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Effective Length of Beam on Elastic Foundation Under a Moving Load

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Abstract: Vibrations induced by high-velocity train passing through environmental sensitive area are important issues to railway authorities. Research has been granted to resolve the environmental impact generated from vehicle-track interaction to the neighbor buildings. Models were established to simulate the vehicle-track interaction system which is the source of vibrations. Among them, Euler beam with finite length on elastic foundation subjected to moving loads is most commonly used to simulate the continuous track on site. The reason of modeling infinite length track with finite length model is to make solving possible by numerical analysis. However, the accuracy of analysis depends on the length of the model. If the length of the model is not long enough, the solution may be interfered by reflective waves from the boundary. Effective length, which is the shortest length that makes the result stable, may be determined with many parameters such as velocity, damping ratio, material stiffness, etc. Determination of "effective length" is the key for reliable results. Lacking of universal and efficient approach, trial-and-error was widely adopted to determine the effective length of models in the past research. A neat and efficient guideline was proposed based on the analytical solution of the model of Euler beam on elastic foundation subjected to a moving load with constant velocity. The accuracy and efficiency were discussed to justify the proposed approach.

Keywords: reflection, wave, railway track, effective length

1 Introduction

The vibration impact induced by railway trains on environment has become an important issue along railway routes in recent decades. The high-velocity railway in Taiwan, which passes through science industrial park, has gained attentions and initiated research projects to analyze impacts of track vibrations on yield of wafers. Vehicle-track interaction models have been noticed and introduced in Taiwan to

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assess the deformations, accelerations, wave propagation characteristics of track, viaduct, and soil [Chen and Huang (2000); Wu and Shih (2000); Zhai (2002); Andersen, Nielsen and Iwankiewicz (2002); Chen and Huang (2003); Hussein and Hunt (2004)].

In railway dynamic analysis, it is common to model the railway track as a beam on elastic foundation. Several beam models were tried depending on various boundary conditions. Kenny (1954) pioneered the study on two types of elastic foundation models, and presented analytical solutions for both models. Sobczyk (1970) established the model considering inhomogeneous elastic foundation. Kerr (1972) further improved the beam model by adding axial forces. Frýba (1976) conducted comprehensive studies about simply supported beam subjected to a random force moving with constant velocity on it. Iwankiewicz and Śniady (1984) studied the behavior of a simply supported beam subjected to a stream of deterministic point forces moving at constant velocity with random interval time.

In 1990's, Ricciardy (1994) started to focus on random vibrations of elastic supported beam subjected to external forces. Zibdeh (1995) studied vibrations caused by random loads moving on an axial loaded simply-supported Euler-Bernoulli beam. Zibdeh and Rackwitz (1995) also studied the effect of loads moving at random velocities. Chen and Sheu (1996) proposed dynamic stiffness method to model railway. Cui and Chew (2000) established a continuous model for floating slab track system.

Generally, models with simply supported Euler beam on elastic foundation were most commonly used in track analysis. However, it is found that the accuracy of numerical analysis depends on the length of the model. Numerical analysis would converge only when the rail model is long enough. Interference by the reflection of propagation wave from the boundary was suspected to be responsible for convergence difficulties if the model is not long enough.

Although the aforementioned studies revealed several methods of modeling railway system from many aspects, there were no vigorous criteria for determining the minimum model length to ensure analysis convergence. It is desirable to find a convenient and universal way of deciding the length of track model for numerical computations. The phenomenon of reflective wave was reviewed in this study. The influences of reflective waves were examined carefully. Finally, a simple index was proposed to determine the required model length for reliable numerical solutions of mechanical problems in railway track engineering.

2 Methods

Continuous infinite beam is usually adopted in analytical models of railway studies for the advantage of derivability. The assumption is considered realistic in recent years as continuous welded rail becomes more and more popular. It is common to depict track responses by solving analytical equations of the rail models with a finite beam resting on various types of supports. However, researchers acknowledge that deformation waves reflected from boundaries of the finite-length beam may interfere the analysis results [Wang, So and Chan (2006)]. Mead (1994) further found that reflective waves might occur in different types depending on the boundaries as shown in Fig. 1. and Fig. 2. The relationship between wave propagation and reflective wave was given as Eq. (1), where A_1, A_3, A_4 represent amplitudes of reflective wave, reflective energy, and incident wave respectively.

