 CMES, vol.65, no.1, pp.95-106, 2010

Application of Residual Correction Method on Error Analysis of Numerical Solution on the non-Fourier Fin Problem

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Abstract: Up to now, solving some nonlinear differential equations is still a challenge to many scholars, by either numerical or theoretical methods. In this paper, the method of the maximum principle applied on differential equations incorporating the Residual Correction Method is brought up and utilized to obtain the upper and lower approximate solutions of nonlinear heat transfer problem of the non-Fourier fin. Under the fundamental of the maximum principle, the monotonic residual relations of the partial differential governing equation are established first. Then, the finite difference method is applied to discretize the equation, converting the differential equation into the mathematical programming problem. Finally, based on the Residual Correction Method, the optimal solution under the constraints of inequalities can be obtained. The methodology of incorporating the Residual Correction Method into the nonlinear iterative procedure of the finite difference will make it easier and faster to obtain upper and lower approximate solutions and can save the computing time, reduce the storage of memory and avoid unnecessary repeated testing.

Keywords: Residual Correction Method, upper and lower approximate solutions, non-Fourier fin, finite difference method, mathematical programming

Nomenclature

T	temperature field of the fin
t	time
x	space coordinate
T_{in}	initial temperature of the fin
T_b	base temperature of the fin
$\bar{T}_b$	mean temperature of the fin base

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τ	relaxation time
ρC	volumetric heat capacity
h	heat transfer coefficient
h_0	referenced heat transfer coefficient
b	thickness of the fin
L	length of the fin
A	amplitude of the input temperature
$\hat{\omega}$	frequency of base temperature oscillation
θ	dimensionless temperature
ξ	dimensionless time
η	dimensionless space coordinate
ω	dimensionless frequency of base temperature oscillation
$\sim$	approximate solution
$\cup, \cap$	lower and upper approximate solution

1 Introduction

1.1 Residual Correction Method

Whether in engineering application or technical research, finding solutions under given initial and boundary conditions is to be concerned. However, it is difficult to find exact solutions for complex geometric shapes or with nonlinear equations and boundary conditions. Given the difficulty in obtaining analytic solutions of such nonlinear problems, it is only possible to find out their approximate solutions with some numerical methods. Residual Correction Method is one of them.

Early in 1967, Protter has brought up the concept of maximum principle which explains the relationship between the solution and the residual of the equation. However, this kind of method must find the optimal solution of a mathematical programming problem. The calculation load is quite heavy and complicated. Therefore, to probe into possibilities of solving some nonlinear problems, Chang and Lee (2004) adopted genetic algorithms, Cheng (2009) and Wang (2010) used cubic spline residual correction method.

In this work, a methodology called the Fast Residual Correction Method based on maximum principles in differential equations is utilized to determine the upper and lower approximate solutions of the non-Fourier fin. This method with error-analysis characteristic can correct the defects resulting from its increasing number of grids or approximate functions when using traditional numerical methods.

1.2 The non-Fourier Fin

In many industrial applications such as electronic components and solar collectors, the analysis of fin performance is quite important. During the past decades, the fin problems with the periodic thermal loading have been investigated by several researchers. Most of them solved the problem in Fourier domain, just a few having solved the problem in non-Fourier domain. Lin (1998) concluded that the non-Fourier effect should be considered in the thermal analysis of a fin when the period of the input temperature frequency is lower than the thermal relaxation time. Yang (2005) proposed a sequential method on the inverse non-Fourier fin problems to determine the base temperature of the fin and estimate the periodic boundary conditions. Huang and Wu (2006) applied an iterative regularization method, CGM, for the solution of the inverse non-Fourier fin problems in estimating the base temperature. It can be shown that the inverse solutions obtained by CGM are always better than the algorithm used in Yang's study (2005) and the reliable inverse solutions can still be obtained when large measurement errors were considered. Further in 2008, Yang, Chien and Chen proposed a double decomposition method for solving the periodic base temperature in convective longitudinal fins. In this article, effects on temperature distribution of several variables, such as the relaxation time, frequency and amplitude of the base temperature oscillation, time, and grid number will be investigated.

