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TVD Finite Element Scheme for Hyperbolic Systems of Conservation Laws

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Abstract: A finite element scheme based on the concept of TVD (total variation diminishing) with a flux-limiter for the hyperbolic systems of conservation laws is presented. The numerical flux is formulated effectively by the weighted integral form using exponential weighting functions. The TVD finite element scheme is applied to a Riemann problem, namely the shock-tube problem, for the Euler system of equations. Numerical results demonstrate the workability and the validity of the present approach through comparison with the exact solutions.

Keywords: Finite element, Petrov-Galerkin method, exponential weighting functions, flux-limiter, TVD.

1 Introduction

Numerical difficulties for solving the hyperbolic systems of conservation laws have been experienced in the solutions of the compressible flow field involving shock waves and contact surfaces. It is well known that the first-order upwind scheme leads to a smearing of the numerical solutions and all second-order centered schemes generate oscillations in the vicinity of discontinuities, e.g. [Godunov (1959);Sod (1978);Hirsch (1990)]. Godunov[Godunov (1959)] has presented significantly a first-order accurate upwind finite difference scheme for solving numerically and monotonously the system of Euler equations with discontinuities (the Riemann problem). The numerical results of various schemes on the Riemann problem, namely a shock-tube one, were compared in detail by Sod [Sod (1978)], and the Sod's problems are also available in the books by Hirsch[Hirsch (1990)], and Toro [Toro (1997)]. To overcome such "smear" nonstationary shocks and spurious oscillatory solutions, various upwind high-resolution schemes have been successfully achieved by using a flux-limiter function in a general framework, e.g. [van Leer (1979);Roe (1981);Harten (1983);Osher and Chakravarthy (1983);Wang and

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Richards (1991)]. The second-order accurate upwind schemes to Godunov's method have been achieved by van Leer[van Leer (1979)], and the scheme was named MUSCL (Monotonic Upstream-centered Scheme for Conservation Laws). Roe[Roe (1981)] has suggested approximate Riemann solvers for obtaining numerical solutions to hyperbolic system of conservation laws. Harten[Harten (1983)] has introduced the concept of explicit second-order accurate TVD schemes for the hyperbolic systems of conservation laws, and the extension of the implicit TVD schemes has been newly developed by Yee, Warming and Harten[Yee, Warming and Harten (1985)]. Osher and Chakravarthy[Osher and Chakravarthy (1983)] have reviewed several upwind shock-capturing schemes for hyperbolic systems of conservation laws, and applied the Osher's scheme to multidimensional Euler system of equations. The high-order accurate ENO (Essentially Non-Oscillatory) schemes have been successfully developed by Harten et al.[Harten, Engquist, Osher and Chakravarthy (1987)] for the approximation of hyperbolic systems of conservation laws.

In our previous work, we have proposed a finite element scheme for solving effectively the non-conservation or conservation form of an incompressible viscous fluid flow [Kakuda and Tosaka (1992);Kakuda, Tosaka and Nakamura (1996);Kakuda, Miura and Tosaka (2006)]. The scheme is based on the Petrov-Galerkin weak formulation using exponential weighting functions. However, it is difficult to solve accurately the hyperbolic system of conservation laws, such as Euler system of equations, in the Petrov-Galerkin finite element framework. Therefore, in order to develop a finite element scheme for solving the problem of flow, we consider that the concept of TVD scheme needs to be introduced in the Petrov-Galerkin finite element formalization [Hughes and Mallet (1985);Arminjon and Dervieux (1993)].

The purpose of this paper is to develop a finite element scheme based on the concept of TVD with a suitable flux-limiter function for the hyperbolic systems of conservation laws. In Section 2 we review finite element schemes based on TVD in the context of the linear steady and unsteady advection-diffusion equations. The TVD-based finite element scheme is also applied to the Burger's equation and hyperbolic systems of conservation laws as the non-linear model equations. In Section 3 a high-precision finite element scheme based on TVD is presented to the linear and non-linear advection model equations. The generalization to systems is described in this section. The workability and the validity of the present approach are demonstrated for a Riemann problem, namely the shock tube problem, through comparison with the exact solutions[Toro (1997)]. The conclusions of this work are draw in Section 4.

Throughout this paper, the summation convention on repeated indices is employed. A comma following a variable is used to denote partial differentiation with respect to the spatial/time variable.

2 A finite element scheme based on TVD

2.1 Steady advection-diffusion equation

2.1.1 Problem statement

Let us first consider the one-dimensional advection-diffusion equation in spatial coordinate, x, given by

$$u\varphi_{,x} = k\varphi_{,xx} \tag{1}$$

with the adequate boundary conditions, where u and k are the given velocity and diffusivity, respectively.
Now, we define the following flux f in Eq. 1 :

$$f = u\varphi \tag{2}$$

With this definition, Eq. 1 is given as follows :

$$f_{,x} = k\varphi_{,xx} \tag{3}$$

2.1.2 Finite element formulation

In order to solve the flux in a stable manner, we shall adopt the Petrov-Galerkin finite element formulation using exponential weighting function [Kakuda and Tosaka (1992)]. On the other hand, the conventional Galerkin finite element formulation can be applied to solve numerically Eq. 3.
First of all, we start with the following weighted integral expression in a subdomain $\Omega_i = [x_{i-1}, x_i]$ with respect to weighting function $\tilde{w}$:

$$\int_{\Omega_i} (f - u\varphi)\tilde{w}dx = 0 \tag{4}$$

The weighting function $\tilde{w}$ can be chosen as a general solution which satisfies

$$u\tilde{w} + \Delta x_i \sigma(u)\tilde{w}_{,x} = 0 \tag{5}$$

where $\Delta x_i = x_i - x_{i-1}$, and $\sigma(u)$ denotes some function described by Yee, Warming and Harten [Yee, Warming and Harten (1985)], which is sometimes referred to as the coefficient of numerical viscosity. The solution of Eq. 5 is

$$\tilde{w} = Ae^{-ax} \tag{6}$$

where A is a constant, and

$$a = \frac{u}{\Delta x_i \sigma(u)} \tag{7}$$

The weighting function can be approximated using the exponential function in the subdomain Ω_i as follows :

$$\tilde{w} = \sum_{\alpha=1}^{2} M_\alpha \tilde{w}_\alpha \tag{8}$$

where $\tilde{w}_1$ and $\tilde{w}_2$ are the weighting at points $x_{i-1}(=x_1)$ and $x_i(=x_2)$, respectively, and

$$M_\alpha = e^{-a(x-x_\alpha)} \qquad (\alpha = 1,2) \tag{9}$$

On the other hand, the flux f and φ are taken approximately as the piecewise linear function associated with node β be denoted by N_β

$$\left.\begin{aligned} f &= \sum_{\beta=1}^{2} N_\beta f_\beta \\ \varphi &= \sum_{\beta=1}^{2} N_\beta \varphi_\beta \end{aligned}\right\} \tag{10}$$

Substituting Eq. 8 and Eq. 10 into Eq. 4, from $\tilde{w}_\alpha \neq 0$ we obtain the following integral form

$$\int_{\Omega_i} M_\alpha N_\beta dx f_\beta - u \int_{\Omega_i} M_\alpha N_\beta dx \varphi_\beta = 0 \tag{11}$$

Here, applying an element-wise mass lumping to the first term of the left-hand side of Eq. 11, and carrying out exactly those integrals in Eq. 11, we have the following finite element equation

$$\tilde{c}\delta_{\alpha\beta} f_\beta = -uH_{\alpha\beta}\varphi_\beta \tag{12}$$

where $\delta_{\alpha\beta}$ is the Kroneker's delta, and

$$\left.\begin{aligned} \tilde{c} &= e^{-\gamma} - e^{\gamma} \\ H_{\alpha\beta} &= \begin{bmatrix} (e^{\gamma} + \frac{\tilde{c}}{2\gamma}) & -(e^{-\gamma} + \frac{\tilde{c}}{2\gamma}) \\ (e^{\gamma} - \frac{\tilde{c}}{2\gamma}) & -(e^{-\gamma} - \frac{\tilde{c}}{2\gamma}) \end{bmatrix} \\ \gamma &= \frac{u}{2\sigma(u)} \end{aligned}\right\} \tag{13}$$

From Eq. 12 we can obtain the following numerical flux $f_{i-1/2}$ in the subdomain Ω_i

$$f_{i-1/2} = f_i + \frac{u}{2}[1 + \{sgn(\gamma)coth|\gamma| - \frac{1}{\gamma}\}](\varphi_{i-1} - \varphi_i) \tag{14}$$

and similarly in another subdomain Ω_{i+1}, we have

$$f_{i+1/2} = f_i + \frac{u}{2}[-1 + \{sgn(\gamma)coth|\gamma| - \frac{1}{\gamma}\}](\varphi_i - \varphi_{i+1}) \tag{15}$$

where $sgn(\gamma)$ denotes the signum function.
Let us now derive the Galerkin finite element model for Eq. 3. The weighted residual equation in Ω_i is given as follows :

$$\int_{\Omega_i} (f_{,x} - k\varphi_{,xx})w dx = 0 \tag{16}$$

Integrating by parts over the subdomain leads to

$$\int_{\Omega_i} (f_{,x}w + k\varphi_{,x}w_{,x})dx = k[\varphi_{,x}w]_{x_{i-1}}^{x_i} \tag{17}$$

