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Fuzzy Optimization of Multivariable Fuzzy Functions

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Abstract: In this paper we define multivariable fuzzy functions (MFF) and corresponding multivariable crisp functions (MCF). Then we give a definition for the maximum value of MFF, which in some cases coincides with the maximum value in Pareto sense. We introduce generalized maximizing and minimizing sets in order to determine the maximum values of MFF. By equating membership functions of a given fuzzy domain set and the corresponding maximizing set, we obtain a curve of equal possibilities. Then we use the method of Lagrange multipliers to solve the resulting nonlinear optimization problem when the membership functions are differentiable. We finally present examples of finding extreme points of MFF.

Keywords: Multivariable Fuzzy Functions, Maximizing and minimizing set, Pareto optimum, Lagrange multipliers, membership function, nonlinear optimization.

1 Introduction

Fuzzy sets were defined by Zadeh(1965). The concept of fuzzy functions was introduced in fuzzy theory by many authors [Sasaki(1993); Demirci(1999); Klir and Yuan(1995); Lee and Lee Kwang(2001); Amrahov and Askerzade(2010);]. Finding extremes of fuzzy functions is important in practice. In early research, fuzzy extremes of single variable crisp functions defined on fuzzy domain were investigated. Maximizing and minimizing sets were introduced for the case of single variable functions [Lee Kwang(2005)]. In this paper we are interested in extremes of multivariable fuzzy functions (MFF). We tackle the problem by extending the concept of maximizing and minimizing sets to the multivariable case.

The paper is organized as follows: In Section 2, we give a precise definition for MFF and corresponding multivariable crisp functions (MCF). In Section 3, we introduce maximizing and minimizing sets for MFF. In Section 4, we consider the curve of equal possibilities by equating the membership functions of fuzzy domain sets and corresponding maximizing sets. On such curves, we define the maximum value of MFF. Then in some sample cases, we discuss whether the maximum value

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defined as such coincides with the Pareto maximum. We then apply the method of Lagrange multipliers to solve the arising nonlinear optimization problem with constraints if the membership functions have partial derivatives. We finally provide examples demonstrating the method. In Section 5, we give suggestions regarding further research on the topic.

2 Multivariable Fuzzy Functions (MFF)

Let D be a fuzzy subset of R^n with a membership function μ_D and let f be a real-valued function defined on D. Then we say f is a multivariable fuzzy function (MFF).

Example1. Consider the fuzzy set

$$D = \left\{x = (x_1, x_2) | x_1^2 + x_2^2 \leq 1, x = (x_1, x_2) \in R^2\right\} \tag{1}$$

with the membership function

$$\mu_D(x) = \mu_D(x_1, x_2) = \begin{cases} x_1^2 + x_2^2, & \text{if } (x_1, x_2) \in D \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

and the function

$$f(x) = f(x_1, x_2) = 3x_1 + 4x_2 \tag{3}$$

defined on D. Then f is an MFF.

Now consider the function g defined on the set $X = D$ with the membership function

$$\mu_X(x) = \begin{cases} 1, & \text{if } x \in X \\ 0, & \text{if } x \notin X \end{cases} \tag{4}$$

Note that X is a crisp set. If for all $x \in X$ the equality $g(x) = f(x)$ holds then the function g is called the multivariable crisp function (MCF) corresponding to f and is denoted by $\bar{f}$.

Example2. Consider the function

$$g(x) = g(x_1, x_2) = 3x_1 + 4x_2 \tag{5}$$

defined on the set $X = \left\{x = (x_1, x_2) | x_1^2 + x_2^2 \leq 1, x = (x_1, x_2) \in R^2\right\}$

Then g is the MCF for f from example 1.

3 Maximizing and Minimizing Sets

Let f be an MFF defined on a fuzzy set $D \subset R^n$ with the membership function μ_D and let $\bar{f}$ be MCF corresponding to f. Suppose that $\bar{f}$ has finite infimum and supremum. Put $b = \sup(\bar{f})$ and $a = \inf(\bar{f})$ respectively. The maximizing set M of f is defined as a fuzzy set with membership function

$$\mu_M(x) = \begin{cases} \frac{\bar{f}(x)-a}{b-a}, & \text{if } x \in M \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

Minimizing set of f is defined as the maximizing set of $-f$.