2 Maximum Principles for Differential Equations

In order to obtain the upper and lower approximate solutions of the nonlinear differential equations, the concept of maximum principle is utilized to establish the residual of differential equations. At First, considering the nonlinear equation as below:

$$R_{\tilde{\theta}}(x) = F(x, \tilde{u}, \tilde{u}_x, \tilde{u}_{xx}) - f(x) \text{ in } D \tag{1}$$

Boundary conditions satisfy

$$R_{\tilde{\theta}}(x) = g(x) - \tilde{\theta}(x) \text{ on } \partial D \tag{2}$$

Here, $R_{\tilde{\theta}}(x)$ is known as the residual of the differential equation. On assumption that the approximate solutions have definition in the calculation domain and are continuous till second derivatives, if

$$\frac{\partial R}{\partial \theta} \leq 0 \text{ in } D \tag{3}$$

Then, when the following equation holds:

$$R_{\check{\theta}}(x) \geq R_{\theta}(x) = 0 \geq R_{\hat{\theta}}(x) \text{ on } D \cup \partial D \tag{4}$$

The approximate solutions will have the following relation with the exact solution:

$$\check{\theta}(x) \leq \theta(x) \leq \hat{\theta}(x) \text{ on } D \cup \partial D \tag{5}$$

where $\check{\theta}(x)$ and $\hat{\theta}(x)$ are called the lower and upper solutions of the exact solution $\theta(x)$, while any differential equation with such relations is considered monotonic in solutions.

3 Residual Correction Steps

Let's take a multi-dimension partial equation as the example to briefly describe how this method works. Discretizing a partial differential equation and transfer a mathematical programming problem into an iterative equation with residual correction:

$$R^n_{r,i,j,k}(t,x,y,z) = -(L[\theta]^{n+1}_{r,i,j,k} + N[\theta]^n_{r,i,j,k}) + f_{r,i,j,k} \tag{6}$$

Here, L is the linear operator and N is the nonlinear operator, the superscript n is the iterative times, and the subscript r,i,j,k is the serial number of the grid points after discretizing.

$$\Delta R_{r,i,j,k}(t,x,y,z) = \sum_{s=1}^{\infty}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{l=1}^{\infty} \frac{\partial R_{r,i,j,k}(t,x,y,z)}{\partial t^s \partial x^p \partial y^q \partial z^r} \frac{(t-t_r)^s (x-x_i)^p (y-y_j)^q (z-z_k)^l}{s!p!q!r!} \tag{7}$$

$$\begin{aligned} &(t_r - \Delta t) \leq t \leq (t_r), \quad (x_i - \Delta x) \leq x \leq (x_i + \Delta x) \\ &(y_j - \Delta y) \leq y \leq (y_j + \Delta y), \quad (z_k - \Delta z) \leq z \leq (z_k + \Delta z) \end{aligned} \tag{8}$$

Let n=0, and assume that the residual correction value at each grid point $R^n_{r,i,j,k} = 0$. Use Eq. (7), find the new $\theta^{n+1}_{i,j,k}$ at every grid point and substitute it into next iterative step to do the residual correction again.
For lower approximate solutions,

$$R^{n+1}_{r,i,j,k} = R^n_{r,i,j,k} - Min(\Delta R^n_{r,i,j,k}) \tag{9}$$

For upper approximate solutions

$$R^{n+1}_{r,i,j,k} = R^n_{r,i,j,k} - Max(\Delta R^n_{r,i,j,k}) \tag{10}$$

Advance to the next n value and repeat step 2 until the results converge. The convergence criterion adopted in this article is the relative error convergence as expressed below:

$$E_\theta = \left| \frac{\tilde{\theta}_i^{n+1} - \tilde{\theta}_i^n}{\tilde{\theta}_i^n} \right| \leq \varepsilon,\ i = 0, 1, ..., N_i \tag{11}$$

Where ε is the tolerant error.

4 Problem Formulation for Lower and Upper Solutions of Non-Fourier Fin

In order to explicitly describe this methodology, we consider an one-dimension insulated non-Fourier fin with uniform thickness b and length L (Fig. 1).The value of ratio is small.

