

Development of Intrinsic Formulation of W.-Z. Chien of the Geometrically Nonlinear Theory of Thin Elastic Shells

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Abstract: Chien Wei-Zhang (1944) derived three equilibrium equations and three compatibility conditions of the nonlinear theory of thin, isotropic elastic shells entirely in terms of the surface stress and strain measures associated with the shell base surface. This allowed Him to divide the complex boundary value problem (BVP) of nonlinear shell analysis into two disjoint and supposedly simpler steps: I) finding the surface stress and strain measures from the intrinsic BVP, and II) establishing position in space of the deformed base surface from already known surface strain measures. In the present paper some achievements of this formulation obtained during the last 66 years are reviewed, with special account of the results obtained by the author.

In the first part, using the error of the constitutive equations, we remind some consistent intrinsic BVPs proposed in the literature. These are, in particular: 1) the intrinsic BVP in terms of the surface strain measures, 2) the refined intrinsic BVP in terms of the surface stress resultants and the surface bendings, 3) the almost inextensional bending BVP, 4) the almost membrane BVP, and 5) the intrinsic bending BVP reduced to two PDE for the stress and deformation functions. The alternative set of the refined intrinsic shell equations formulated in the rotated surface base is also presented. All discussed BVPs contain the corresponding natural intrinsic and deformational boundary conditions.

In the second part, recent achievements on determination of position in space of the deformed shell base surface from the surface strains and bendings are reviewed. Three methods of finding such a position are presented: a) by direct solving some vector ODE, b) through establishing the surface deformation gradient, and c) applying the right polar decomposition to the deformation gradient and then solving some ODE for the rotation tensor field.

Finally, we briefly discuss some problems related to the intrinsic formulation of thin shell theory. These are: i) determination of position of a surface in space from components of two fundamental forms, ii) bifurcation buckling of the axially com-

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pressed circular cylinder, iii) determination of position of the deformed shell base surface from the surface strains and a height function, and iv) basic assumptions of the special class of flexible shells.

In conclusion, two open problems of the intrinsic nonlinear theory of shells are pointed out.

Keywords: thin shell, nonlinear theory, intrinsic formulation, equilibrium equations, compatibility conditions, position of surface.

1 Introduction

The term “intrinsic theory of shells” was introduced by Synge and Chien (1941) for the formulation of nonlinear boundary value problem (BVP) of thin, isotropic elastic shells expressed in terms of two-dimensional (2D) strains and bendings of the shell base surface alone. In this way the solution process of the nonlinear shell BVP was divided into two steps, in which the solution of intrinsic shell equations for the surface strains and bendings was disjoined from finding translations of the shell base surface. Roots of such formulation within the linear shell theory can be traced back to the paper by Reissner (1912) on the spherical shell and to Lur’e (1940), who formulated the equilibrium equations and compatibility conditions in the invariant tensor notation.

In the following paper Chien (1944) expanded all 3D fields of nonlinear elasticity into the normal coordinate and used order-of-magnitude estimates valid under assumptions of small 3D strains and small shell thickness. As a result, consistently approximated three non-linear equilibrium equations and three compatibility conditions were derived in terms of the surface strains and bendings of the base surface alone. These two sets of equations were then considered by Chien (1944) on an equal footing. Under additional assumptions about orders of curvatures, strains and bendings of the shell midsurface relative to the small shell thickness, 35 types of simplified approximate versions of the intrinsic shell equations and 12 simplified versions of intrinsic plate equations were given. Some of them became linear, but some remained nonlinear in the surface strain measures. This formal mathematical classification revealed that the behavior of thin shell structures may be governed by a variety of systems of PDE depending on the type of shell problem under consideration.

Generality of the intrinsic formulation of the nonlinear theory of shells gained considerable attention among leading shell specialists of those post-war times. But it was soon recognized that only some shell problems can be formulated and solved directly in the intrinsic form, primarily because the corresponding intrinsic boundary conditions were not formulated by Chien (1944). In engineering applications

it was difficult to predict in advance the orders of 2D strains and bendings in the whole shell region and to choose correctly one of 35 approximate versions of the shell equations most appropriate for the solution of problem at hands. Moreover, Goldenveiser and Lur'e (1947) noted that when the shell thickness tends to zero some 3D fields in the limit may change their orders upon surface differentiation (for example, in the boundary zone or at asymptotic lines of the midsurface). Such an assumption was not used by Chien (1944). Mushtari (1949a) proposed simpler classification using notions of the small or medium bending and the small, moderate, or finite curvature of the midsurface, which led to only six versions of nonlinear intrinsic shell equations more understandable to the engineering shell community. Mushtari (1949a,b) also proposed intrinsic relations describing the nonlinear edge effect in thin shells under small and large bending. Finally, many shell problems require the translations of the shell midsurface as the final outcome of the solution, not only the strains and bendings. But the problem how to recover the translation field from known surface strains and bendings was not discussed by Chien (1944) as well.