Example 3. For the function in example 2, by the Cauchy-Schwarz inequality we have

$$|3x_1 + 4x_2| \leq \sqrt{3^2+4^2}\sqrt{x_1^2+x_2^2} \leq 5 \tag{7}$$

From here and taking into account that $f(-0.6,\ -0.8) = -5$ and $f(0.6,\ 0.8) = 5$ we have $a = -5,\ b = 5,\ \mu_M(x) = \frac{3x_1+4x_2+5}{10}$.

4 Fuzzy optimization

Firstly, we will consider the maximum value of $\bar{f}$ on a crisp domain. Assume that $x^0 = (x_1^0, x_2^0, \ldots x_n^0)$ is the point at which $\bar{f}$ attain the maximum value on a crisp domain $X \subset R^n$. Hence $x^0 = (x_1^0, x_2^0, \ldots x_n^0)$ gives $\mu_M(x_1, x_2, \ldots x_n)$ its maximum value. In other words,

$\mu_M(x_1^0, x_2^0, \ldots x_n^0) \geq \mu_M(x_1, x_2, \ldots x_n)$ for all $x = (x_1, x_2, \ldots x_n) \in X$,

where $\mu_M(x)$ is the membership function of the maximizing set.

Therefore we have

$$\mu_M(x^0) = \max_{x \in X} \mu_M(x) = \max_{x \in R^n} \min[\mu_M(x), \mu_X(x)] \tag{8}$$

Here

$$\mu_X(x) = \begin{cases} 1, & \text{if } x \in X \\ 0, & \text{if } x \notin X \end{cases} \tag{9}$$

Now we can consider optimizing f on a fuzzy set $D \subset R^n$ with the membership function μ_D. Similar to the crisp case, we can maximize the function

$$\mu_{M \cap D} = \min\left[\mu_M, \mu_D\right] \tag{10}$$

Let us define

$$S = \{s \in D | \mu_{M\cap D}(x) \leq \mu_{M\cap D}(s), \forall x \in D\} \tag{11}$$

and

$$S' = \{x \in R^n | \mu_M(x) = \mu_D(x)\} \tag{12}$$

We define the maximum value of MFF f in the fuzzy set D as the maximum value of the MCF $\bar{f}$ in crisp set S'. If $S \subset S'$, that f attains its maximum at $x = x^0$ means the following:

There does not exist an $x \in D$ that satisfies both of the following inequalities such that at least one of them is strict.

$$\bar{f}(x) \geq \bar{f}(x^0) \tag{13}$$

$$\mu_D(x) \geq \mu_D(x^0) \tag{14}$$

This actually means that in this case the maximum value we defined is the same as the maximum value in Pareto sense. We note that in many cases we can expect $S \subset S'$. The following graph of single-variable functions demonstrates that.

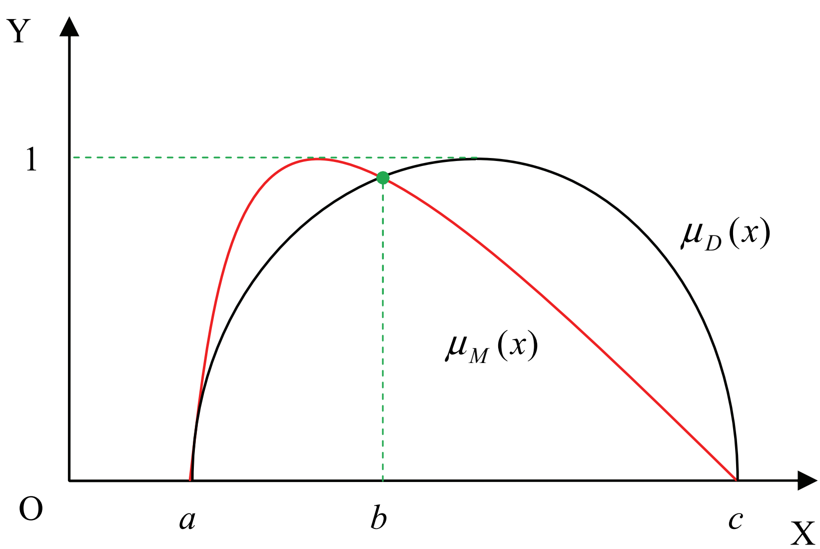

Figure 1: Some membership functions $\mu_D(x)$ and $\mu_M(x)$.

