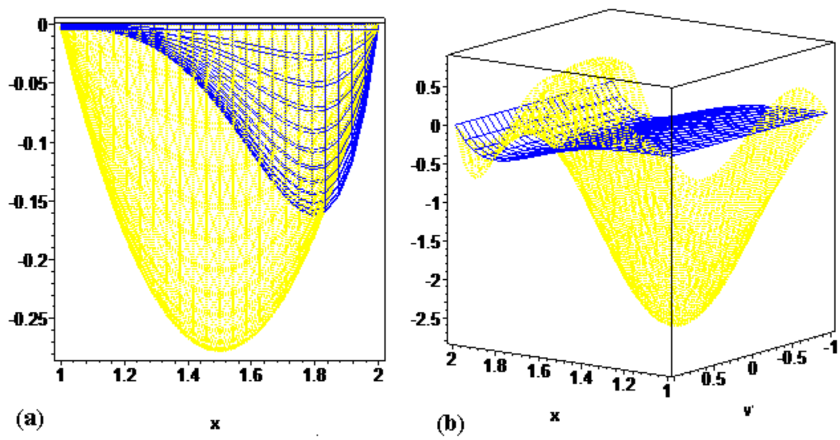

Figure 1: The exact solution and approximate solution of the problem at $N = 1$: (a) for $u(x, v, t)$ (b) for $\rho(x, v, t)$.

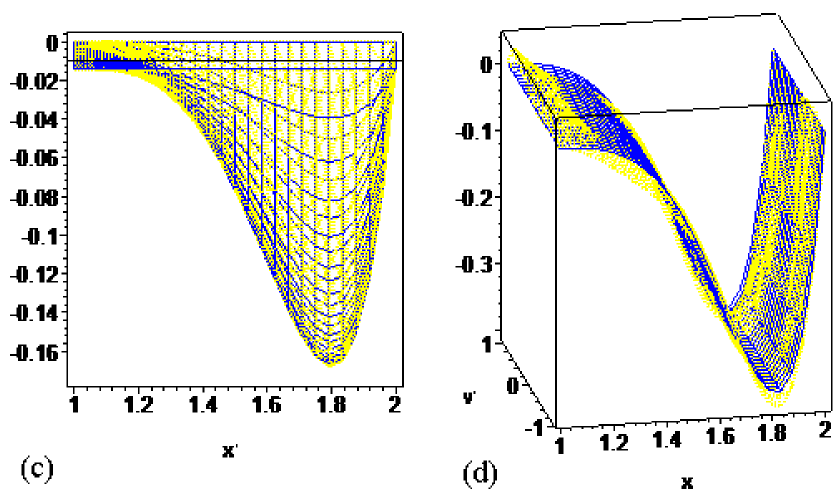

Figure 2: The exact solution and approximate solution of the problem at $N = 4$: (c) for $u(x, v, t)$ (d) for $\rho(x, v, t)$.

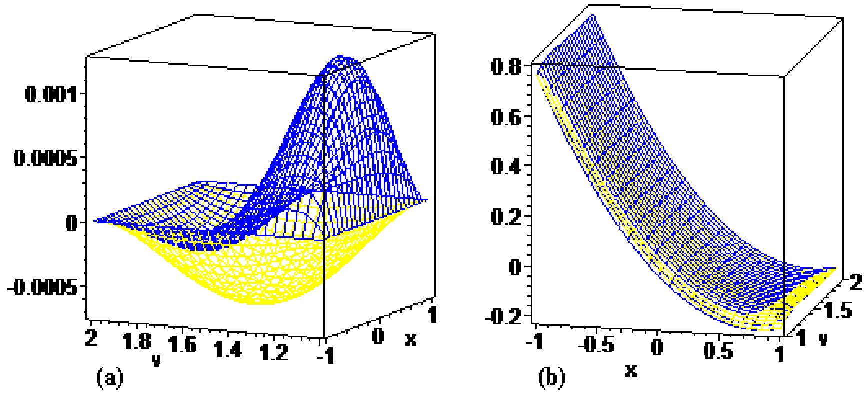

Figure 3: Aproximate solution and exact solution of the problem for $N=1$: (a) for $u(x,v,t)$ (b) for $\rho(x,v,t)$.