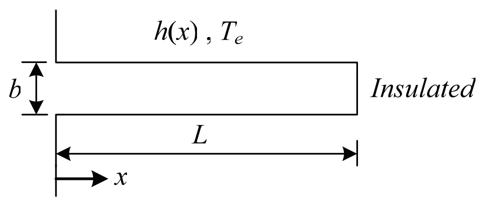

Figure 1: The fin configuration

Its governing equation can be formulated as

$$\tau\rho C \frac{\partial^2 T}{\partial t^2} + \rho C \frac{\partial T}{\partial t} + 2\tau \frac{h}{b} \frac{\partial}{\partial t} (T - T_e) = k \frac{\partial^2 T}{\partial x^2} - 2 \frac{h}{b} (T - T_e),\ t > 0,\ 0 < x < L \tag{12}$$

subject to the initial conditions and boundary conditions as below:

$$T(x,0) = T_{in}\ ,\ 0 \leq x \leq L \tag{13}$$

$$\frac{\partial T(x,t)}{\partial t} = 0 \text{ at } t = 0 \text{ and } 0 \leq x \leq L \tag{14}$$

$$T(0,t) = T_b = \bar{T}_b + A\cos(\hat{\omega} t)(\bar{T}_b - T_{in}) \text{ at } x = 0 \text{ and } t > 0 \tag{15}$$

$$\frac{\partial T(x,t)}{\partial x} = 0 \text{ at } x = L \text{ and } t > 0 \tag{16}$$

The relaxation time τ is defined as

$$\tau = \frac{\alpha}{\upsilon^2} \tag{17}$$

where is the thermal diffusivity and is the propagation speed of the thermal wave. The heat transfer coefficient which varies with x takes the form

$$h(x) = h_0 H\left(\frac{x}{L}\right) \tag{18}$$

where is the referenced heat transfer coefficient that is defined as $h_0 = bk/2L^2$. For the convenience of numerical analysis, introducing the dimensionless parameters as follows:

$$\theta = \frac{T - T_{in}}{T_b - T_{in}}\ ,\ \theta_e = \frac{T_e - T_{in}}{T_b - T_{in}}\ ,\ \eta = \frac{x}{L}\ ,\ \xi = \frac{\alpha t}{L^2}\ ,\ \beta = \frac{\alpha\tau}{L^2}\ ,\ \omega = \frac{\hat{\omega}L}{\alpha} \tag{19}$$

Here, represents the dimensionless temperature and is the dimensionless relaxation time. Eq. (12)–(16) can be rewritten into dimensionless form as

$$\beta\frac{\partial^2\theta}{\partial\xi^2} + (1+\beta H)\frac{\partial\theta}{\partial\xi} = \frac{\partial^2\theta}{\partial\eta^2} - H\theta + H\theta_e\ ,\ 0 < \eta < 1\ ,\ \xi = 0 \tag{20}$$

$$\theta(\eta,0) = 0\ ,\ 0 < \eta < 1\ ,\ \xi = 0 \tag{21}$$

$$\frac{\partial\theta(\eta,0)}{\partial\xi} = 0,\ 0 < \eta < 1,\ \xi = 0 \tag{22}$$

$$\theta(0,\xi) = 1 + A\cos(\omega\xi)\ ,\ \eta = 0\ ,\ \xi > 0 \tag{23}$$

$$\frac{\partial\theta(1,\xi)}{\partial\eta} = 0,\ \eta = 1,\ \xi > 0 \tag{24}$$

In order to analyze the error of approximate solutions, the residual correction relation is established first.

$$R = \theta_{\eta\eta} - H\theta + H\theta_e - \beta\theta_{\xi\xi} - (1+\beta H)\theta_\xi \tag{25}$$

Before continuing our calculation steps, applying the maximum principle which is mentioned before to judge if this equation has the monotonic characteristic.

$$\frac{\partial R}{\partial\theta} = \frac{\partial}{\partial\theta}[\theta_{\eta\eta} - H\theta + H\theta_e - \beta\theta_{\xi\xi} - (1+\beta H)\theta_\xi] = -H \tag{26}$$

If the maximum principle is satisfied, the required condition is:

$$\frac{\partial R}{\partial\theta} \le 0 \tag{27}$$

Because the value of H in Eq. (26) is positive, Eq. (27) holds. Thus, the monotonicity exists.