$$w(x) = A_1 e^{kx} + A_3 e^{kx} + A_4 e^{-kx} \tag{1}$$

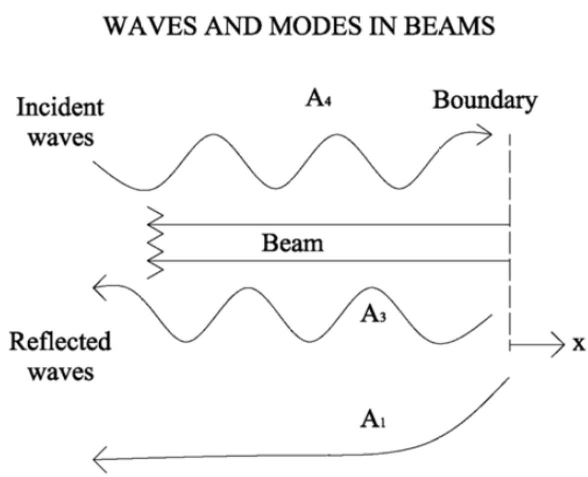

Figure 1: Wave propagating on a finite beam with boundary [Mead (1994)]

As mentioned above, the behavior of a finite beam might depend on boundary conditions. The annoyance due to reflective waves from boundaries could be avoided if the beam is long enough to allow deformation waves decay to negligible level. In other words, the analytical steady-state solutions of an infinite beam can be approached with the numerical analysis as long as the finite model is long enough.

Among numerous analytical methods, a simply supported beam resting on elastic foundation is often used to model the railway track. In 1954, Kenny (1954) studied

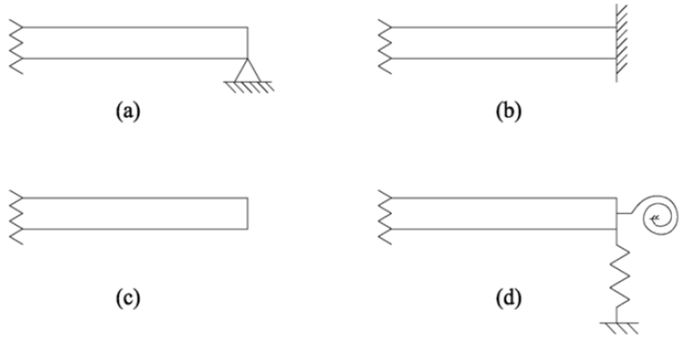

Figure 2: Four different types of common boundary of a beam [Mead (1994)]

the problem of an infinite Euler-Bernoulli beam on elastic foundation subjected to a moving load with constant velocity along the beam, shown as Fig. 3., and proposed the steady-state solution, shown as Eq. (2), by fixing the origin of the coordinate to the moving load. Hence, in the solution, positive x represents the beam beyond the loading and negative value means behind the loading position.

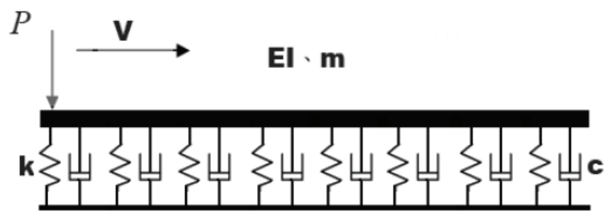

Figure 3: Infinite beam on elastic foundation under a moving load with constant velocity

$x < 0$

$$y = \frac{P\lambda}{2k}\left[\frac{\eta e^{\eta\lambda x}}{\eta^4 + (\eta\theta)^2 + \frac{1}{2}\left(\frac{\theta\beta}{\eta}\right)^2}\right]$$
$$\left\{A\sin\left[\left(2\theta^2 + \eta^2 - \frac{2\theta\beta}{\eta}\right)^{1/2}\lambda x\right] + \cos\left[\left(2\theta^2 + \eta^2 - \frac{2\theta\beta}{\eta}\right)^{1/2}\lambda x\right]\right\}$$

$x > 0$

$$y = \frac{P\lambda}{2k}\left[\frac{\eta e^{-\eta\lambda x}}{\eta^4 + (\eta\theta)^2 + \frac{1}{2}\left(\frac{\theta\beta}{\eta}\right)^2}\right]$$

$$\left\{B\sin\left[\left(2\theta^2 + \eta^2 + \frac{2\theta\beta}{\eta}\right)^{1/2}\lambda x\right] + \cos\left[\left(2\theta^2 + \eta^2 + \frac{2\theta\beta}{\eta}\right)^{1/2}\lambda x\right]\right\} \quad (2)$$