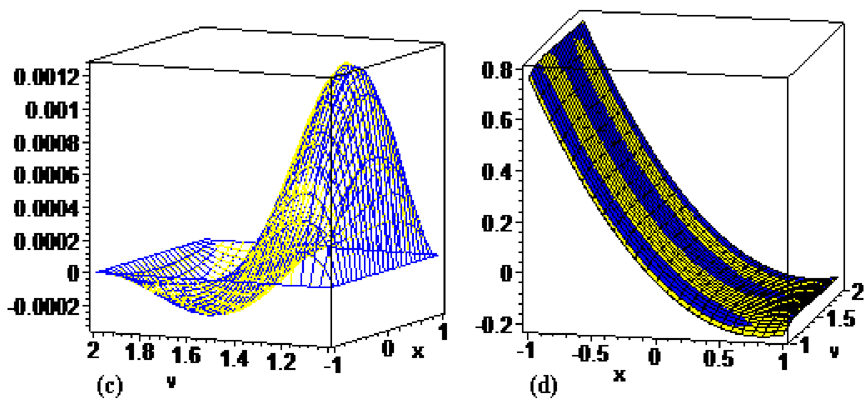

Figure 4: Aproximate solution and exact solution of the problem for $N=4$: (c) for $u(x,v,t)$ (d) for $\rho(x,v,t)$.

Example 1 *In the domain* $\Omega = \{(x,v,t)|\ x\in(1,2),\ v\in(-1,1),\ t\in(0,1)\}$, *according to the given functions*
$F(x,v,t) = -(x-1)(x-2)(1-v^2)x^5 ln(x)t + (x-1)(x-2)(1-v^2)x^5 ln(x)(1-t)$
$+v((x-2)(1-v^2)x^5 ln(x)(1-t)t + (x-1)(1-v^2)x^5 ln(x)(1-t)t$
$+5(x-1)(x-2)(1-v^2)x^4 ln(x)(1-t)t + (x-1)(x-2)(1-v^2)x^4(1-t)t)$
$-2x^8(x-1)(x-2)v ln(x)(1-t)t,\ f_1(x,v,t) = x^3,\ K(x,v,v') = x+v,$

approximate solution of the problem at $N = 1$ *is*

$$U_1 = (\frac{4288}{209}ln(2) - \frac{46040419}{23173920})(1-x)(2-x)(1-v^2)(1-t)t,$$

where the exact solution is

$$u(x,v,t) = (x-1)(x-2)(1-v^2)x^5 ln(x)(1-t)t.$$

A comparison between the approximate solution (dotted, yellow graph) and the exact solution (solid, blue graph) of the problem is presented for $t = 0.1$ *in Figure 1 (a)-(b) and Figure 2 (c)-(d).*

Example 2 *In the domain* $\Omega = \{(x,v,t)|\ x \in (-1,1),\ v \in (1,2),\ t \in (0,1)\}$, *according to the given functions*
$F(x,v,t) = \frac{-1}{48}(xvt(72(-t+x^2t+1-tvx-vx-x^2)-48v^3(1-t)+96t(x+v)) +x^2v^3e^{-v}(96+48v^2-144v)+tv^3x^2e^{-v}(-192+528v-336v^2)-48x^2v+72x^2v^2 +96xt(-1+x^2+t-x^2t-v^2)+48v^3xte^{-v}(-v+vt-5v^2+5v^2t+2v^3-2v^3t+5 -5x^2+5x^2v-x^2v^2+v^3x-5tvx+5tv^2x-5t+5x^2t-5x^2tv+x^2tv^2-tv^3x))/v^3$, $f_1(x,v,t) = x-v$ *and* $K(x,v,v') = \frac{1}{6}x^2v^3$ *computed approximate solutions* (U_1,ρ_1), (U_4,ρ_4) *at* $t = 0.2$ *and the exact solution of the problem are represented in Figure 3 (a)-(b) and Figure 4 (c)-(d) respectively, where the exact solution pair is*

$$u = (1-x^2)(1-v)(2-v)(\frac{-1}{2v^2}+e^{-v})(t-t^2),$$

$$\begin{aligned}\rho \ = \ & -1/48(24v(2-3v-x^2v^2+v^2)+v^3xt(-144t+144+xt+x^3) \\ & +48tv^3e^{-v}(4-v^2+7v^2)+3552x^2te^{-2}v^3(1-t-x^2+x^2t) \\ & +47x^2tv^3-x^4t^2v^3+tv^4e^{-v}(-528+240t+48tv-240tv) \\ & +1304x^2te^{-1}v^3(t+x^2-x^2t-1)+24tv(3v-2v^2-4t+3tv) \\ & +48e^{-v}v^3(-2+3v-v^2))/v^3.\end{aligned}$$

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