In the present paper we review and summarize some achievements of intrinsic formulation of the nonlinear theory of thin, isotropic elastic shells which have been obtained during the last 66 years, with special reference to the results obtained by the author.

2 Notation and basic shell relations

Let \mathcal{P} be a region of the three-dimensional (3D) Euclidean point space \mathcal{E} occupied by the shell in its undeformed configuration. In \mathcal{P} we introduce the normal system of curvilinear coordinates (θ^α, ζ) , $\alpha = 1, 2$, such that $-h/2 \leq \zeta \leq h/2$ is the distance from the shell middle surface \mathcal{M} to points in \mathcal{P} , and h is the undeformed shell thickness assumed here to be constant. The surface \mathcal{M} is described by the position vector $\mathbf{r} = \mathbf{r}(\theta^\alpha)$ relative to a point $O \in \mathcal{E}$. With each point $M \in \mathcal{M}$ we associate the natural covariant base vectors $\mathbf{a}_\alpha = \partial/\partial\theta^\alpha(\mathbf{r}) \equiv \mathbf{r}_{,\alpha}$, the covariant components $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ of the surface metric tensor \mathbf{a} with $a = \det(a_{\alpha\beta}) > 0$, the contravariant components $\boldsymbol{\varepsilon}^{\alpha\beta}$ of the surface permutation tensor $\boldsymbol{\varepsilon}$ such that $\boldsymbol{\varepsilon}^{12} = -\boldsymbol{\varepsilon}^{21} = 1/\sqrt{a}$, $\boldsymbol{\varepsilon}^{11} = \boldsymbol{\varepsilon}^{22} = 0$, the unit normal vector $\mathbf{n} = 1/2\boldsymbol{\varepsilon}^{\alpha\beta}\mathbf{a}_\alpha \times \mathbf{a}_\beta$ orienting \mathcal{M} , and the covariant components $b_{\alpha\beta} = -\mathbf{a}_\alpha \cdot \mathbf{n}_{,\beta}$ of the surface curvature tensor \mathbf{b} . The contravariant components $a^{\alpha\beta}$ of \mathbf{a} satisfying the relations $a^{\alpha\gamma}a_{\beta\gamma} = \delta_\beta^\alpha$ are used to raise indices of components of the surface vectors and tensors.

The boundary contour $\partial\mathcal{M}$ of \mathcal{M} consists of a finite number of piecewise smooth Jordan curves given by $\mathbf{r}(s) = \mathbf{r}^\alpha[\theta(s)]$, where s is the arc length along $\partial\mathcal{M}$. With each regular point $M \in \partial\mathcal{M}$ we associate the unit tangent vector $\boldsymbol{\tau} = d\mathbf{r}/ds \equiv$

$\mathbf{r}' = \tau^\alpha \mathbf{a}_\alpha$ and the outward unit normal vector $\mathbf{v} \equiv \mathbf{r}_{, \nu} = \boldsymbol{\tau} \times \mathbf{n} = \nu^\alpha \mathbf{a}_\alpha$. For other geometric definitions and relations we refer to Eisenhart (1947); Green and Zerna (1968); Chernykh (1964); Pietraszkiewicz (1977,1980a).

The deformed configuration $\overline{\mathcal{M}}$ of the surface \mathcal{M} is described by the position vector $\overline{\mathbf{r}}(\theta^\alpha) = \chi[\mathbf{r}(\theta^\alpha)] = \mathbf{r}(\theta^\alpha) + \mathbf{u}(\theta^\alpha)$ relative to the same point $O \in \mathcal{E}$, where θ^α are the same surface curvilinear convected coordinates, and \mathbf{u} is the translation field. In the convected coordinates all geometric quantities and relations on the deformed surface $\overline{\mathcal{M}}$ are defined analogously as their counterparts in the undeformed configuration; they will be marked here by an additional dash, for example $\overline{\mathbf{a}}_\alpha, \overline{a}^{\alpha\beta}, \overline{b}^{\alpha\beta}, \overline{\mathbf{n}}, \overline{\mathbf{v}}, \boldsymbol{\tau}$, etc. The dashed quantities on $\overline{\mathcal{M}}$ can be expressed through analogous undashed quantities defined on \mathcal{M} and the translation field \mathbf{u} with the help of formulae given, for example, in Pietraszkiewicz (1980a,1989).