Here, the weighting function w can be linearly interpolated as follows :

$$w = \sum_{\alpha=1}^{2} N_\alpha w_\alpha \tag{18}$$

Substituting Eq. 18 into Eq. 17 leads to

$$\int_{\Omega_i} N_\alpha N_{\beta,x} dx f_\beta + k\int_{\Omega_i} N_{\alpha,x} N_{\beta,x} dx \varphi_\beta = k[\varphi_{,x}N_\alpha]_{x_{i-1}}^{x_i} \tag{19}$$

The finite element equation for Eq. 19 is given as follows :

$$G_{\alpha\beta} f_\beta + K_{\alpha\beta}\varphi_\beta = kq_\alpha \tag{20}$$

where

$$G_{\alpha\beta} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \; K_{\alpha\beta} = \frac{k}{\Delta x_i}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \; q_\alpha = \begin{bmatrix} -\varphi_{,x}(x_{i-1}) \\ \varphi_{,x}(x_i) \end{bmatrix} \tag{21}$$

In this stage, we assume a uniform mesh $\Delta x_i = \Delta x$ for simplicity of the formulation. Taking into consideration the continuity of $\varphi_{,x}$ at nodal point i, we can obtain the following discrete form

$$f_{i-1/2} - f_{i+1/2} + \frac{k}{\Delta x}(\varphi_{i-1} - 2\varphi_i + \varphi_{i+1}) = 0 \tag{22}$$

Substituting Eq. 14 and Eq. 15 into Eq. 22 and after some manipulations, we obtain the following finite difference form

$$\frac{u}{2\Delta x}(\varphi_{i+1}-\varphi_{i-1}) = (k+\tilde{k})\frac{\varphi_{i-1}-2\varphi_i+\varphi_{i+1}}{\Delta x^2} \tag{23}$$

where for any velocity u

$$\tilde{k} = \frac{|u|\Delta x}{2}\{coth|\gamma| - \frac{1}{|\gamma|}\} \tag{24}$$

There exist some cases for possible choice of $\sigma(u)$ in Eq. 5. Now if we assume $\sigma(u)$ to be

$$\sigma(u) = \frac{|u|}{\alpha} \tag{25}$$

in which an ad hoc parameter α is also chosen as follows :

$$\alpha = \frac{\Delta x|u|}{k} \tag{26}$$

then γ in Eq. 24 is given as

$$\begin{aligned} \gamma &= \frac{u}{2\sigma(u)} \\ &= Pe \quad (\equiv \frac{\Delta x u}{2k} : element\, Peclet\, number) \end{aligned} \tag{27}$$

Using the element Peclet number *Pe* as γ, we reduce Eq. 23 to the following form

$$\{sgn(Pe) - coth|Pe|\}\varphi_{i+1} + 2coth|Pe|\varphi_i - \{sgn(Pe) + coth|Pe|\}\varphi_{i-1} = 0 \tag{28}$$

This equation has the same structure as the SUPG scheme developed by Hughes et al. [Brooks and Hughes (1982)], and it leads to nodally exact solutions for all values of *Pe* [Christie, Griffiths, Mitchell and Zienkiewicz (1976)].

2.2 Unsteady advection-diffusion equation

2.2.1 Problem statement

Following the formulation of previous section we consider the one-dimensional unsteady problem in space ,x, and time ,t, given by

$$\varphi_{,t} + f_{,x} = k\varphi_{,xx} \tag{29}$$

with Eq. 2 as the flux ,f, and the adequate boundary conditions and the initial one.

2.2.2 Finite element formulation

In the following, let us construct the finite element model for Eq. 29. The weak form with respect to Eq. 29 is given as follows :

$$\int_{\Omega_i} (\varphi_{,t} w + f_{,x} w + k\varphi_{,x} w_{,x}) dx = k[\varphi_{,x} w]_{x_{i-1}}^{x_i} \tag{30}$$

Substituting Eq. 10 and Eq. 18 into Eq. 30, we obtain the following integral form

$$\begin{aligned} \int_{\Omega_i} N_\alpha N_\beta dx \varphi_{\beta,t} \quad &+ \quad \int_{\Omega_i} N_\alpha N_{\beta,x} dx f_\beta \\ &+ \quad k \int_{\Omega_i} N_{\alpha,x} N_{\beta,x} dx \varphi_\beta = k[\varphi_{,x} N_\alpha]_{x_{i-1}}^{x_i} \end{aligned} \tag{31}$$

and the finite element equation as follows :

$$\tilde{M}_{\alpha\beta} \varphi_{\beta,t} + G_{\alpha\beta} f_\beta + K_{\alpha\beta} \varphi_\beta = k q_\alpha \tag{32}$$

where $\tilde{M}_{\alpha\beta} \equiv \Delta x_i \delta_{\alpha\beta}/2$ is the lumped mass matrix.

Assuming a uniform mesh Δx for simplicity and taking into consideration the continuity of $\varphi_{,x}$ at nodal point i, we obtain the following form

$$\Delta x \varphi_{i,t} = f_{i-1/2} - f_{i+1/2} + \frac{k}{\Delta x}(\varphi_{i-1} - 2\varphi_i + \varphi_{i+1}) \tag{33}$$

Approximating the above equation using forward in time and substituting Eq. 14 and Eq. 15 we have

$$\varphi_i^{n+1} = \varphi_i^n + C_{+,i+1/2}(\varphi_{i+1}^n - \varphi_i^n) - C_{-,i-1/2}(\varphi_i^n - \varphi_{i-1}^n) \tag{34}$$

where φ_i^n is a numerical solution of Eq. 34 at $x_i = i\Delta x$ and $t_n = n\Delta t$, Δt is the time step, and

$$\left. \begin{array}{l} C_{+,i+1/2} = \frac{c}{2}[-1 + \frac{1}{Pe} + \{sgn(\gamma)coth|\gamma| - \frac{1}{\gamma}\}] \\ C_{-,i-1/2} = \frac{c}{2}[1 + \frac{1}{Pe} + \{sgn(\gamma)coth|\gamma| - \frac{1}{\gamma}\}] \end{array} \right\} \tag{35}$$

where $c = u\Delta t/\Delta x$ is the Courant number.

In this stage, let us design a sufficient condition that Eq. 34 be total variation diminishing (TVD) [Harten (1983)]. A sufficient condition for Eq. 34 with Eq. 35 to be a TVD scheme is that

$$\left. \begin{array}{l} C_{-,i+1/2} \geq 0 \quad , \quad C_{+,i+1/2} \geq 0 \\ C_{-,i+1/2} + C_{+,i+1/2} \leq 1 \end{array} \right\} \tag{36}$$

The resulting condition for γ is as follows :

$$1-\frac{1}{|Pe|}\leq coth|\gamma|-\frac{1}{|\gamma|}\leq\frac{1}{|c|}-\frac{1}{|Pe|} \tag{37}$$

with $|\gamma|=\alpha/2$. From this condition we can determine the bounds of γ or α by using the given c and Pe.

2.2.3 Remarks

(1) If $\alpha=\Delta x|u|/k$ (or $\gamma=Pe$, i.e., the case of nodal exact solution for steady advection-diffusion equation), then the condition of Eq. 37 is reduced to

$$1\leq coth|Pe|\leq\frac{1}{|c|} \tag{38}$$

The above restriction can also be derived from applying the von Neumann stability strategy to Eq. 34.

(2) If $k=0$ (i.e., the case of advection equation), then the condition of Eq. 37 is given as follows :

$$1\leq coth|\gamma|-\frac{1}{|\gamma|}\leq\frac{1}{|c|} \tag{39}$$

The solution which satisfies the above condition is only given by (see Fig. 1)

$$coth|\gamma|-\frac{1}{|\gamma|}=1 \tag{40}$$

2.3 Burger's equation

2.3.1 Problem statement

As an application of the concept of previous section to non-linear equation, we consider the following Burger's equation

$$u_{,t}+uu_{,x}=\nu u_{,xx} \tag{41}$$

where ν denotes a parameter which corresponds to the Reynolds number in viscous fluid flow problems.

Now, linearizing the non-linear equation of Eq. 41 in the subdomain Ω_i, we obtain the following system of equations

$$\hat{f}=\hat{u}u \tag{42}$$

$$u_{,t}+\hat{f}_{,x}=\nu u_{,xx} \tag{43}$$

where $\hat{u}$ is assumed to be the mean value of the end points of each subdomain.