Where

E: modulus of elasticity of beam,

I: moment of inertia of beam,

v: moving velocity of applied load,

m: mass per unit length of beam,

k, c: spring constant, damping coefficient per unit length of beam,

β: damping ratio to critical damping,

θ: velocity ratio to critical velocity,

$\lambda = \sqrt[4]{\frac{k}{4EI}}$,

P: loading force,

critical damping: $2\sqrt{km}$, critical velocity: $\sqrt[4]{\frac{4kEI}{m^2}}$,

η: the positive real root of $\eta^6 + 2\theta\eta^4 + (\theta^4 - 1)\eta^2 - \theta^2\beta^2 = 0$, shown as Fig. 4.

$$A = \frac{-\left(\frac{\theta\beta}{\eta} + \eta^2\right)}{\eta\left(2\theta^2 + \eta^2 - \frac{2\theta\beta}{\eta}\right)^{1/2}}, \quad B = \frac{-\left(\frac{\theta\beta}{\eta} - \eta^2\right)}{\eta\left(2\theta^2 + \eta^2 + \frac{2\theta\beta}{\eta}\right)^{1/2}}$$

Eq. (2) can be reorganized to Eq. (3)

$x < 0$

$$y = \frac{P\lambda}{2k}\left[\frac{\eta e^{\eta\lambda x}}{\eta^4 + (\eta\theta)^2 + \frac{1}{2}\left(\frac{\theta\beta}{\eta}\right)^2}\right]\sqrt{A^2+1}$$

$$\left\{\frac{A}{\sqrt{A^2+1}}\sin\left[\left(2\theta^2 + \eta^2 - \frac{2\theta\beta}{\eta}\right)^{1/2}\lambda x\right] + \frac{1}{\sqrt{A^2+1}}\cos\left[\left(2\theta^2 + \eta^2 - \frac{2\theta\beta}{\eta}\right)^{1/2}\lambda x\right]\right\}$$

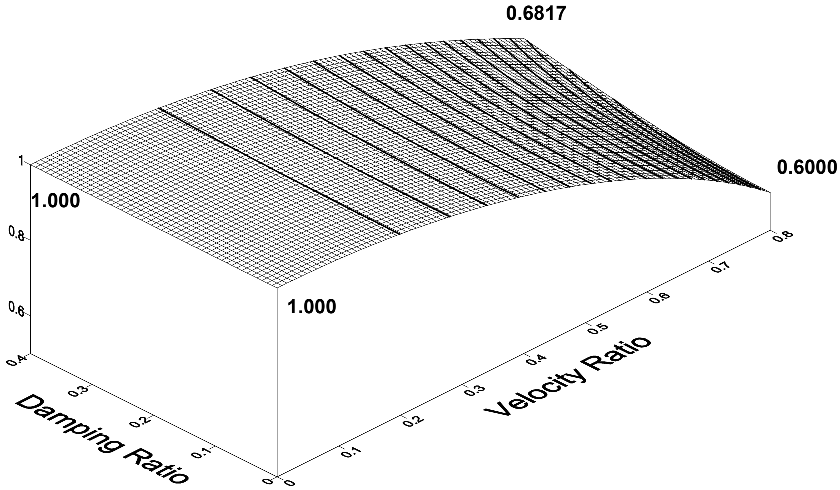

Figure 4: Solutions of the parameter η in Eq. (2)

$x > 0$

$$y = \frac{P\lambda}{2k}\left[\frac{\eta e^{-\eta\lambda x}}{\eta^4 + (\eta\theta)^2 + \frac{1}{2}\left(\frac{\theta\beta}{\eta}\right)^2}\right]\sqrt{B^2+1}$$

$$\left\{\frac{B}{\sqrt{B^2+1}}\sin\left[\left(2\theta^2+\eta^2+\frac{2\theta\beta}{\eta}\right)^{1/2}\lambda x\right] + \frac{1}{\sqrt{B^2+1}}\cos\left[\left(2\theta^2+\eta^2+\frac{2\theta\beta}{\eta}\right)^{1/2}\lambda x\right]\right\} \quad (3)$$

Eq. (3) can be derived into Eq. (4). It should be noted that C and D in Eq. (4) would be constants if the material properties and moving velocity of the loading were constants.