The deformation of the shell base surface from \mathcal{M} to $\overline{\mathcal{M}}$ is described by the surface Green type strain tensor $\boldsymbol{\gamma}$ and the bending tensor $\boldsymbol{\kappa}$ defined through their covariant components by

$$\gamma_{\alpha\beta} = \frac{1}{2} (\overline{a}_{\alpha\beta} - a_{\alpha\beta}) , \quad \kappa_{\alpha\beta} = - (\overline{b}_{\alpha\beta} - b_{\alpha\beta}) . \tag{1}$$

The surface strain measures (1) have to satisfy the following exact compatibility conditions, see Chien (1944); Galimov (1953); Koiter (1966):

$$\begin{aligned} \varepsilon^{\alpha\beta} \varepsilon^{\lambda\mu} [\kappa_{\beta\lambda|\mu} + \overline{a}^{\kappa\rho} (b_{\kappa\lambda} - \kappa_{\kappa\lambda}) \gamma_{\rho\beta\mu}] &= 0 , \\ \varepsilon^{\alpha\beta} \varepsilon^{\lambda\mu} \left[\gamma_{\alpha\mu|\beta\lambda} - b_{\alpha\mu} \kappa_{\beta\lambda} + \frac{1}{2} (\kappa_{\alpha\mu} \kappa_{\beta\lambda} + \overline{a}^{\kappa\rho} \gamma_{\kappa\alpha\mu} \gamma_{\rho\beta\lambda}) \right] + K \gamma_\kappa^\kappa &= 0 , \end{aligned} \tag{2}$$

where K is the Gaussian curvature of \mathcal{M} , $(\cdot)_{|\alpha}$ is the covariant derivative in the metric $a_{\alpha\beta}$, and $\gamma_{\rho\beta\mu} = \gamma_{\rho\beta|\mu} + \gamma_{\rho\mu|\beta} - \gamma_{\beta\mu|\rho}$.

Applying order estimates of many small terms, Chien (1944) confirmed the assumption used already by Aron (1874) that within the geometrically nonlinear range of deformation the behavior of an interior domain of a thin elastic shell can be described with a sufficient accuracy by the behavior of the shell middle surface alone. This physically well understood property of shell deformation was used in many later papers to formulate various versions of the nonlinear theory of thin elastic shells, see for example Galimov (1951); Sanders (1963); Koiter (1966); Pietraszkiewicz (1974,1980a,1984), and the survey by Pietraszkiewicz (1989).

Let the base surface $\overline{\mathcal{M}}$ of a deformed shell in an equilibrium state be loaded by the surface force $\mathbf{p}(\theta^\alpha) = p^\alpha \overline{\mathbf{a}}_\alpha + p \overline{\mathbf{n}}$ and static moment $\mathbf{h} = m^\alpha \overline{\mathbf{a}}_\alpha$ vectors, both per unit area of $\overline{\mathcal{M}}$, and by the boundary force $\mathbf{N}^*(s) = N_\nu^* \overline{\mathbf{v}} + N_\tau^* \boldsymbol{\tau} + N^* \overline{\mathbf{n}}$ and static moment $\mathbf{H}^*(s) = M_\nu^* \overline{\mathbf{v}} + M_\tau^* \boldsymbol{\tau}$ vectors, both per unit length of $\partial \overline{\mathcal{M}}_f$. Then for all

kinematically admissible virtual translations $\delta \mathbf{u}$ the equilibrium conditions for $\overline{\mathcal{M}}$ are given by the principle of virtual work (PVW),

$$\begin{aligned} & \iint_{\overline{\mathcal{M}}} \left(N^{\alpha\beta} \delta\gamma_{\alpha\beta} + M^{\alpha\beta} \delta\kappa_{\alpha\beta} \right) dA \\ &= \iint_{\overline{\mathcal{M}}} (\mathbf{p} \cdot \delta \mathbf{u} + \mathbf{h} \cdot \delta \bar{\mathbf{n}}) dA + \int_{\partial \mathcal{M}_f} (\mathbf{N}^* \cdot \delta \mathbf{u} + \mathbf{H}^* \cdot \delta \bar{\mathbf{n}}) ds. \end{aligned} \quad (3)$$

Here $N^{\alpha\beta}$ and $M^{\alpha\beta}$ are components of the symmetric surface stress resultants and couples of the Kirchhoff type, while $\delta\gamma_{\alpha\beta}$ and $\delta\kappa_{\alpha\beta}$ are virtual changes of the surface strain measures (1).