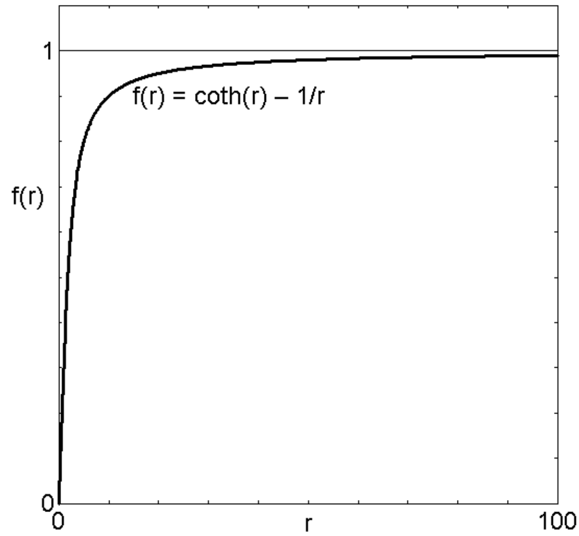


Figure 1: A function f(r) for different values of r

2.3.2 Finite element formulation

Following the approach of previous section, the finite element approximations for Eq. 42 and Eq. 43 are given as follows :

$$\left.\begin{aligned}\hat{f}_{i-1/2} &= \hat{f}_i + \frac{\hat{u}_{i-1/2}}{2}[1 + \{sgn(\gamma_{i-1/2})coth|\gamma_{i-1/2}| - \frac{1}{\gamma_{i-1/2}}\}](u_{i-1} - u_i)\\ \hat{f}_{i+1/2} &= \hat{f}_i + \frac{\hat{u}_{i+1/2}}{2}[-1 + \{sgn(\gamma_{i+1/2})coth|\gamma_{i+1/2}| - \frac{1}{\gamma_{i+1/2}}\}](u_i - u_{i+1})\end{aligned}\right\} \quad (44)$$

and

$$\Delta x u_{i,t} = \hat{f}_{i-1/2} - \hat{f}_{i+1/2} + \frac{\nu}{\Delta x}(u_{i-1} - 2u_i + u_{i+1}) \quad (45)$$

respectively. Substituting Eq. 44 into Eq. 45 and using the forward difference in time, we obtain the following explicit form in terms of u_i

$$u_i^{n+1} = u_i^n + \hat{C}_{+,i+1/2}(u_{i+1}^n - u_i^n) - \hat{C}_{-,i-1/2}(u_i^n - u_{i-1}^n) \quad (46)$$

where

$$\left.\begin{aligned}\hat{C}_{+,i+1/2} &= \frac{c_{i+1/2}}{2}[-1 + \frac{1}{Re_{i+1/2}} + \{sgn(\gamma_{i+1/2})coth|\gamma_{i+1/2}| - \frac{1}{\gamma_{i+1/2}}\}]\\ \hat{C}_{-,i-1/2} &= \frac{c_{i-1/2}}{2}[1 + \frac{1}{Re_{i-1/2}} + \{sgn(\gamma_{i-1/2})coth|\gamma_{i-1/2}| - \frac{1}{\gamma_{i-1/2}}\}]\end{aligned}\right\} \quad (47)$$

and $c_{i\pm1/2} = \hat{u}_{i\pm1/2}\Delta t/\Delta x$ is the Courant number, $Re_{i\pm1/2} = \hat{u}_{i\pm1/2}\Delta x/2\nu$ is the cell Reynolds number, and $\gamma_{i\pm1/2} = \alpha\hat{u}_{i\pm1/2}/2Q(\hat{u}_{i\pm1/2})$ in which $Q(\hat{u}_{i\pm1/2})$ is defined consistently by Harten [Harten (1983)].

The sufficient condition for $\gamma_{i+1/2}$ in Eq. 47 to be a TVD scheme is given as follows:

$$1-\frac{1}{|Re_{i+1/2}|} \le coth|\gamma_{i+1/2}| - \frac{1}{|\gamma_{i+1/2}|} \le \frac{1}{|c_{i+1/2}|} - \frac{1}{|Re_{i+1/2}|} \quad (48)$$

2.4 Hyperbolic systems of conservation laws

A generalization of the preceding ideas to one-dimensional hyperbolic systems of conservation laws is constructed as follows.

2.4.1 Problem statement

Let us consider a hyperbolic systems of conservation laws

$$\boldsymbol{U}_{,t} + \boldsymbol{F}(\boldsymbol{U})_{,x} = \boldsymbol{0} \quad (49)$$

where $\boldsymbol{U}$ is a $m \times 1$ vector, and

$$\boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{A}(\boldsymbol{U})\boldsymbol{U} \quad (50)$$

with $\boldsymbol{A}(\boldsymbol{U}) = \partial\boldsymbol{F}(\boldsymbol{U})/\partial\boldsymbol{U}$ is the Jacobian of $\boldsymbol{F}(\boldsymbol{U})$. If an 'average' value of $\boldsymbol{A}(\boldsymbol{U})$ is assumed in each subdomain, then we obtain

$$\boldsymbol{A}(\boldsymbol{U}) = \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S}^{-1} \quad (51)$$

where $\boldsymbol{\Lambda}$ is a $m \times m$ diagonal matrix which consists of real eigenvalues $\hat{a}^k$ $(k = 1,2,...,m)$ of $\boldsymbol{A}$, and $\boldsymbol{S}$ denotes a complete set of right eigenvectors $\boldsymbol{R}^k$.

Now, we define the characteristic variables $\boldsymbol{W}$ with respect to $\boldsymbol{U}$ as follows :

$$\boldsymbol{W} = \boldsymbol{S}^{-1}\boldsymbol{U} \quad (52)$$

Substituting Eq. 51 and Eq. 52 into Eq. 50 leads to

$$\boldsymbol{F} = \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{W} \quad (53)$$

We also define as

$$\hat{\boldsymbol{F}} = \boldsymbol{\Lambda}\boldsymbol{W} \quad (54)$$

Consequently, we adopt the following four systems of equations instead of the original Eq. 49.

$$\boldsymbol{W} = \boldsymbol{S}^{-1}\boldsymbol{U} \quad (55)$$

$$\hat{\boldsymbol{F}} = \boldsymbol{\Lambda}\boldsymbol{W} \quad (56)$$

$$\boldsymbol{F} = \boldsymbol{S}\hat{\boldsymbol{F}} \quad (57)$$

$$\boldsymbol{U}_{,t} + \boldsymbol{F}_{,x} = \boldsymbol{0} \quad (58)$$

2.4.2 Finite element formulation

The key factor for solving the system of Eq. 55 to Eq. 58 is to calculate accurately and in a stable manner Eq. 56 with respect to the component $\hat{f}_k$ of $\hat{\boldsymbol{F}}$. We apply the preceding exponential weighting strategy to solve Eq. 56. On the other hand, the conservation law form of Eq. 58 can be formulated by the Galerkin finite element method.

The weighted residual form for Eq. 56 in the subdomain $\Omega_i = [x_{i-1}, x_i]$ is given by

$$\int_{\Omega_i} \{\hat{f}_k - \hat{a}^k_{i-1/2} w_k\} \tilde{w}^k dx = 0 \quad , \quad (\textit{no sum on } k) \tag{59}$$

Applying the generation of Eq. 8 and Eq. 10 to the above equation, we obtain the following integral form.

$$\left.\begin{array}{l} \int_{\Omega_i} M^k_\alpha N_\beta dx (\hat{f}_\beta)_k - \int_{\Omega_i} M^k_\alpha \hat{a}^k_{i-1/2} N_\beta dx (w_\beta)_k = 0 \quad , \quad (\textit{no sum on } k) \\ M^k_\alpha = e^{-a^k_{i-1/2}(x-x_\alpha)} \quad , \quad a^k_{i-1/2} = \dfrac{(\sigma^k_{i-1/2})^{-1} \hat{a}^k_{i-1/2}}{h_{i-1/2}} \end{array}\right\} \tag{60}$$

By implementing the integrals in Eq. 60 and using the mass lumping technique, we find the solutions of $\hat{f}_k$ in the subdomain Ω_i as follows :

$$\begin{aligned} (\hat{f}_{i-1/2})_k &= \frac{\hat{a}^k_{i-1/2}}{2} \{(w_{i-1})_k + (w_i)_k\} \\ &+ \frac{\hat{a}^k_{i-1/2}}{2} \{sgn(\gamma^k_{i-1/2}) coth|\gamma^k_{i-1/2}| - \frac{1}{\gamma^k_{i-1/2}}\} \{(w_{i-1})_k - (w_i)_k\} \end{aligned} \tag{61}$$

and similarly in an adjacent subdomain Ω_{i+1}

$$\begin{aligned} (\hat{f}_{i+1/2})_k &= \frac{\hat{a}^k_{i+1/2}}{2} \{(w_i)_k + (w_{i+1})_k\} \\ &+ \frac{\hat{a}^k_{i+1/2}}{2} \{sgn(\gamma^k_{i+1/2}) coth|\gamma^k_{i+1/2}| - \frac{1}{\gamma^k_{i+1/2}}\} \{(w_i)_k - (w_{i+1})_k\} \end{aligned} \tag{62}$$