$x < 0$

$$y = Ce^{\eta\lambda x}\sin\left[\left(2\theta^2+\eta^2-\frac{2\theta\beta}{\eta}\right)^{1/2}\lambda x + \cos^{-1}\left(\frac{A}{\sqrt{A^2+1}}\right)\right]$$

$x > 0$

$$y = De^{-\eta\lambda x}\sin\left[\left(2\theta^2+\eta^2+\frac{2\theta\beta}{\eta}\right)^{1/2}\lambda x + \cos^{-1}\left(\frac{B}{\sqrt{B^2+1}}\right)\right] \quad (4)$$

Where

$$C = \frac{P\lambda}{2k}\left[\frac{\eta\sqrt{A^2+1}}{\eta^4+(\eta\theta)^2+\frac{1}{2}\left(\frac{\theta\beta}{\eta}\right)^2}\right], \quad D = \frac{P\lambda}{2k}\left[\frac{\eta\sqrt{B^2+1}}{\eta^4+(\eta\theta)^2+\frac{1}{2}\left(\frac{\theta\beta}{\eta}\right)^2}\right]$$

3 Determination of the effective length for an continuously supported infinite beam

In order to investigate the reflective waves on a finite-length beam, the transient-state behavior of a continuously supported beam carrying a moving load was calculated by solving the vehicle-railway coupled equations via numerical approach (Kuo, Huang and Chen 2008). The beam ends were set to be hinge and roller to achieve computation convergence. The deflected shapes while the moving load passing the beam center, shown as Fig. 5., differ a lot with various beam lengths. The cases of L=10 m and L=60 m experienced severe displacements along the entire beam. Whereas, the deformation wave of L=100 m case has a very different pattern illustrating wave decay and little displacements in the vicinity of beam ends. This comparison demonstrates that the analyzed beam behavior depends tremendously on the beam length.

Fig. 6. shows the time history of vibration at the central point of the beam with various lengths. Not only the vibration frequencies but also the peak deflections vary with length of the analyzed beam models. It again justified the effect of beam length on analysis results.

Referring to Eq. (4), it can be found that the beam displacement is calculated by the product of a sine function and the "decay factor", $e^{-\eta\lambda x}$. Since sine functions vary in the range between -1 and 1, the beam displacements decay along the beam away from the loading point with the "decay factor" as x increases. The decay factor is only related to the traveling distance of deformation waves as long as the material properties were kept constant. Hence, if the length of the beam is long enough, the annoyance on beam vibration shapes due to waves reflected from boundaries could be neglected.

Based on the thinking, the disturbance of reflective deformation waves can be observed with the deflection at a distance x from the loading position. The authors proposed that the "effective length" of an infinite beam resting on continuous supports can be determined by choosing appropriate x to have $e^{-\eta\lambda x}$ smaller than the prescribed accuracy tolerance, "*error*".