The principle (3) can be transformed with the help of Stokes' theorem in \mathcal{M} and integration by parts along $\partial \mathcal{M}$, which leads to, see Galimov (1951); Pietraszkiewicz (1989),

$$\begin{aligned} & - \iint_{\overline{\mathcal{M}}} \left(\mathbf{T}^\beta |_\beta + \mathbf{p} + (m^\beta \bar{\mathbf{n}}) |_\beta \right) \cdot \delta \mathbf{u} dA + \sum_{M_n \in \partial \mathcal{M}_f} (\Phi_n - \Phi_n^*) \cdot \delta \mathbf{u}_n \\ & + \int_{\partial \mathcal{M}_f} \left[\left(\mathbf{T}^\beta \mathbf{v}_\beta + \Phi' + m^\beta \mathbf{v}_\beta \bar{\mathbf{n}} - \mathbf{N}^* - \Phi'^* \right) \cdot \delta \mathbf{u} + (K - M_v^*) \bar{\mathbf{v}} \cdot \delta \bar{\mathbf{n}} \right] ds = 0, \end{aligned} \quad (4)$$

where

$$\begin{aligned} \mathbf{T}^\beta &= \left(N^{\alpha\beta} - \bar{b}_\lambda^\alpha M^{\lambda\beta} \right) \bar{\mathbf{a}}_\alpha + \left(M^{\alpha\beta} |_\alpha + \bar{a}^{\beta\kappa} \gamma_{\kappa\lambda\mu} M^{\lambda\mu} \right) \bar{\mathbf{n}}, \\ \Phi &= \frac{1}{a_\tau^2} \mathbf{v}_\alpha M^{\alpha\beta} \bar{a}_{\beta\lambda} \boldsymbol{\tau}^\lambda \bar{\mathbf{n}}, \quad \Phi^* = \frac{1}{a_\tau} M_\tau^* \bar{\mathbf{n}}, \quad K = \frac{1}{a_\tau} \sqrt{\frac{\bar{a}}{a}} \mathbf{v}_\alpha M^{\alpha\beta} \mathbf{v}_\beta, \\ \Phi_n &= \Phi(s_n + 0) - \Phi(s_n - 0), \quad \delta \mathbf{u}_n = \delta \mathbf{u}(s_n), \\ a_\tau &= \sqrt{1 + 2\gamma_{\tau\tau}}, \quad \gamma_{\tau\tau} = \gamma_{\alpha\beta} \boldsymbol{\tau}^\alpha \boldsymbol{\tau}^\beta. \end{aligned} \quad (5)$$

For any kinematically admissible $\delta \mathbf{u}$ from (4) the following vector equilibrium equations, natural boundary conditions, and natural conditions at the boundary corners can be derived, see Pietraszkiewicz (1989); Opoka and Pietraszkiewicz (2004):

$$\begin{aligned} \mathbf{T}^\beta |_\beta + \mathbf{p} + (m^\beta \bar{\mathbf{n}}) |_\beta &= \mathbf{0} \quad \text{in } \mathcal{M}, \\ \mathbf{T}^\beta \mathbf{v}_\beta + \Phi' + m^\beta \mathbf{v}_\beta \bar{\mathbf{n}} - \mathbf{N}^* - \Phi'^* &= \mathbf{0}, \quad K - M_v^* = 0 \quad \text{along } \partial \mathcal{M}_f, \\ \Phi_n - \Phi_n^* &= \mathbf{0} \quad \text{at each corner } M_n \in \partial \mathcal{M}_f. \end{aligned} \quad (6)$$