The local matrix form for Eq. 61 and Eq. 62 is also given by respectively

$$\hat{\boldsymbol{F}}_{i-1/2} = \frac{\boldsymbol{\Lambda}_{i-1/2}}{2} (\boldsymbol{W}_{i-1} + \boldsymbol{W}_i) + \frac{\boldsymbol{\Lambda}_{i-1/2}}{2} \tilde{\boldsymbol{\zeta}}_{i-1/2} (\boldsymbol{W}_{i-1} - \boldsymbol{W}_i) \tag{63}$$

$$\hat{\boldsymbol{F}}_{i+1/2} = \frac{\boldsymbol{\Lambda}_{i+1/2}}{2} (\boldsymbol{W}_i + \boldsymbol{W}_{i+1}) + \frac{\boldsymbol{\Lambda}_{i+1/2}}{2} \tilde{\boldsymbol{\zeta}}_{i+1/2} (\boldsymbol{W}_i - \boldsymbol{W}_{i+1}) \tag{64}$$

where

$$\left.\begin{array}{l}\tilde{\boldsymbol{\zeta}}_{i\pm1/2}=diag\{sgn(\boldsymbol{\gamma}_{i\pm1/2})coth|\boldsymbol{\gamma}_{i\pm1/2}|-\boldsymbol{\gamma}_{i\pm1/2}^{-1}\}\\ \boldsymbol{\gamma}_{i\pm1/2}=\dfrac{\boldsymbol{\Sigma}_{i\pm1/2}^{-1}\boldsymbol{\Lambda}_{i\pm1/2}}{2}\\ \boldsymbol{\Lambda}_{i\pm1/2}=diag\{\hat{\boldsymbol{a}}_{i\pm1/2}\}\\ \boldsymbol{\Sigma}_{i\pm1/2}=diag\{\boldsymbol{\sigma}_{i\pm1/2}\}\end{array}\right\} \tag{65}$$

The finite element approximations for Eq. 55 to Eq. 58 can be eventually constituted as follows :

(a) The forms of Eq. 55 in Ω_i are given by

$$\boldsymbol{W}_{i-1}=\boldsymbol{S}_{i-1/2}^{-1}\boldsymbol{U}_{i-1} \quad , \quad \boldsymbol{W}_i=\boldsymbol{S}_{i-1/2}^{-1}\boldsymbol{U}_i \tag{66}$$

and in Ω_{i+1}

$$\boldsymbol{W}_i=\boldsymbol{S}_{i+1/2}^{-1}\boldsymbol{U}_i \quad , \quad \boldsymbol{W}_{i+1}=\boldsymbol{S}_{i+1/2}^{-1}\boldsymbol{U}_{i+1} \tag{67}$$

(b) The forms for Eq. 56 are rewritten as

$$\hat{\boldsymbol{F}}_{i-1/2}=\boldsymbol{\Lambda}_{i-1/2}\boldsymbol{W}_i+\frac{\boldsymbol{\Lambda}_{i-1/2}}{2}\{\boldsymbol{I}+\tilde{\boldsymbol{\zeta}}_{i-1/2}\}(\boldsymbol{W}_{i-1}-\boldsymbol{W}_i) \tag{68}$$

$$\hat{\boldsymbol{F}}_{i+1/2}=\boldsymbol{\Lambda}_{i+1/2}\boldsymbol{W}_i-\frac{\boldsymbol{\Lambda}_{i+1/2}}{2}\{\boldsymbol{I}-\tilde{\boldsymbol{\zeta}}_{i+1/2}\}(\boldsymbol{W}_i-\boldsymbol{W}_{i+1}) \tag{69}$$

where $\boldsymbol{I}$ denotes an $m\times m$ unit matrix.

(c) The numerical flux for Eq. 57 is given by

$$\boldsymbol{F}_{i\pm1/2}=\boldsymbol{S}_{i\pm1/2}\hat{\boldsymbol{F}}_{i\pm1/2} \tag{70}$$

(d) Applying the finite element approximation to Eq. 58 leads to

$$\boldsymbol{U}_i^{n+1}=\boldsymbol{U}_i^n+\lambda(\boldsymbol{F}_{i-1/2}^n-\boldsymbol{F}_{i+1/2}^n) \tag{71}$$

where $\lambda=\Delta t/\Delta x$.

By substituting Eq. 66 to Eq. 70 into Eq. 71, we obtain the following final system of equations to solve explicitly the unknown vector $\boldsymbol{U}_i^{n+1}$:

$$\boldsymbol{U}_i^{n+1}=\boldsymbol{U}_i^n-\lambda\left[\begin{array}{c}\boldsymbol{S}_{i-1/2}\boldsymbol{\Lambda}_{i-1/2}\tilde{\boldsymbol{\Sigma}}_{i-1/2}\boldsymbol{W}_{i-1/2}^n\\ +\boldsymbol{S}_{i+1/2}\boldsymbol{\Lambda}_{i+1/2}\tilde{\boldsymbol{\Sigma}}_{i+1/2}^*\boldsymbol{W}_{i+1/2}^n\end{array}\right] \tag{72}$$

where $\tilde{\boldsymbol{\Sigma}}_{i\pm1/2} = (\boldsymbol{I} + \tilde{\boldsymbol{\zeta}}_{i\pm1/2})/2$, $\tilde{\boldsymbol{\Sigma}}^*_{i\pm1/2} = (\boldsymbol{I} - \tilde{\boldsymbol{\zeta}}_{i\pm1/2})/2$, and

$$\begin{aligned} \boldsymbol{W}^n_{i-1/2} &= \boldsymbol{W}^n_i - \boldsymbol{W}^n_{i-1} \\ &= \boldsymbol{S}^{-1}_{i-1/2}\{\boldsymbol{U}^n_i - \boldsymbol{U}^n_{i-1}\} \end{aligned} \tag{73}$$

$$\begin{aligned} \boldsymbol{W}^n_{i+1/2} &= \boldsymbol{W}^n_{i+1} - \boldsymbol{W}^n_i \\ &= \boldsymbol{S}^{-1}_{i+1/2}\{\boldsymbol{U}^n_{i+1} - \boldsymbol{U}^n_i\} \end{aligned} \tag{74}$$

3 A high-precision finite element scheme based on TVD

In the following, we shall construct a high-precision explicit scheme for hyperbolic system of equations which is based on the concept into the finite element framework [Brooks and Hughes (1982);Hughes and Mallet (1985)].

3.1 Linear advection equation

3.1.1 Problem statement

As before, the linear advection equation is given as follows :

$$\left.\begin{aligned} &f = u\varphi \\ &\varphi_{,t} + f_{,x} = 0 \end{aligned}\right\} \tag{75}$$

In order to construct an explicit scheme with a support of five mesh points, we shall adopt the following governing equations modified by adding a term with a 'limiter', $\tilde{\sigma}$, to the flux f

$$\tilde{f} = f + \tilde{\sigma} g \tag{76}$$

$$\varphi_{,t} + \tilde{f}_{,x} = 0 \tag{77}$$

in which

$$g = h\varphi_{,x} \tag{78}$$

where h is a characteristic length.

3.1.2 Finite element formulation

Let us consider the finite element models for Eq. 76 with Eq. 78 and Eq. 77 in subdomains Ω_i and Ω_{i+1}.

The finite element approximations for the modified numerical flux $\tilde{f}$ are given by using the weighting function $\tilde{w}$ of Eq. 8 as follows :

$$\tilde{f}_{i-1/2} = f_{i-1/2} - \frac{\tilde{\sigma}_{i-1/2}}{\tilde{c}}\{(e^{\gamma} + \frac{\tilde{c}}{2\gamma})g_{i-1} - (e^{-\gamma} + \frac{\tilde{c}}{2\gamma})g_i\} \tag{79}$$

$$\tilde{f}_{i+1/2} = f_{i+1/2} - \frac{\tilde{\sigma}_{i+1/2}}{\tilde{c}}\{(e^{\gamma} + \frac{\tilde{c}}{2\gamma})g_{i} - (e^{-\gamma} + \frac{\tilde{c}}{2\gamma})g_{i+1}\} \tag{80}$$

The finite element form at a point i of Eq. 77 can also be rewritten by assuming a uniform mesh as

$$\Delta x \varphi_{i,t} = \tilde{f}_{i-1/2} - \tilde{f}_{i+1/2} \tag{81}$$

On the other hand, applying the Galerkin finite element strategy to Eq. 78 leads easily to

$$\left.\begin{array}{l} g_{i-1} = \frac{1}{2}\{(\varphi_{i-1} - \varphi_{i-2}) + (\varphi_i - \varphi_{i-1})\} \\ g_i = \frac{1}{2}\{(\varphi_i - \varphi_{i-1}) + (\varphi_{i+1} - \varphi_i)\} \\ g_{i+1} = \frac{1}{2}\{(\varphi_{i+1} - \varphi_i) + (\varphi_{i+2} - \varphi_{i+1})\} \end{array}\right\} \tag{82}$$

By substituting Eq. 79 and Eq. 80 with Eq. 82 into Eq. 81, we obtain

$$\begin{aligned} \Delta x \varphi_{i,t} &= f_{i-1/2} - f_{i+1/2} \\ &- \frac{1}{2}\left[\begin{array}{l} (\frac{e^{\gamma}}{\tilde{c}} + \frac{1}{2\gamma})\tilde{\sigma}_{i-1/2}\{(\varphi_{i-1} - \varphi_{i-2}) + (\varphi_i - \varphi_{i-1})\} \\ -(\frac{e^{-\gamma}}{\tilde{c}} + \frac{1}{2\gamma})\tilde{\sigma}_{i-1/2}\{(\varphi_i - \varphi_{i-1}) + (\varphi_{i+1} - \varphi_i)\} \\ -(\frac{e^{\gamma}}{\tilde{c}} + \frac{1}{2\gamma})\tilde{\sigma}_{i+1/2}\{(\varphi_i - \varphi_{i-1}) + (\varphi_{i+1} - \varphi_i)\} \\ +(\frac{e^{-\gamma}}{\tilde{c}} + \frac{1}{2\gamma})\tilde{\sigma}_{i+1/2}\{(\varphi_{i+1} - \varphi_i) + (\varphi_{i+2} - \varphi_{i+1})\} \end{array}\right] \end{aligned} \tag{83}$$