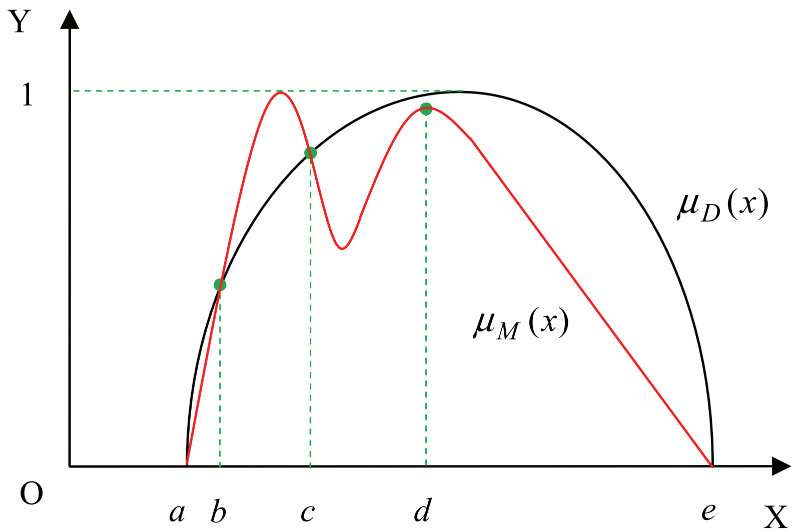

Figure 2: Some other membership functions $\mu_D(x)$ and $\mu_M(x)$.

Note, that in Fig.1, $S = \{b\}$; $\quad S' = \{a,\ b,\ c\}$.
But it is not always the case that $S \subset S'$ as shown in the graph below. (see Fig.2)
Note, that in Fig. 2, $S = \{d\}$; $\quad S' = \{a,\ b,\ c,\ e\}$
We notice that the maximum value here is not the same as the Pareto maximum. In fact, in this case it would be preferable to maximize the function $\mu_{M\cap D} = \min[\mu_M, \mu_D]$. In other words, in order to obtain the maximum of f in Pareto sense, we need to maximize

$$\mu_{M\cap D}(x) = \min[\mu_M(x), \mu_D(x)] = (\mu_M(x) + \mu_D(x) - |\mu_M(x) - \mu_D(x)|)/2 \tag{15}$$

In this paper, we confine our attention to the case $S \subset S'$. Hence we have an optimization problem with constraints. We are supposed to find the maximum value of $\bar{f}$ under the constraint

$$\mu_M(x) = \mu_D(x) \tag{16}$$

We can use the method of Lagrange multipliers [Giordano, Weir and Fox (2003)]. We introduce a new variable λ, so-called the Lagrange multiplier, and study the Lagrange function defined by

$$L(x,\lambda) = \bar{f}(x) + \lambda(\mu_M(x) - \mu_D(x)) \tag{17}$$

If x is an extreme for the original problem, then there exists a λ such that (x,λ) is a stationary point for the Lagrange function (stationary points are those where the partial derivatives of L vanish).

Theorem. Let f be an MFF defined on a fuzzy set $D \subset R^n$ with differentiable membership function μ_D and let $\bar{f}$ be the MCF corresponding to f defined on crisp set $X = D \subset R^n$. Further, suppose that $\bar{f}$ is differentiable and bounded. Let $(x_1^0, x_2^0, ...x_n^0, \lambda^0)$ be a stationary point for the Lagrange function and let the largest and the smallest values of $\bar{f}$ be $b = \sup(\bar{f})$, $a = \inf(\bar{f})$. Then the following equalities hold:

$$\frac{\partial \bar{f}(x_1^0, x_2^0, ...x_n^0)}{\partial x_i}\left(1+\frac{\lambda^0}{b-a}\right) = \lambda^0 \frac{\partial \mu_D(x_1^0, x_2^0, ...x_n^0)}{\partial x_i}, i = 1, 2, ...n \tag{18}$$

and

$$\mu_M(x_1^0, x_2^0, ...x_n^0) = \mu_D(x_1^0, x_2^0, ...x_n^0) \tag{19}$$

Proof. According to the methods of Lagrange multipliers, at point $(x_1^0, x_2^0, ...x_n^0, \lambda^0)$ partial derivatives of $L(x,\lambda) = \bar{f}(x) + \lambda(\mu_M(x) - \mu_D(x))$ vanish. Then for each $i \in \{1, 2, ...n\}$
we have

$$\frac{\partial \bar{f}}{\partial x_i} + \lambda \frac{\partial \mu_M}{\partial x_i} - \lambda \frac{\partial \mu_D}{\partial x_i} = 0 \tag{20}$$

at point $(x_1^0, x_2^0, ...x_n^0, \lambda^0)$. Then from (6) and (20) the equality

$$\frac{\partial \bar{f}}{\partial x_i} + \frac{\lambda}{b-a}\frac{\partial \bar{f}}{\partial x_i} - \lambda \frac{\partial \mu_D}{\partial x_i} = 0 \tag{21}$$

holds at point $(x_1^0, x_2^0, ...x_n^0, \lambda^0)$. Thus we have (18). Finally, the equality (19) holds because of the condition $\frac{\partial L}{\partial \lambda} = 0$ at the point $(x_1^0, x_2^0, ...x_n^0, \lambda^0)$.