With (5)₁ the equilibrium equations (6)₁, when written in components relative to

the deformed basis $\bar{\mathbf{a}}_\alpha, \bar{\mathbf{n}}$, become

$$\begin{aligned} & \left(N^{\alpha\beta} - \bar{b}_\kappa^\alpha M^{\kappa\beta} \right) |_\beta + \bar{a}^{\alpha\kappa} \gamma_{\kappa\lambda\beta} \left(N^{\lambda\beta} - \bar{b}_\mu^\lambda M^{\mu\beta} \right) \\ & - \bar{b}_\beta^\alpha \left(M^{\lambda\beta} |_\lambda + \bar{a}^{\beta\kappa} \gamma_{\kappa\lambda\mu} M^{\lambda\mu} \right) + p^\alpha - \bar{b}_\beta^\alpha m^\beta = 0, \tag{7} \\ & M^{\alpha\beta} |_{\alpha\beta} + \left(\bar{a}^{\beta\kappa} \gamma_{\kappa\lambda\mu} M^{\lambda\mu} \right) |_\beta + \bar{b}_{\alpha\beta} \left(N^{\alpha\beta} - \bar{b}_\kappa^\alpha M^{\kappa\beta} \right) + p + m^\beta |_\beta = 0. \end{aligned}$$

Please note that the equilibrium equations (7) and components of the natural boundary conditions (6)_{2,3} in the deformed boundary base $\bar{\mathbf{v}}, \bar{\boldsymbol{\tau}}, \bar{\mathbf{n}}$ are expressible entirely through the surface stress and strain measures as well as known load components. It can easily be shown that all 35 versions of the equilibrium equations of Chien (1944) are just appropriately simplified versions of (7).

The natural boundary quantities (6)_{2,3} perform in (4) the virtual work on $\delta \mathbf{u}$ and $\bar{\mathbf{v}} \cdot \delta \bar{\mathbf{n}}$, respectively, not on kinematic quantities expressed through $\delta \gamma_{\alpha\beta}$ and $\delta \kappa_{\alpha\beta}$ alone. Pietraszkiewicz (1980a,b) applied twice the integration by parts to the boundary terms in (4). After transformations given in Pietraszkiewicz (1989) and Opoka and Pietraszkiewicz (2004) the boundary integral in (4) was replaced by

$$\begin{aligned} & - \int_{\partial \mathcal{M}_f} \left[(\mathbf{B} - \mathbf{B}^*) \cdot \boldsymbol{\omega}'_\tau + \frac{1}{a_\tau} (\mathbf{A} - \mathbf{A}^*) \cdot \bar{\boldsymbol{\tau}} \delta \gamma_{\tau\tau} \right] ds \\ & + \sum_{M_n \in \partial \mathcal{M}_f} \{ [(\boldsymbol{\Phi}_n - \boldsymbol{\Phi}_n^*) - (\mathbf{A}_n - \mathbf{A}_n^*)] \cdot \delta \mathbf{u}_n - (\mathbf{B}_n - \mathbf{B}_n^*) \cdot \boldsymbol{\omega}_{\tau n} \}, \tag{8} \end{aligned}$$

where

$$\begin{aligned} \mathbf{A} &= \mathbf{A}_0 + \int_{s_0}^s \left(\mathbf{T}^\beta \mathbf{v}_\beta + \boldsymbol{\Phi}' + m^\beta \mathbf{v}_\beta \bar{\mathbf{n}} \right) ds, \quad \boldsymbol{\omega}_{\tau n} = \boldsymbol{\omega}_\tau(s_n), \\ \mathbf{B}(O) &= \mathbf{B}_0(O) + \int_{s_0}^s \left[\bar{\mathbf{r}} \times \left(\mathbf{T}^\beta \mathbf{v}_\beta + \boldsymbol{\Phi}' + m^\beta \mathbf{v}_\beta \bar{\mathbf{n}} \right) + K \bar{\boldsymbol{\tau}} \right] ds - \bar{\mathbf{r}} \times \mathbf{A}. \tag{9} \end{aligned}$$

In (9), \mathbf{A} and $\mathbf{B}(O)$ are the total force and the total couple about the origin $O \in \mathcal{E}$ of internal interactions along $\partial \mathcal{M}$, while \mathbf{A}_0 and $\mathbf{B}_0(O)$ are their initial values at $s_0 \in \partial \mathcal{M}$, respectively.