In order to obtain a relationship between the $\tilde{\sigma}$ and B-functions as described in Hughes and Mallet [Hughes and Mallet (1985)], we assume $u > 0, \gamma \to \infty$, and the forward difference in time. With these assumptions Eq. 83 is reduced to

$$\begin{aligned} \varphi_i^{n+1} &= \varphi_i^n + c(\varphi_{i-1}^n - \varphi_i^n) \\ &- \frac{\lambda}{2}\left[\begin{array}{l} (-\tilde{\sigma}_{i-1/2} + \tilde{\sigma}_{i+1/2} + r_i^n \tilde{\sigma}_{i+1/2})(\varphi_i^n - \varphi_{i-1}^n) \\ -\tilde{\sigma}_{i-1/2}(\varphi_{i-1}^n - \varphi_{i-2}^n) \end{array}\right] \end{aligned} \tag{84}$$

where the smoothness moniter r_i^n at node i, n-th time level is defined by

$$r_i^n = \frac{\varphi_{i+1}^n - \varphi_i^n}{\varphi_i^n - \varphi_{i-1}^n} \tag{85}$$

Here, if $\tilde{\sigma}_{i-1/2}$ and $\tilde{\sigma}_{i+1/2}$ in Eq. 84 are given as follows :

$$\left.\begin{array}{l}\tilde{\sigma}_{i-1/2} = u(1-c)B_\nu(r^n_{i-1}) \\ \tilde{\sigma}_{i+1/2} = \dfrac{u(1-c)\{B_\nu(r^n_{i-1}) + B_\nu(r^n_i)\}}{1+r^n_i}\end{array}\right\} \tag{86}$$

then we derive the following expression from Eq. 84

$$\begin{aligned}\varphi^{n+1}_i &= \varphi^n_i + c(\varphi^n_{i-1} - \varphi^n_i) \\ &- \frac{c(1-c)}{2}\{B_\nu(r^n_i)(\varphi^n_i - \varphi^n_{i-1}) - B_\nu(r^n_{i-1})(\varphi^n_{i-1} - \varphi^n_{i-2})\}\end{aligned} \tag{87}$$

Repeating the process for $u < 0$ and $\gamma \to -\infty$ we obtain

$$\begin{aligned}\varphi^{n+1}_i &= \varphi^n_i - |c|(\varphi^n_i - \varphi^n_{i+1}) \\ &- \frac{\lambda}{2}\left[\begin{array}{l}(\tilde{\sigma}_{i-1/2} - \tilde{\sigma}_{i+1/2} + \frac{1}{r^n_i}\tilde{\sigma}_{i-1/2})(\varphi^n_i - \varphi^n_{i+1}) \\ -\tilde{\sigma}_{i+1/2}(\varphi^n_{i+1} - \varphi^n_{i+2})\end{array}\right]\end{aligned} \tag{88}$$

and we put

$$\left.\begin{array}{l}\tilde{\sigma}_{i-1/2} = \dfrac{|u|(1-|c|)\{B_\nu(\frac{1}{r^n_i}) + B_\nu(\frac{1}{r^n_{i+1}})\}}{1+\dfrac{1}{r^n_i}} \\ \tilde{\sigma}_{i+1/2} = |u|(1-|c|)B_\nu(\frac{1}{r^n_{i+1}})\end{array}\right\} \tag{89}$$

Then we lead to

$$\begin{aligned}\varphi^{n+1}_i &= \varphi^n_i - |c|(\varphi^n_i - \varphi^n_{i+1}) \\ &- \frac{|c|(1-|c|)}{2}\{B_\nu(\frac{1}{r^n_i})(\varphi^n_i - \varphi^n_{i+1}) - B_\nu(\frac{1}{r^n_{i+1}})(\varphi^n_{i+1} - \varphi^n_{i+2})\}\end{aligned} \tag{90}$$

The general cases of $\tilde{\sigma}_{i\pm1/2}$ are summarized as follows :

$$\left.\begin{array}{l}\tilde{\sigma}_{i-s/2} = |u|(1-|c|)B_\nu((r^n_{i-s})^s) \\ \tilde{\sigma}_{i+s/2} = \dfrac{|u|(1-|c|)\{B_\nu((r^n_i)^s) + B_\nu((r^n_{i-s})^s)\}}{1+(r^n_i)^s}\end{array}\right\} \tag{91}$$

where $s = sgn(u)$.

3.2 Nonlinear advection equation

3.2.1 Problem statement

Let us consider the following nonlinear advection equation

$$u_{,t} + uu_{,x} = 0 \tag{92}$$

The linearization of above Eq. 92 is given by

$$\left.\begin{array}{l} \hat{f} = \hat{u}u \\ u_{,t} + \hat{f}_{,x} = 0 \end{array}\right\} \tag{93}$$

The modified equations in order to derive a high-precision scheme are of forms

$$g = hu_{,x} \tag{94}$$

$$\tilde{f}^* = \hat{f} + \tilde{\sigma}(\hat{u})g \tag{95}$$

$$u_{,t} + \tilde{f}^*_{,x} = 0 \tag{96}$$

3.2.2 Finite element formulation

As before, we assume a uniform mesh Δx. The finite element approximations for Eq. 94 to Eq. 96 are given respectively as follows :

(a) The finite element model for Eq. 94 is of form

$$\left.\begin{array}{l} g_{i-1} = \frac{1}{2}\{(u_{i-1} - u_{i-2}) + (u_i - u_{i-1})\} \\ g_i = \frac{1}{2}\{(u_i - u_{i-1}) + (u_{i+1} - u_i)\} \\ g_{i+1} = \frac{1}{2}\{(u_{i+1} - u_i) + (u_{i+2} - u_{i+1})\} \end{array}\right\} \tag{97}$$

(b) The finite element model for Eq. 95 is of form

$$\left.\begin{array}{l} \tilde{f}^*_{i-1/2} = \hat{f}_{i-1/2} \\ -\frac{\tilde{\sigma}(\hat{u}_{i-1/2})}{\tilde{c}_{i-1/2}}\{(e^{\gamma_{i-1/2}} + \frac{\tilde{c}_{i-1/2}}{2\gamma_{i-1/2}})g_{i-1} - (e^{-\gamma_{i-1/2}} + \frac{\tilde{c}_{i-1/2}}{2\gamma_{i-1/2}})g_i\} \\ \tilde{f}^*_{i+1/2} = \hat{f}_{i+1/2} \\ -\frac{\tilde{\sigma}(\hat{u}_{i+1/2})}{\tilde{c}_{i+1/2}}\{(e^{\gamma_{i+1/2}} + \frac{\tilde{c}_{i+1/2}}{2\gamma_{i+1/2}})g_i - (e^{-\gamma_{i+1/2}} + \frac{\tilde{c}_{i+1/2}}{2\gamma_{i+1/2}})g_{i+1}\} \end{array}\right\} \tag{98}$$

(c) The finite element form for Eq. 96 is

$$\Delta x u_{i,t} = \tilde{f}^*_{i-1/2} - \tilde{f}^*_{i+1/2} \tag{99}$$

The substitution of Eq. 97 and Eq. 98 into Eq. 99 leads to

$$\Delta x u_{i,t} = \hat{f}_{i-1/2} - \hat{f}_{i+1/2} - \frac{1}{2}\left[\begin{array}{l} (\frac{e^{\gamma_{i-1/2}}}{\tilde{c}_{i-1/2}} + \frac{1}{2\gamma_{i-1/2}})\tilde{\sigma}(\hat{u}_{i-1/2})\{(u_{i-1}-u_{i-2})+(u_i-u_{i-1})\} \\ -(\frac{e^{-\gamma_{i-1/2}}}{\tilde{c}_{i-1/2}} + \frac{1}{2\gamma_{i-1/2}})\tilde{\sigma}(\hat{u}_{i-1/2})\{(u_i-u_{i-1})+(u_{i+1}-u_i)\} \\ -(\frac{e^{\gamma_{i+1/2}}}{\tilde{c}_{i+1/2}} + \frac{1}{2\gamma_{i+1/2}})\tilde{\sigma}(\hat{u}_{i+1/2})\{(u_i-u_{i-1})+(u_{i+1}-u_i)\} \\ +(\frac{e^{-\gamma_{i+1/2}}}{\tilde{c}_{i+1/2}} + \frac{1}{2\gamma_{i+1/2}})\tilde{\sigma}(\hat{u}_{i+1/2})\{(u_{i+1}-u_i)+(u_{i+2}-u_{i+1})\} \end{array}\right] \quad (100)$$