Corollary. Let $(x_1^0, x_2^0, ...x_n^0, \lambda^0)$ be a stationary point for the Lagrange function. Suppose that for all $i \in \{1, 2, ..., n\}$

$$\frac{\partial \mu_D(x_1^0, x_2^0, ...x_n^0)}{\partial x_i} \neq 0 \tag{22}$$

Then for all $i \neq j$ and $i, j \in \{1, 2, ..., n\}$

$$\frac{\frac{\partial \bar{f}}{\partial x_i}(x_1^0, x_2^0, ...x_n^0)}{\frac{\partial \mu_D}{\partial x_i}(x_1^0, x_2^0, ...x_n^0)} = \frac{\frac{\partial \bar{f}}{\partial x_j}(x_1^0, x_2^0, ...x_n^0)}{\frac{\partial \mu_D}{\partial x_j}(x_1^0, x_2^0, ...x_n^0)} \tag{23}$$

Proof.

Under (22) from the equalities (18) we have

$$\frac{\frac{\partial \bar{f}(x_1^0, x_2^0, \ldots x_n^0)}{\partial x_i}}{\frac{\partial \mu_D(x_1^0, x_2^0, \ldots x_n^0)}{\partial x_i}} = \frac{\lambda^0(b-a)}{b-a+\lambda^0} \tag{24}$$

which proves the equality (23) for all $i \neq j, i, j \in \{1, 2, \ldots n\}$

Example 4. Find the maximum value of the MFF $f(x_1, x_2) = 1 + x_1 x_2$ defined on the fuzzy set $D = \left\{(x_1, x_2) \in R^2 | 0 \le x_1 \le 2, 0 \le x_2 \le 2\right\}$ with the membership function

$$\mu_D(x_1, x_2) = \begin{cases} \frac{-x_1^2 + 2x_1 - x_2^2 + 2x_2}{2}, & \text{if } (x_1, x_2) \in D \\ 0, & \text{otherwise} \end{cases} \tag{25}$$

Solution.

Consider the MCF $\bar{f}(x_1, x_2)$. We have $a = \inf(\bar{f}) = 1$ and $b = \sup(\bar{f}) = 5$. Then $\mu_M(x_1, x_2) = \frac{x_1 x_2}{4}$. If the $\frac{\partial \mu_D}{\partial x_i} \neq 0$ for $i = 1, 2$ from the (23) we have $\frac{x_2}{1-x_1} = \frac{x_1}{1-x_2}$. Therefore $x_1 = x_2$ or $x_1 + x_2 = 1$. If $x_1 = x_2$ from the condition (19) we have $-2x_1^2 + 4x_1 - 2x_2^2 + 4x_2 = x_1 x_2$. Hence $8x_1 = 5x_1^2$. For $x_1 = 0$ the function $\bar{f}$ attains the minimum value. Then $x_1 = x_2 = \frac{8}{5}$. Therefore $f(x_1, x_2) = 1 + x_1 x_2$ has the value $\frac{89}{25}$ with the possibility $\frac{16}{25}$. If $x_1 + x_2 = 1$ from the condition (19) we have $3x_2^2 - 3x_2 - 2 = 0$. But in this case, either x_1 or x_2 is negative, that is outside of D. If $\frac{\partial \mu_D}{\partial x_1} = -x_1 + 1 = 0$ then from (18) we have $\frac{\partial \bar{f}}{\partial x_1} = x_2 = 0$ or $\lambda = a - b = -4$. In the first case $x_1 = 1, x_2 = 0, \bar{f}(1, 0) = 1$ is the minimum value. In second case again from (18) we have $\frac{\partial \mu_D}{\partial x_2} = -x_2 + 1 = 0$ and $\bar{f}(1, 1) = 2$. But in this case, $\mu_M(1, 1) = \frac{1}{4} \neq 1 = \mu_D(1, 1)$.The case in which $\frac{\partial \mu_D}{\partial x_2} = 0$ is handled in a similar fashion. Therefore f has its maximum value $\frac{89}{25}$ at the point $\left(\frac{8}{5}, \frac{8}{5}\right)$ with the possibility $\frac{16}{25}$.