In (8), the virtual rotation vector $\boldsymbol{\omega}_\tau$ along $\partial \mathcal{M}$ follows from the relations, see Pietraszkiewicz (1989),

$$\begin{aligned} \mathbf{R}_\tau &= \bar{\mathbf{v}} \otimes \mathbf{v} + \bar{\boldsymbol{\tau}} \otimes \boldsymbol{\tau} + \bar{\mathbf{n}} \otimes \mathbf{n}, \quad \mathbf{R}_\tau^T \mathbf{R}_\tau = \mathbf{k}_\tau \times \mathbf{1}, \quad \delta \mathbf{R}_\tau^T \mathbf{R}_\tau = \boldsymbol{\omega}_\tau \times \mathbf{1}, \\ \mathbf{k}_\tau &= -k_{\tau\tau} \mathbf{v} + k_{v\tau} \boldsymbol{\tau} - k_{n\tau} \mathbf{n}, \quad \boldsymbol{\omega}'_\tau = -\delta k_{\tau\tau} \bar{\mathbf{v}} + \delta k_{v\tau} \bar{\boldsymbol{\tau}} - \delta k_{n\tau} \bar{\mathbf{n}}, \tag{10} \\ k_{v\tau} &= k_{\alpha\beta} \mathbf{v}^\alpha \boldsymbol{\tau}^\beta, \text{ etc.} \end{aligned}$$

The vector \mathbf{k}_τ of change of curvature of the shell boundary contour was first introduced by Novozhilov and Shamina (1975), and with components expressed through physical components of $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ along $\partial\mathcal{M}$ by Pietraszkiewicz (1977,1980a). It is expressed entirely through the surface strain measures. Now the deformational boundary conditions appropriate for the intrinsic shell equations take the form

$$k_{\tau\tau} = k_{\tau\tau}^*, \quad k_{\nu\tau} = k_{\nu\tau}^*, \quad k_{n\tau} = k_{n\tau}^*, \quad \gamma_{\tau\tau} = \gamma_{\tau\tau}^* \quad \text{along } \partial\mathcal{M}_d. \quad (11)$$

All the relations (3) to (11) are two-dimensionally exact for the shell base surface. An alternative intrinsic theory for nonlinear dynamics of double curved shells was proposed by Libai (1981,1983).

3 Constitutive equations of the first-approximation theory of shells

The geometrically nonlinear first-approximation theory of thin elastic shells is applicable when:

1. the shell is made of a homogeneous, isotropic and elastic material;
2. the shell is thin, i.e. $h/R \ll 1$, where h is the constant thickness of the undeformed shell and R is the smallest radius of curvature of its middle surface;
3. the undeformed shell middle surface is smooth, i.e. $(h/l)^2 \ll 1$, where l is the smallest wave length of geometric patterns of \mathcal{M} ;
4. the shell deformation is smooth, i.e. $(h/L)^2 \ll 1$, where L is the smallest wave length of deformation patterns on \mathcal{M} ;
5. the strains are small everywhere, i.e. $\eta \ll 1$, where η is the largest strain in the shell space.

Within the assumptions given above, the strain energy density Σ , per unit area of \mathcal{M} , is given by the sum of two quadratic functions describing the stretching and bending energies associated with the shell middle surface. The accuracy of such an approximation was discussed in a number of papers, among others by Novozhilov and Finkel'shtein (1943) and Koiter (1960), leading to the strain energy density of the form

$$\Sigma = \frac{h}{2} H^{\alpha\beta\lambda\mu} \left(\gamma_{\alpha\beta} \gamma_{\lambda\mu} + \frac{h^2}{12} \kappa_{\alpha\beta} \kappa_{\lambda\mu} \right) + O(Eh\eta^2\theta^2). \quad (12)$$

Here the symbol $O(\dots)$ describes the order of maximal error and

$$H^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)} \left(a^{\alpha\lambda} a^{\beta\mu} + a^{\alpha\mu} a^{\beta\lambda} + \frac{2\nu}{1-\nu} a^{\alpha\beta} a^{\lambda\mu} \right), \quad (13)$$

$$\theta = \max \left(\frac{h}{d}, \frac{h}{l}, \frac{h}{L}, \sqrt{\frac{h}{R}}, \sqrt{\eta} \right),$$

where E is the Young modulus and ν is the Poisson ratio of the isotropic linearly-elastic material, while d is the distance from the lateral shell boundary. From (12) follow the constitutive equations

$$\begin{aligned} N^{\alpha\beta} &= C \left[(1 - \nu) \gamma^{\alpha\beta} + \nu a^{\alpha\beta} \gamma_\lambda^\lambda \right] + O(Eh\eta\theta^2), \quad C = \frac{Eh}{1 - \nu^2}, \\ M^{\alpha\beta} &= D \left[(1 - \nu) \kappa^{\alpha\beta} + \nu a^{\alpha\beta} \gamma_\lambda^\lambda \right] + O(Eh^2\eta\theta^2), \quad D = \frac{Eh^3}{12(1 - \nu^2)}. \end{aligned} \tag{14}$$