In the following, the above equation is formulated within the same assumptions as previous strategy. For $\hat{u}_{i\pm1/2} > 0$ and $\gamma_{i\pm1/2} \to \infty$ we obtain

$$u_i^{n+1} = u_i^n + c_{i-1/2}^n(u_{i-1}^n - u_i^n) - \frac{\lambda}{2}\left[\begin{array}{l} \{-\tilde{\sigma}(\hat{u}_{i-1/2}^n) + \tilde{\sigma}(\hat{u}_{i+1/2}^n) + \hat{r}_i^n \frac{\hat{u}_{i-1/2}^n}{\hat{u}_{i+1/2}^n}\tilde{\sigma}(\hat{u}_{i+1/2}^n)\}(u_i^n - u_{i-1}^n) \\ -\tilde{\sigma}(\hat{u}_{i-1/2}^n)(u_{i-1}^n - u_{i-2}^n) \end{array}\right] \quad (101)$$

where $c_{i\pm1/2}^n = \hat{u}_{i\pm1/2}^n \Delta t/\Delta x$, and

$$\hat{r}_i^n = \frac{\hat{u}_{i+1/2}^n(u_{i+1}^n - u_i^n)}{\hat{u}_{i-1/2}^n(u_i^n - u_{i-1}^n)} \quad (102)$$

From the following $\tilde{\sigma}(\hat{u}_{i\pm1/2}^n)$ associated with the B-functions

$$\left.\begin{array}{l} \tilde{\sigma}(\hat{u}_{i-1/2}^n) = \hat{u}_{i-1/2}^n(1-c_{i-1/2}^n)B_\nu(\hat{r}_{i-1}^n) \\ \tilde{\sigma}(\hat{u}_{i+1/2}^n) = \dfrac{\hat{u}_{i-1/2}^n(1-c_{i-1/2}^n)B_\nu(\hat{r}_{i-1}^n) + \hat{u}_{i+1/2}^n(1-c_{i+1/2}^n)B_\nu(\hat{r}_i^n)}{1+\hat{r}_i^n\dfrac{\hat{u}_{i-1/2}^n}{\hat{u}_{i+1/2}^n}} \end{array}\right\} \quad (103)$$

we can obtain

$$u_i^{n+1} = u_i^n + c_{i-1/2}^n(u_{i-1}^n - u_i^n) - \frac{1}{2}\left[\begin{array}{l} c_{i+1/2}^n(1-c_{i+1/2}^n)B_\nu(\hat{r}_i^n)(u_i^n - u_{i-1}^n) \\ -c_{i-1/2}^n(1-c_{i-1/2}^n)B_\nu(\hat{r}_{i-1}^n)(u_{i-1}^n - u_{i-2}^n) \end{array}\right] \quad (104)$$

On the other hand, the assumptions of $\hat{u}_{i\pm1/2} < 0$ and $\gamma_{i\pm1/2} \to -\infty$ lead to

$$\begin{aligned} u_i^{n+1} &= u_i^n - |c_{i+1/2}^n|(u_i^n - u_{i+1}^n) \\ &- \frac{\lambda}{2}\left[\begin{array}{l} \{\tilde{\sigma}(\hat{u}_{i-1/2}^n) - \tilde{\sigma}(\hat{u}_{i+1/2}^n) + \frac{1}{\hat{r}_i^n}\frac{\hat{u}_{i+1/2}^n}{\hat{u}_{i-1/2}^n}\tilde{\sigma}(\hat{u}_{i-1/2}^n)\}(u_i^n - u_{i+1}^n) \\ -\tilde{\sigma}(\hat{u}_{i+1/2}^n)(u_{i+1}^n - u_{i+2}^n) \end{array}\right] \end{aligned} \tag{105}$$

and from the definitions

$$\left.\begin{array}{l} \tilde{\sigma}(\hat{u}_{i-1/2}^n) = \dfrac{|\hat{u}_{i-1/2}^n|(1-|c_{i-1/2}^n|)B_\nu(\frac{1}{\hat{r}_i^n}) + |\hat{u}_{i+1/2}^n|(1-|c_{i+1/2}^n|)B_\nu(\frac{1}{\hat{r}_{i+1}^n})}{1 + \frac{1}{\hat{r}_i^n}\frac{\hat{u}_{i+1/2}^n}{\hat{u}_{i-1/2}^n}} \\ \tilde{\sigma}(\hat{u}_{i+1/2}^n) = |\hat{u}_{i+1/2}^n|(1-|c_{i+1/2}^n|)B_\nu(\frac{1}{\hat{r}_{i+1}^n}) \end{array}\right\} \tag{106}$$

we obtain the following expression

$$\begin{aligned} u_i^{n+1} &= u_i^n - |c_{i+1/2}^n|(u_i^n - u_{i+1}^n) \\ &- \frac{1}{2}\left[\begin{array}{l} |c_{i-1/2}^n|(1-|c_{i-1/2}^n|)B_\nu(\frac{1}{\hat{r}_i^n})(u_i^n - u_{i+1}^n) \\ -|c_{i+1/2}^n|(1-|c_{i+1/2}^n|)B_\nu(\frac{1}{\hat{r}_{i+1}^n})(u_{i+1}^n - u_{i+2}^n) \end{array}\right] \end{aligned} \tag{107}$$

The general cases of $\tilde{\sigma}(\hat{u}_{i\pm1/2}^n)$ are also summarized in form :

$$\left.\begin{array}{l} \tilde{\sigma}(\hat{u}_{i-s/2}^n) = |\hat{u}_{i-s/2}^n|(1-|c_{i-s/2}^n|)B_\nu((\hat{r}_{i-s}^n)^s) \\ \tilde{\sigma}(\hat{u}_{i+s/2}^n) = \dfrac{|\hat{u}_{i-s/2}^n|(1-|c_{i-s/2}^n|)B_\nu((\hat{r}_{i-s}^n)^s) + |\hat{u}_{i+s/2}^n|(1-|c_{i+s/2}^n|)B_\nu((\hat{r}_i^n)^s)}{1 + (\hat{r}_i^n)^s\frac{\hat{u}_{i-s/2}^n}{\hat{u}_{i+s/2}^n}} \end{array}\right\} \tag{108}$$

where $s = sgn(\hat{u}_{i\pm1/2}^n)$.

3.3 Hyperbolic systems of conservation laws

3.3.1 Problem statement

In this stage, in order to develop a high-resolution scheme based on the TVD we add an ad hoc function $\boldsymbol{G}$ to Eq. 56. As a result, we obtain the following modified hyperbolic systems of conservation laws :

$$\boldsymbol{W} = \boldsymbol{S}^{-1}\boldsymbol{U} \tag{109}$$

$$\boldsymbol{G} = h\boldsymbol{W}_{,x} \tag{110}$$

$$\tilde{\boldsymbol{F}} = \boldsymbol{\Lambda W} + \hat{\boldsymbol{\Sigma}}\boldsymbol{G} \tag{111}$$

$$\tilde{\boldsymbol{F}}^* = \boldsymbol{S}\tilde{\boldsymbol{F}} \tag{112}$$

$$\boldsymbol{U}_{,t} + \tilde{\boldsymbol{F}}^*_{,x} = \boldsymbol{0} \tag{113}$$

where $\hat{\boldsymbol{\Sigma}}$ is a $m \times m$ diagonal matrix associated with the limiter functions.

3.3.2 Finite element formulation

As before, the weighted integral form of Eq. 111 in the subdomain $\Omega_i = [x_{i-1}, x_i]$ is given by

$$\int_{\Omega_i} \{\tilde{\boldsymbol{F}} - \boldsymbol{\Lambda}_{i-1/2}\boldsymbol{W} - \hat{\boldsymbol{\Sigma}}_{i-1/2}\boldsymbol{G}\}\boldsymbol{M}_\alpha dx = \boldsymbol{0} \tag{114}$$

in which $\boldsymbol{M}_\alpha$ is the exponential weighting functions as follows :

$$\left.\begin{aligned} \boldsymbol{M}_\alpha &= e^{-\boldsymbol{a}_{i-1/2}(x-x_\alpha)} \\ \boldsymbol{a}_{i-1/2} &= \frac{\boldsymbol{\Sigma}^{-1}_{i-1/2}\boldsymbol{\Lambda}_{i-1/2}}{h_{i-1/2}} \end{aligned}\right\} \tag{115}$$

By calculating the integrals in Eq. 114 and using the flux lumping technique such as the mass lumping one, we find the solutions of $\tilde{\boldsymbol{F}}_{i-1/2}$ in Ω_i as follows :

$$\tilde{\boldsymbol{F}}_{i-1/2} = \hat{\boldsymbol{F}}_{i-1/2} + \hat{\boldsymbol{\Sigma}}_{i-1/2}[\boldsymbol{G}_i + \frac{1}{2}\{\boldsymbol{I} + \tilde{\boldsymbol{\zeta}}_{i-1/2}\}(\boldsymbol{G}_{i-1} - \boldsymbol{G}_i)] \tag{116}$$

and similarly in an adjacent subdomain Ω_{i+1}

$$\tilde{\boldsymbol{F}}_{i+1/2} = \hat{\boldsymbol{F}}_{i+1/2} + \hat{\boldsymbol{\Sigma}}_{i+1/2}[\boldsymbol{G}_i - \frac{1}{2}\{\boldsymbol{I} - \tilde{\boldsymbol{\zeta}}_{i+1/2}\}(\boldsymbol{G}_i - \boldsymbol{G}_{i+1})] \tag{117}$$

where $\boldsymbol{I}$ is the unit matrix, and $\hat{\boldsymbol{F}}_{i\pm1/2}$ denotes the first-order accurate numerical flux of Eq. 68 and Eq. 69.