Then employing residual correction method with finite-difference method to discretize Eq. (20), and further to add one residual correction value at every calculation grid point, the iterative relation can be formed as:

$$\beta \frac{\theta_i^{n+1} - 2\theta_i^n + \theta_i^{n-1}}{\Delta\xi^2} + (1+\beta H)\frac{\theta_i^{n+1} - \theta_i^n}{\Delta\xi} = \frac{\theta_{i+1}^{n+1} - 2\theta_i^{n+1} + \theta_{i-1}^{n+1}}{\Delta\eta^2} - H\left(\theta_i^{n+1} - \theta_e\right) - \begin{matrix} Min \\ Max \end{matrix}\left(0, -R_\xi^n \Delta\xi\right) - \begin{matrix} Min \\ Max \end{matrix}\left(R_\eta^n \Delta\eta, -R_\eta^n \Delta\eta\right) \quad (28)$$

or

$$\left[\frac{1}{\Delta\eta^2}\right]\theta_{i-1,j}^{n+1} + \left[-\frac{2}{\Delta\eta^2} - H - \frac{\beta}{\Delta\xi^2} - \frac{1+\beta H}{\Delta\xi}\right]\theta_{i,j}^{n+1} + \left[\frac{1}{\Delta\eta^2}\right]\theta_{i+1,j}^{n+1} = -\frac{1+\beta H}{\Delta\xi}\theta_{i,j}^n - H\theta_e + \beta\frac{-2\theta_{i,j}^n + \theta_{i,j}^{n-1}}{\Delta\xi^2} - \begin{matrix} Min \\ Max \end{matrix}\left(0, -R_\xi^n \Delta\xi\right) - \begin{matrix} Min \\ Max \end{matrix}\left(R_\eta^n \Delta\eta, -R_\eta^n \Delta\eta\right) \quad (29)$$

The selected residual correction value at each grid point is either the minimum or the maximum in this grid to ensure $R_\theta(x,t) \geq 0$ or $R_\theta(x,t) \leq 0$. In Eq. (25), space term and time term exist simultaneously, therefore, take derivative toward η and ξ separately to get the slope of residual on every grid point:

$$R_\eta = \beta\theta_{\xi\xi\eta} + (1+\beta H)\theta_{\xi\eta} + H\theta_\mu \quad (30)$$

$$R_\xi = (1+\beta H)\theta_{\xi\xi} + H\theta_\xi - \theta_{\xi\eta\eta} \quad (31)$$

Next, making use of residual correction method together with finite-difference method to discretize the initial and boundary conditions Eq. (21) $\sim$ Eq. (24) as follows:

$$\theta_i^1 = 0 \quad (32)$$

$$\frac{\theta_i^1 - \theta_i^0}{\Delta\xi} = 0 \rightarrow \theta_i^0 = \theta_i^1 \quad (33)$$

$$\theta_0^n = 1 + A\cos(\omega \cdot i\Delta\xi) \quad (34)$$

$$\frac{\theta_{Ni}^{n}-\theta_{Ni-1}^{n}}{\Delta\eta}=0\rightarrow\theta_{Ni}^{n}=\theta_{Ni-1}^{n} \quad (35)$$

Finally, calculate iteratively to get numerical solutions while they meet the convergence criterion. Here, set the relative tolerant error ε as 10^{-6}. In the iterative process, the residual corrections are added together and the calculation processes are done repeatedly until $\tilde{\theta}^{n}$ and $\tilde{\theta}^{n+1}$satisfy the convergence criterion to find the upper and lower bound of the approximate solution.

5 Results and Discussions

The upper and lower approximate solutions obtained through residual correction method under various variables of the non-Fourier fin are shown as in Fig. 2 $\sim$ Fig. 6, which show clearly that the upper approximate solutions are always distributed on the upper side of the lower approximate solutions in the whole calculation domain, that is, no matter how the parameters change, the mean approximate solutions always locate between the upper and lower approximate solutions and it is clear that the upper and lower approximate solutions all gradually approximate to a certain value and always satisfy the requirement for monotonic residual relation.

Fig. 2 shows dimensionless temperature distributions for various values β at $\xi=0.5$. It can be found that the thermal wave travels a shorter distance for a larger value of β. The phenomena can be explained by observing the definition of the relaxation time in Eq. (17) which states that a larger value of relaxation time implies a slower propagation speed of the thermal wave. Comparing the mean approximate solution $\tilde{\theta}$ with the numerical solution found by Huang and Wu (2006), the relative error between both is only within 10^{-2}.

Fig. 3 shows the dimensionless temperature distributions for various values of ω when the relaxation time $\beta=1$. The result shows that the speed of the thermal wave propagation is independent on the oscillation frequency of the periodic input. Moreover, the tendency of the results in Fig. 3 is nearly the same with the solution shown in the study of Huang and Wu (2006).