The inverse of (14) leads to

$$\begin{aligned} \gamma_{\alpha\beta} &= A \left[(1 + \nu) N_{\alpha\beta} - \nu a_{\alpha\beta} N_\lambda^\lambda \right] + O(\eta\theta^2), \quad A = \frac{1}{Eh}, \\ \kappa_{\alpha\beta} &= \frac{12}{Eh^3} \left[(1 + \nu) M_{\alpha\beta} - \nu a_{\alpha\beta} M_\lambda^\lambda \right] + O\left(\frac{\eta\theta^2}{h}\right). \end{aligned} \tag{15}$$

Within the indicated error of (12) one can modify the bending tensor $\kappa_{\alpha\beta}$ by adding subtracting terms of the order of η/R , for example $b_\alpha^\lambda \gamma_{\lambda\beta}$ or $b_{\alpha\beta} \gamma_\lambda^\lambda$. This allows one to formulate other energetically equivalent versions of the theory of thin elastic shells, see for example Budiansky and Sanders (1963); Koiter (1960,1966); Budiansky (1968); Koiter and Simmonds (1973); Pietraszkiewicz and Szwabowicz (1981); etc.

4 Intrinsic shell equations in terms of the surface strain measures

In the case of bending strain state in the shell, the small strains η caused by stretching and bending of its base surface are regarded to be of comparable order in the interior shell domain, $\gamma_{\alpha\beta} \sim h\kappa_{\alpha\beta} \sim \eta$, where \sim means “of the same order as”. To obtain the shell relations in terms of $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ as suggested by Chien (1944), one has to introduce (14) into (7) and omit in the resulting equilibrium equations as well as in the compatibility conditions (2) all small terms of the order of errors indicated in (14) and (15). Then we obtain the following consistently reduced set

of six intrinsic bending shell equations in \mathcal{M} , see Pietraszkiewicz (1977):

$$\begin{aligned}
 C \left[(1 - \mathbf{v}) \gamma_{\alpha}^{\beta} |_{\beta} + \mathbf{v} \gamma_{\beta}^{\alpha} |_{\alpha} \right] + p_{\alpha} &= O \left(Eh \frac{\eta \theta^2}{\lambda} \right), \\
 D \kappa_{\alpha}^{\alpha} |_{\beta} + C (b_{\beta}^{\alpha} - \kappa_{\beta}^{\alpha}) \left[(1 - \mathbf{v}) \gamma_{\alpha}^{\beta} + \mathbf{v} \delta_{\alpha}^{\beta} \gamma_{\lambda}^{\lambda} \right] + p + m^{\alpha} |_{\alpha} &= O \left(Eh^2 \frac{\eta \theta^2}{\lambda^2} \right), \\
 \kappa_{\alpha}^{\beta} |_{\beta} - \kappa_{\beta}^{\alpha} |_{\alpha} &= O \left(\frac{\eta \theta^2}{h \lambda} \right), \\
 \gamma_{\alpha}^{\beta} |_{\beta} - \gamma_{\alpha}^{\alpha} |_{\beta} - (b_{\alpha}^{\beta} \kappa_{\beta}^{\alpha} - b_{\alpha}^{\alpha} \kappa_{\beta}^{\beta}) + \frac{1}{2} (\kappa_{\alpha}^{\beta} \kappa_{\beta}^{\alpha} - \kappa_{\alpha}^{\alpha} \kappa_{\beta}^{\beta}) &= O \left(\frac{\eta \theta^2}{\lambda^2} \right).
 \end{aligned} \tag{16}$$

In the estimation procedure used above covariant surface derivatives of the surface fields are estimated by dividing their maximal value by a large parameter λ defined by

$$\lambda = \frac{h}{\theta} = \min \left(d, l, L, \sqrt{hR}, \frac{1}{\sqrt{\eta}} \right). \tag{17}$$