$$e^{-\eta\lambda x} \leq error \tag{5}$$

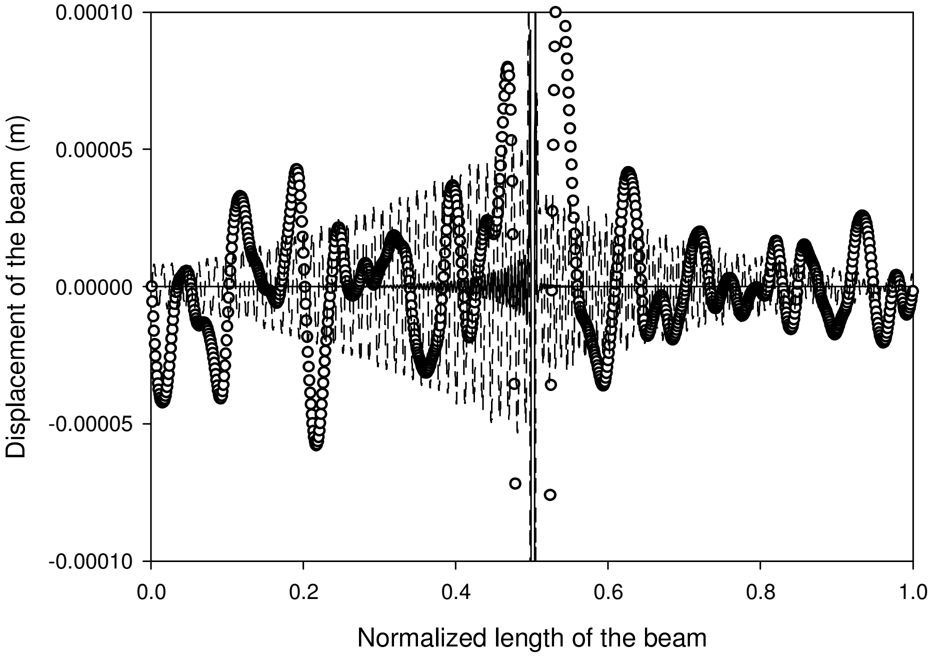

Figure 5: Vibration of the beam when external load applied at the center of the beam (◯ 10m, - - 60m, — 100m)

$$x \geq \frac{\ln(\text{error})}{-\eta\lambda} \tag{6}$$

Since *x* is the distance from loading position to one end with little displacement, the length of the beam model should be at least two times of *x*.

$$\text{Effective length} \geq 2 \cdot \frac{\ln(\text{error})}{-\eta\lambda} \tag{7}$$

Fig. 7. shows that normalized effective length increases with velocity ratio and moderately decreases with damping ratio for high velocity cases (with *error*=10^{-5}). It is obvious that velocity of moving load matters even more significantly than foundation damping does. If *EI* of the beam is 6.12×10^6 $N{\cdot}m^2$ and foundation stiffness k is 16 MPa, the parameter of relative stiffness between beam and foundation, λ, was calculated to be 0.899m^{-1}. The effective length of beam is shown in Fig. 8.. For cases with low moving velocity ratio, damping ratio has almost no effect on effective length. On the other hand, cases with high moving velocity require considerably longer model to achieve stable responses. The damping effects become a

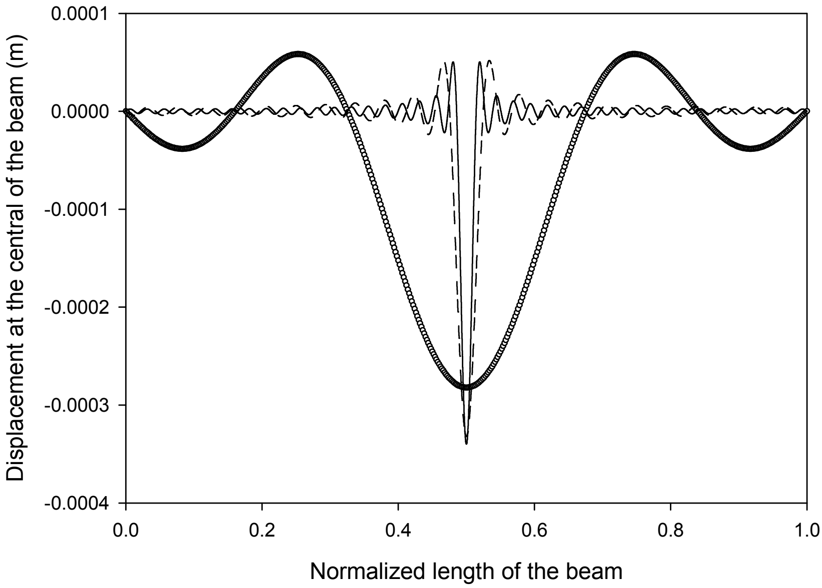

Figure 6: Vibration at the central point of the beam (◯ 10m, - - 60m, — 100m)

little bit significant as load moving velocity increases. The effective length remains almost constant for damping ratio of the railway foundation in the common range in real world, say 0.02 - 0.3. Even in very high velocity cases, the figure illustrates that up to 15% increase of effective length might be possible caused by damping properties of railway foundation.

With the help of effective length, researcher no longer need to determine the beam length of railway track models by iterative runs of the track model until convergence. Tedious loops of computations with incremental beam length can be replaced with just a second of a straightforward calculation by input material properties and load moving velocity. For general conditions, β=0∼0.4 and θ=0∼0.5, modeling a continuous rail with a 30-meter beam should be satisfactory to avoid reflection from boundaries.