The dimensionless temperature distributions for various values of ξ are shown in Fig. 4. According to Fig. 4, at any dimensionless time ξ, the maximum error range occurs at where the sharp decrease is located. Near the end of the fin, the error range between the upper and lower approximate solutions becomes smaller gradually. While getting close to the end tip, the error range almost approaches to zero. It can be further seen from Fig. 4 that the greatest temperature variation at rapid inclination becomes larger as ξ increases. It's because when the time increases, the error of time term increases too.

The number of the grid points is crucial in the numerical analysis. It affects the ac-

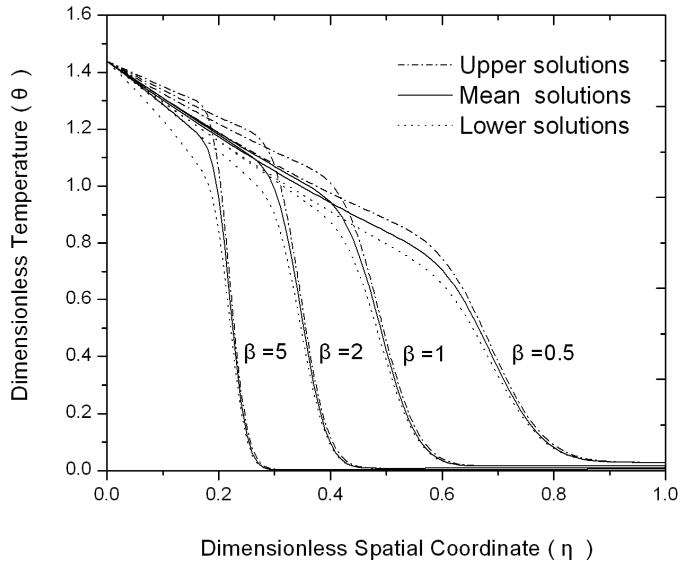

Figure 2: Upper and lower solutions of temperature distributions for various values of β ($\xi = 0.5$ and $\Delta\eta$ = 0.001)

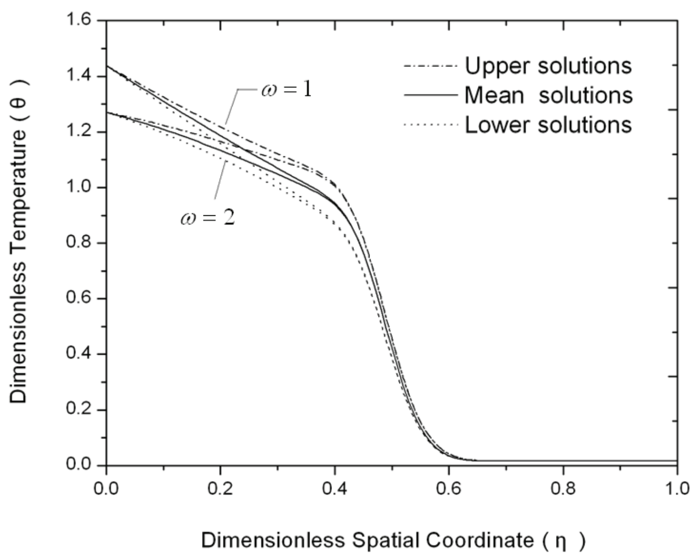

Figure 3: Upper and lower solutions of temperature distributions for various values of ω ($\xi = 0.5, \Delta\eta = 0.001$ and $\beta = 1$)

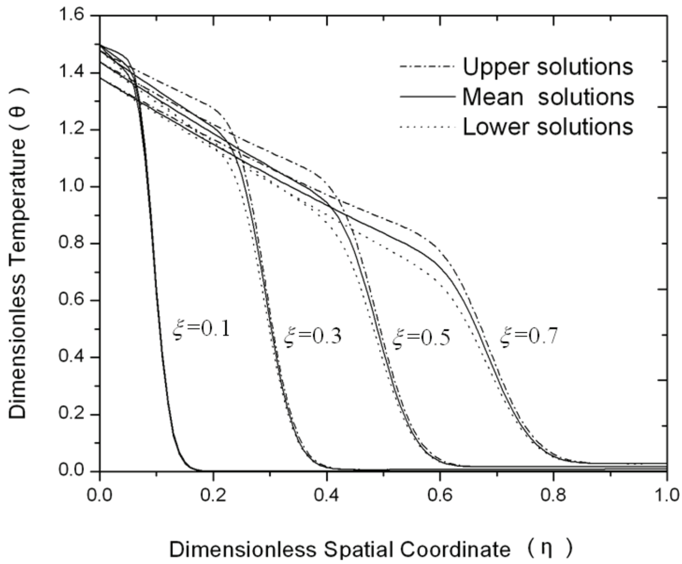

Figure 4: Upper and lower solutions of temperature distributions for various values of ξ ($\Delta\eta$ = 0.001 and $\beta = 1$)