On the other hand, the finite element approximations for Eq. 109, Eq. 110, Eq. 112 and Eq. 113 are constituted formally as follows :

$$\begin{aligned} \boldsymbol{W}_{i-1/2} &= \boldsymbol{W}_i - \boldsymbol{W}_{i-1} \\ &= \boldsymbol{S}^{-1}_{i-1/2}\{\boldsymbol{U}_i - \boldsymbol{U}_{i-1}\} \end{aligned} \tag{118}$$

$$\boldsymbol{G}_i = \frac{1}{2}\{(\boldsymbol{W}_i - \boldsymbol{W}_{i-1}) + (\boldsymbol{W}_{i+1} - \boldsymbol{W}_i)\} \tag{119}$$

$$\tilde{\boldsymbol{F}}^*_{i\pm1/2} = \boldsymbol{S}_{i\pm1/2}\tilde{\boldsymbol{F}}_{i\pm1/2} \tag{120}$$

$$\boldsymbol{U}_i^{n+1} = \boldsymbol{U}_i^n + \lambda(\tilde{\boldsymbol{F}}_{i-1/2}^{*n} - \tilde{\boldsymbol{F}}_{i+1/2}^{*n}) \tag{121}$$

By substituting Eq. 116 to Eq. 120 into Eq. 121 we obtain the final system of equations as follows :

$$\begin{aligned} \boldsymbol{U}_i^{n+1} &= \boldsymbol{U}_i^n - \lambda \begin{bmatrix} \boldsymbol{S}_{i-1/2}\boldsymbol{\Lambda}_{i-1/2}\tilde{\boldsymbol{\Sigma}}_{i-1/2}\boldsymbol{W}_{i-1/2}^n \\ +\boldsymbol{S}_{i+1/2}\boldsymbol{\Lambda}_{i+1/2}\tilde{\boldsymbol{\Sigma}}_{i+1/2}^*\boldsymbol{W}_{i+1/2}^n \end{bmatrix} \\ &+ \frac{\lambda}{2}\begin{bmatrix} \boldsymbol{S}_{i-1/2}\hat{\boldsymbol{\Sigma}}_{i-1/2}\left\{ \begin{array}{c} \tilde{\boldsymbol{\Sigma}}_{i-1/2}(\boldsymbol{I}+\boldsymbol{r}_{i-1}\dfrac{\boldsymbol{\Lambda}_{i-3/2}}{\boldsymbol{\Lambda}_{i-1/2}})\boldsymbol{W}_{i-3/2}^n \\ +\tilde{\boldsymbol{\Sigma}}_{i-1/2}^*(\boldsymbol{I}+\dfrac{1}{\boldsymbol{r}_i}\dfrac{\boldsymbol{\Lambda}_{i+1/2}}{\boldsymbol{\Lambda}_{i-1/2}})\boldsymbol{W}_{i+1/2}^n \end{array}\right\} \\ -\boldsymbol{S}_{i+1/2}\hat{\boldsymbol{\Sigma}}_{i+1/2}\left\{ \begin{array}{c} \tilde{\boldsymbol{\Sigma}}_{i+1/2}(\boldsymbol{I}+\boldsymbol{r}_{i}\dfrac{\boldsymbol{\Lambda}_{i-1/2}}{\boldsymbol{\Lambda}_{i+1/2}})\boldsymbol{W}_{i-1/2}^n \\ +\tilde{\boldsymbol{\Sigma}}_{i+1/2}^*(\boldsymbol{I}+\dfrac{1}{\boldsymbol{r}_{i+1}}\dfrac{\boldsymbol{\Lambda}_{i+3/2}}{\boldsymbol{\Lambda}_{i+1/2}})\boldsymbol{W}_{i+3/2}^n \end{array}\right\} \end{bmatrix} \end{aligned} \tag{122}$$

3.3.3 Application to Euler equations

For the Euler equations of gasdynamics, $\boldsymbol{U}$ and $\boldsymbol{F}(\boldsymbol{U})$ in Eq. 49 are given as follows:

$$\boldsymbol{U} = \begin{pmatrix} \rho \\ m \\ E \end{pmatrix} \quad , \quad \boldsymbol{F}(\boldsymbol{U}) = \begin{pmatrix} m \\ mu+p \\ Eu+pu \end{pmatrix} \tag{123}$$

where $\rho, u, p = (\kappa-1)(E-mu/2)$, and $E = \rho e$ are the density, the velocity, the pressure, and the total energy, respectively, and $m = \rho u$ is the momentum, and κ denotes the gas constant $(= 1.4)$. The Jacobian matrix of Eq. 51 is given by Harten [Harten (1983)]

$$\boldsymbol{A}(\boldsymbol{U}) = \begin{pmatrix} 0 & 1 & 0 \\ (\bar{\kappa}/2-1)u^2 & (2-\bar{\kappa})u & \bar{\kappa} \\ (\bar{\kappa}u^2-\kappa e)u & \varepsilon-\bar{\kappa}u^2 & \kappa u \end{pmatrix} \tag{124}$$

The set of eigenvectors and the eigenvalues of the above Jacobian matrix are also

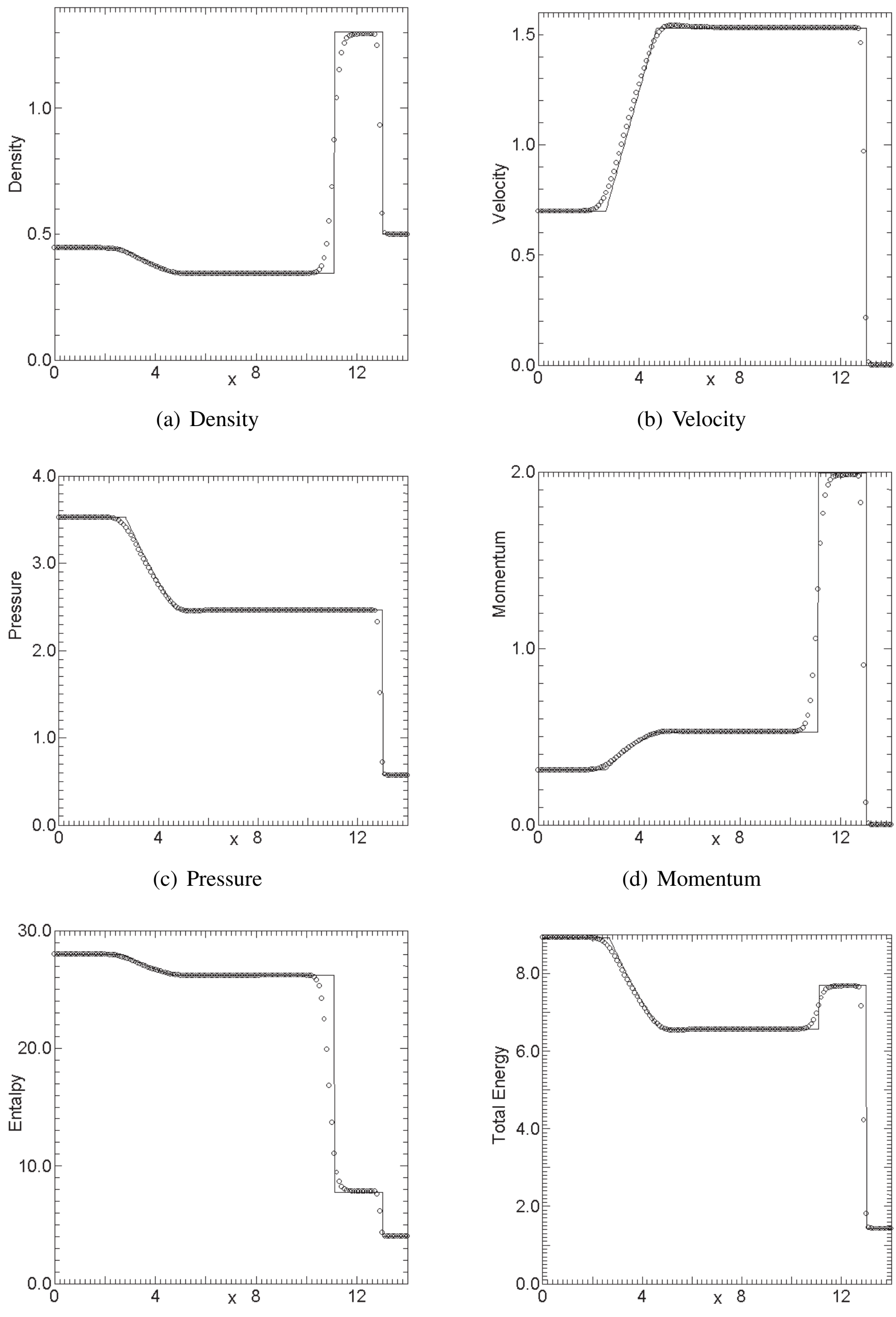
$$\boldsymbol{S} = \begin{pmatrix} 1 & 1 & 1 \\ u-c & u & u+c \\ H-uc & u^2/2 & H+uc \end{pmatrix} \tag{125}$$

$$\boldsymbol{\Lambda} = diag\left(\ u-c, \quad u, \quad u+c\ \right) \tag{126}$$

where $\bar{\kappa} = \kappa - 1$, $\varepsilon = \kappa e - \bar{\kappa}u^2/2$, $H = e + p/\rho$ is the enthalpy, and c is the sound speed, $c = (\kappa p/\rho)^{1/2}$.