4 Justification of effective length on railway track deflections

In order to examine if it is appropriate to adopt the computed effective length in lieu of iterative analysis for convergence to determine the required model size, a simple model was established to justify the proposed criteria of effective length for

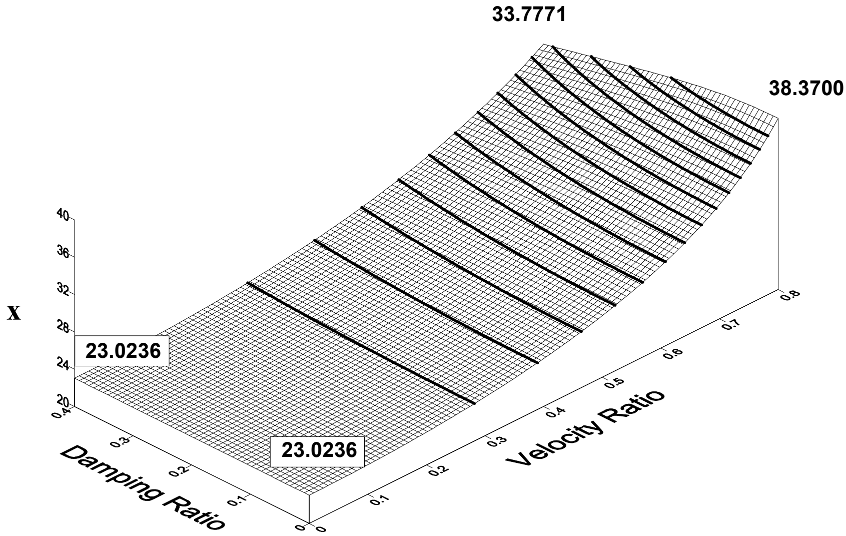

Figure 7: Normalized effective length for different damping ratios and velocity ratios

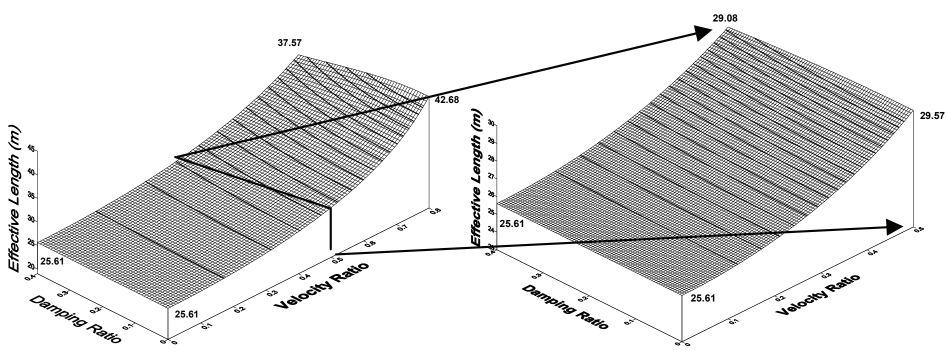

Figure 8: Effective length for different velocity ratios and damping ratios at λ=0.899m^{-1}

railway models. A simply supported Euler-Bernoulli beam resting on a very soft elastic foundation and subjected to a moving load with constant velocity is considered, as shown in Fig. 9.. By assuming *EI* of the beam as 6×10^7 $N{\cdot}m^2$, unit mass of the beam m =60 *kg/m*, loading force P=10^5 N, coefficient of support reaction k=200 *kPa*, moving velocity of loading v=200 m/s, and boundary deflection tolerance *error*=10^{-5}, the effective length was computed to be 167 m (θ=0.5886, β=0,

η=0.8084, λ=0.1699).

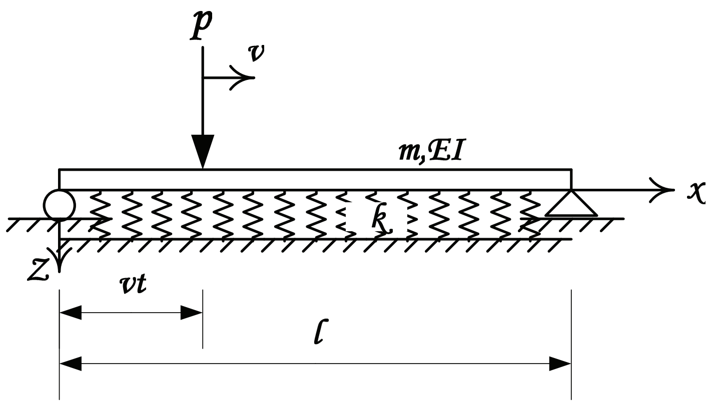

Figure 9: Model of simply supported beam on elastic foundation without damping under moving load