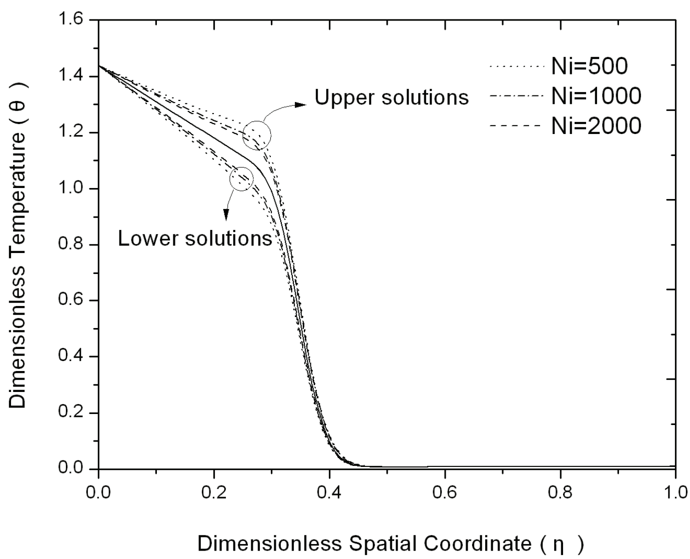

Figure 5: Upper and lower solutions of temperature distributions for various values of Ni (ξ = 0.5 and $\beta = 2$)

curacy and efficiency a lot. The dimensionless temperature distributions for various number of grid points are illustrated in Fig. 5. As expected, it can be observed that when the more grid points it has, the higher accuracy it will reach. However, the difference between Ni=500 and Ni=2000 isn't distinct, and the mean approximate solutions of them nearly overlap. Take the calculation efficiency for consideration, fewer grid points could be adopted during the calculation process.

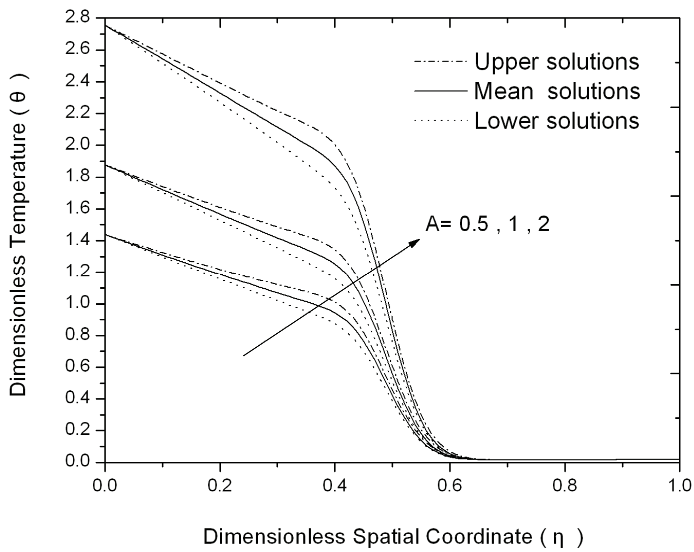

Figure 6: Upper and lower solutions of temperature distributions for various values of A ($\xi = 0.5$, $\Delta\eta = 0.001$ and $\beta = 1$)

Fig. 6 indicates the influence of the amplitude A of the input temperature on the temperature distributions. The temperature at the fin base differs with A; however, the monoticity still holds in the whole calculation domain.

6 Conclusions

By introducing the Fast Residual Correction Method, the upper and lower approximate solutions can be obtained successfully and the monotonic relation between them holds regardless of the various variables. The results of temperature distributions of the non-Fourier fin in this work coincide with the studies which have been published, so the methodology can be verified successfully. The Fast Residual Correction can solve the inequality constraint mathematical programming problems in less time, compared to other numerical methods, and it can save the computing time, reduce the storage of memory and avoid unnecessary repeated testing. To sum up, the Fast Residual Correction Method is a great numerical method which helps to solve the various complicated nonlinear problems with preciseness and effectiveness.

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