Table 1: Comparisons with the present and the exact solutions

x	Density		Velocity		Pressure		Momentum		Entalpy		Total energy	
	Present	Exact	Present	Exact	Present	Exact	Present	Exact	Present	Exact	Present	Exact
1.0	0.4450	0.4450	0.6989	0.6990	3.528	3.528	0.3110	0.3110	27.99	27.99	8.928	8.928
2.0	0.4450	0.4450	0.7014	0.6990	3.524	3.528	0.3119	0.3110	27.98	27.99	8.919	8.928
2.1	0.4443	0.4450	0.7038	0.6990	3.520	3.528	0.3127	0.3110	27.98	27.99	8.911	8.928
2.2	0.4438	0.4450	0.7080	0.6990	3.514	3.528	0.3142	0.3110	27.97	27.99	8.897	8.928
2.3	0.4429	0.4450	0.7148	0.6990	3.504	3.528	0.3166	0.3110	27.95	27.99	8.874	8.928
2.4	0.4415	0.4450	0.7250	0.6990	3.489	3.528	0.3201	0.3110	27.92	27.99	8.839	8.928
2.5	0.4396	0.4450	0.7394	0.6990	3.468	3.528	0.3251	0.3110	27.88	27.99	8.790	8.928
2.6	0.4371	0.4450	0.7585	0.6990	3.440	3.528	0.3315	0.3110	27.84	27.99	8.727	8.928
2.7	0.4340	0.4437	0.7823	0.7089	3.406	3.513	0.3395	0.3145	27.77	27.97	8.648	8.895
2.8	0.4303	0.4382	0.8107	0.7501	3.365	3.453	0.3488	0.3287	27.70	27.86	8.554	8.756
2.9	0.4260	0.4328	0.8433	0.7913	3.319	3.393	0.3593	0.3425	27.62	27.75	8.449	8.619
3.0	0.4214	0.4275	0.8794	0.8325	3.269	3.335	0.3706	0.3559	27.53	27.65	8.335	8.485
3.2	0.4111	0.4169	0.9605	0.9150	3.158	3.220	0.3949	0.3814	27.34	27.45	8.084	8.225
3.4	0.4010	0.4066	1.042	0.9974	3.049	3.109	0.4179	0.4055	27.16	27.26	7.841	7.974
3.6	0.3913	0.3964	1.122	1.080	2.947	3.001	0.4390	0.4280	26.98	27.08	7.613	7.733
3.8	0.3820	0.3865	1.199	1.162	2.849	2.896	0.4582	0.4492	26.82	26.90	7.398	7.501
4.0	0.3732	0.3768	1.275	1.245	2.757	2.794	0.4758	0.4689	26.67	26.73	7.197	7.278
4.1	0.3690	0.3720	1.312	1.286	2.714	2.745	0.4839	0.4783	26.60	26.65	7.102	7.169
4.2	0.3649	0.3672	1.347	1.327	2.672	2.696	0.4916	0.4873	26.54	26.58	7.011	7.063
4.3	0.3610	0.3625	1.382	1.368	2.632	2.648	0.4987	0.4960	26.47	26.50	6.924	6.959
4.4	0.3573	0.3579	1.414	1.409	2.595	2.600	0.5053	0.5044	26.41	26.43	6.844	6.856
4.5	0.3540	0.3533	1.444	1.451	2.561	2.554	0.5111	0.5125	26.36	26.35	6.771	6.756
4.6	0.3510	0.3487	1.471	1.492	2.531	2.508	0.5162	0.5202	26.32	26.28	6.707	6.657
4.7	0.3485	0.3446	1.493	1.529	2.506	2.467	0.5205	0.5270	26.28	26.22	6.653	6.570
4.8	0.3465	0.3446	1.512	1.529	2.486	2.467	0.5238	0.5270	26.25	26.22	6.610	6.570
4.9	0.3451	0.3446	1.525	1.529	2.471	2.467	0.5262	0.5270	26.23	26.22	6.578	6.570
5.0	0.3441	0.3446	1.534	1.529	2.461	2.467	0.5278	0.5270	26.21	26.22	6.558	6.570
6.0	0.3439	0.3446	1.536	1.529	2.459	2.467	0.5282	0.5270	26.21	26.22	6.553	6.570
7.0	0.3442	0.3446	1.533	1.529	2.463	2.467	0.5276	0.5270	26.21	26.22	6.561	6.570
8.0	0.3443	0.3446	1.532	1.529	2.463	2.467	0.5275	0.5270	26.21	26.22	6.562	6.570
9.0	0.3443	0.3446	1.532	1.529	2.463	2.467	0.5274	0.5270	26.22	26.22	6.562	6.570
10.0	0.3443	0.3446	1.532	1.529	2.464	2.467	0.5274	0.5270	26.21	26.22	6.563	6.570
11.0	0.6879	0.3446	1.532	1.529	2.464	2.467	1.054	0.5270	13.71	26.22	6.966	6.570
11.1	0.8729	1.304	1.532	1.529	2.464	2.467	1.337	1.994	11.05	7.789	7.183	7.692
11.5	1.257	1.304	1.532	1.529	2.464	2.467	1.926	1.994	8.031	7.789	7.634	7.692
12.0	1.294	1.304	1.532	1.529	2.464	2.467	1.981	1.994	7.839	7.789	7.677	7.692
12.5	1.295	1.304	1.532	1.529	2.464	2.467	1.983	1.994	7.834	7.789	7.678	7.692
12.6	1.294	1.304	1.531	1.529	2.463	2.467	1.982	1.994	7.834	7.789	7.676	7.692
12.7	1.292	1.304	1.529	1.529	2.457	2.467	1.975	1.994	7.825	7.789	7.652	7.692
12.8	1.249	1.304	1.463	1.529	2.329	2.467	1.827	1.994	7.597	7.789	7.159	7.692
12.9	0.9313	1.304	0.9702	1.529	1.512	2.467	0.9035	1.994	6.151	7.789	4.217	7.692
13.0	0.5824	1.304	0.2153	1.529	0.7203	2.467	0.1254	1.994	4.351	7.789	1.814	7.692
13.1	0.5063	0.5000	0.0159	0.0000	0.5811	0.5710	0.0080	0.0000	4.018	3.997	1.4529	1.4275
13.2	0.5003	0.5000	0.0007	0.0000	0.5715	0.5710	0.0004	0.0000	3.998	3.997	1.4287	1.4275
13.3	0.5000	0.5000	0.0000	0.0000	0.5710	0.5710	0.0000	0.0000	3.997	3.997	1.4275	1.4275
13.4	0.5000	0.5000	0.0000	0.0000	0.5710	0.5710	0.0000	0.0000	3.997	3.997	1.4275	1.4275
14.0	0.5000	0.5000	0.0000	0.0000	0.5710	0.5710	0.0000	0.0000	3.997	3.997	1.4275	1.4275



(a) Density (b) Velocity

(c) Pressure (d) Momentum

(e) Entalpy (f) Total energy

Figure 2: Numerical results

3.3.4 Numerical example

Let us now consider a Riemann problem, namely the shock-tube problem, for the above Euler system of equations in order to demonstrate the workability and the validity of the present approach. The initial data in a field $\{x|0<x<14\}$ is given as follows :

$$\boldsymbol{U}(x,0)=\begin{cases}\boldsymbol{U}_L & if\ x<8\\ \boldsymbol{U}_R & if\ x>8\end{cases} \tag{127}$$

where $\boldsymbol{U}_L=(0.445,0.311,8.928)^T$, $\boldsymbol{U}_R=(0.5,0.0,1.4275)^T$.
In Fig. 2 and the corresponding data of Tab. 1, we show the numerical results obtained by the scheme of Eq. 122 with the Harten's limiter [Harten (1983)] and the Roe's linearization [Roe (1981)]. The calculations were performed with 100 time steps under the CFL restriction of 0.95. The number of elements is 140. The agreement between the present results and the exact solutions shown by the solid line[Toro (1997)] appears satisfactory.

4 Conclusions

We have presented a high-resolution TVD finite element scheme for solving numerically the hyperbolic systems of conservation laws. The numerical flux was formulated by the weighted integral system using exponential weighting functions. The numerical results for the shock-tube problem demonstrated that the approach was capable of solving accurately and in a stable manner the hyperbolic systems of conservation laws in comparison with the exact solutions.

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