Example5. Let D be defined as $D = \left\{(x_1, x_2) \in R^2 | 0 \le x_1 \le 1,\ 0 \le x_2 \le 1\right\}$. Consider the following example.

$$f(x) = f(x_1, x_2) = 2x_1 + 4x_2 + 5, \quad (x_1, x_2) \in D \tag{26}$$

$$\mu_D(x) = \mu_D(x_1, x_2) = \begin{cases} \frac{x_1^2 + x_2^2}{2}, & \text{if } (x_1, x_2) \in D \\ 0, & \text{otherwise} \end{cases} \tag{27}$$

Solution.

We find the membership function of maximizing set as $\mu_M(x) = \frac{2x_1+4x_2+5-5}{11-5} = \frac{1}{3}x_1 + \frac{2}{3}x_2$. From the following equation $\mu_M(x) = \mu_D(x)$ for $0 \le x_1 \le 1, 0 \le x_2 \le 1$ we have $\frac{x_1^2+x_2^2}{2} = \frac{1}{3}x_1 + \frac{2}{3}x_2$. Hence $x_1^2 + x_2^2 = \frac{2}{3}x_1 + \frac{4}{3}x_2$. Therefore $\left(x_1 - \frac{1}{3}\right)^2 + \left(x_2 - \frac{2}{3}\right)^2 = \frac{5}{9}$. Hence the curve of equal possibilities is the circle with radius $\frac{\sqrt{5}}{3}$ and centered at the point$\left(\frac{1}{3}, \frac{2}{3}\right)$.

The intersection of this circle with the set D is shown in the Fig. 3. The maximum value of the function is 11 and the function attains this value at the point (1,1) with the possibility 1.

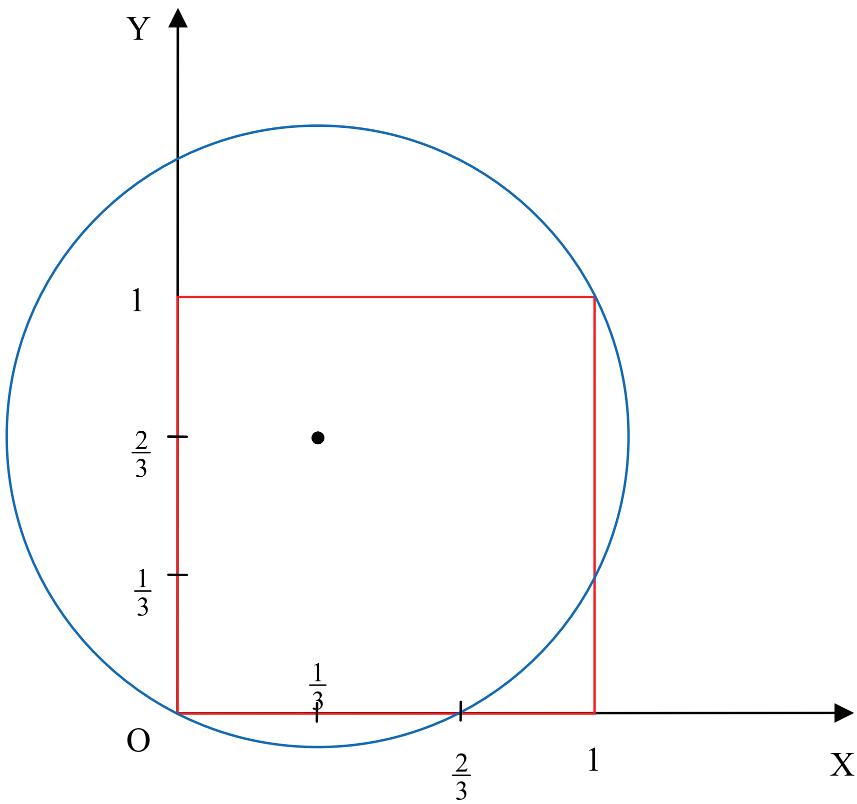

Figure 3: Intersection the curve of equal possibilities with the domain set

5 Conclusions

In conclusion, in this paper we have introduced the concept of maximum point for MFF. This concept in some cases coincides with the concept of maximum point in Pareto sense. We have shown how to determine the maximum values of partially differentiable MFF using the method of Lagrange multipliers. We demonstrated the suggested method on examples. In the future, one can investigate when the maximum point defined in this paper is equivalent to the Pareto maximum. In addition, one can look into the problem of maximizing (15) when they are not equivalent.

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