Alternatively, the model size was determined by iterative runs of models with incremental length of beam. The target response, normally the deflection of beam center, may converge as beam length increases, as shown in Fig. 10.. The convergence rate was defined as Eq. (8). Although the converge rate data fluctuate back and forth as model length increases, the fitted trend line apparently illustrate definite convergence. The convergent trend line justifies the calculated effective length with the proposed criteria, 167 m, within prescribed error.

$$\text{convergence rate} = \frac{|\delta_{\mathrm{i}+1} - \delta_i|}{\delta_i} \times 100\% \tag{8}$$

where δ_i: beam center deflection of the model with the length of the i^{th} increment
δ_{i+1}: beam center deflection of the model with the length of the $(i+1)^{th}$ increment

5 Conclusions

With literature reviews and simple derivation, a simple criterion was developed to decide the cut-off length of railway models built with Euler-Bernoulli beam on elastic foundation subjected to a moving load. The proposed criterion correlated material properties, moment of inertia of cross section of beam, stiffness and damping of elastic foundation, and velocity of the moving load. The "effective length" can be calculated quickly. This method outperformed the traditional procedure of determining model size with iterative model analysis by increasing beam length step

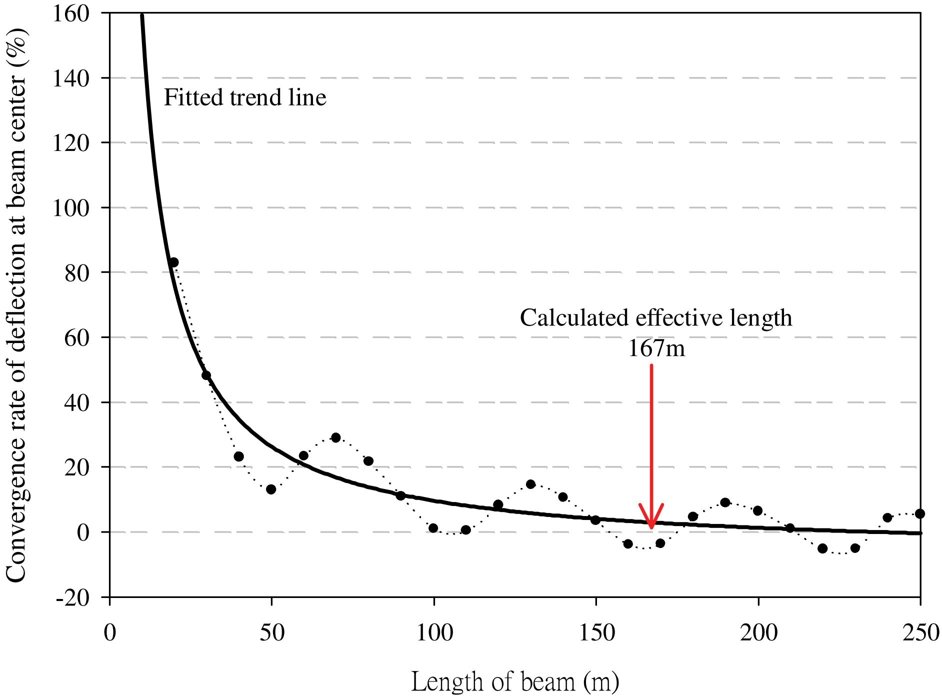

Figure 10: Convergence of calculated beam deflections with various length of models

by step. The suggested length of railway model was verified with the convergence trend with satisfactory accuracy. The effective length depends on beam parameters, stiffness and damping of foundation, and moving velocity of loading.

The success of the proposed method also implies that track responses of continuous welded railways can be estimated with numerical analysis of models of sufficient length allowing deformation waves to damp out and not to reflect from boundaries.

It was found that the effective length of railway model not only depends on rail properties and foundation stiffness, but also varies with velocity of moving load and foundation damping. Fortunately, unless in extreme high speed lines or special foundation designs, the proposed criterion estimated that 30-meter is an appropriate length for most railways with general conditions of track design and train velocity.

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