Appropriately reduced four natural intrinsic boundary conditions along $\partial \mathcal{M}_f$ take the form, see Pietraszkiewicz (1989),

$$\begin{aligned}
 C (\gamma_{\nu\nu} + \mathbf{v} \gamma_{\tau\tau}) &= N_{\nu}^* + O(Eh\eta\theta^2), \quad C (1 - \mathbf{v}) \gamma_{\nu\tau} = N_{\tau}^* + O(Eh\eta\theta^2), \\
 D [\kappa_{\nu\nu, \nu} + \kappa_{\tau\tau, \nu} + (1 - \mathbf{v}) \kappa'_{\nu\tau}] + m_{\nu} &= N^* + M_{\tau}^* + O \left(Eh^2 \frac{\eta \theta^2}{\lambda} \right), \\
 D (\kappa_{\nu\nu} + \mathbf{v} \kappa_{\tau\tau}) &= M_{\nu}^* + O(Eh^2 \eta \theta^2), \\
 (\cdot)_{, \nu} \equiv \frac{\partial}{\partial \theta^{\alpha}} (\cdot) \mathbf{v}^{\alpha}, \quad \gamma_{\nu\nu} &= \gamma_{\alpha\beta} \mathbf{v}^{\alpha} \mathbf{v}^{\beta}, \quad \kappa_{\nu\tau} = \kappa_{\alpha\beta} \mathbf{v}^{\alpha} \mathbf{v}^{\beta}, \quad \text{etc.},
 \end{aligned} \tag{18}$$

and the corresponding reduced four deformational boundary conditions along $\partial \mathcal{M}_d$ are

$$\begin{aligned}
 \kappa_{\tau\tau} &= k_{\tau\tau}^* + O \left(\frac{\eta \theta^2}{h} \right), \quad \kappa_{\nu\tau} = k_{\nu\tau}^* + O \left(\frac{\eta \theta^2}{h} \right), \\
 2\gamma'_{\nu\tau} - \gamma_{\tau\tau, \nu} + 2\rho_{\nu} \gamma_{\nu\tau} + \rho_{\tau} (\gamma_{\nu\nu} - \gamma_{\tau\tau}) &= k_{n\tau}^* + O \left(\frac{\eta \theta^3}{h} \right), \\
 \gamma_{\tau\tau} &= \gamma_{\tau\tau}^* + O(\eta \theta^2).
 \end{aligned} \tag{19}$$

In (19), $\rho_{\tau} = \boldsymbol{\tau}_{\alpha} \mathbf{v}^{\alpha} |_{\beta} \boldsymbol{\tau}^{\beta}$ and $\rho_{\nu} = \mathbf{v}_{\alpha} \boldsymbol{\tau}^{\alpha} |_{\beta} \mathbf{v}^{\beta}$ are the geodesic curvatures of $\partial \mathcal{M}$ and the surface curve orthogonal to it, respectively.

The resulting set of intrinsic relations (16) to (19) is very simple. Four field equations (16)_{1,3} are linear while the remaining two (16)_{2,4} are quadratic in terms of $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$. All intrinsic boundary conditions (18) and (19) are linear in the surface strain measures.

5 Refined intrinsic shell equations

In some nonlinear problems of shells the small strains caused by membrane stress resultants may be of essentially different order (higher or smaller by the factor θ^2) from those caused by the stress couples. In those cases the tangential intrinsic shell equations (16)_{1,3} should be approximated with a greater accuracy, because within the accuracy indicated in the constitutive equations (14) and (15) these tangential equations contain only terms of one kind: either the surface strains $\gamma_{\alpha\beta}$ or the surface bendings $\kappa_{\alpha\beta}$, respectively. In particular, the use of equations (16) in the buckling problem of long axially compressed circular cylinder leads to overestimated buckling load, see for example Opoka and Pietraszkiewicz (2009b).

The refinement of intrinsic bending shell equations (16) may be achieved by selecting the stress resultants $N^{\alpha\beta}$ and bendings $\kappa_{\alpha\beta}$ as the basic independent field variables of the shell BVP. The estimation procedure was originally proposed by Danielson (1970), and developed by Koiter and Simmonds (1973) with the help of concrete error estimates of some shell fields and their surface derivatives given by John (1965). In the papers by Danielson (1970) and Koiter and Simmonds (1973) alternative energetically modified definitions of stress resultants and bendings were used in order to obtain in the limit the “best” formulation of the linear theory of shells proposed by Budiansky and Sanders (1963). But Pietraszkiewicz (1989) found that the same “best” version of the linear shell theory in the limit may be recovered from the two-dimensionally exact shell relations formulated in the rotated basis, with appropriate modification of all field variables. This modified version of the refined intrinsic shell equations (RISEs) will be concisely presented in section 7.

If $N^{\alpha\beta}$ and $\kappa_{\alpha\beta}$ are chosen as the basic independent field variables, then using results of Pietraszkiewicz (1977,1980a,1989) and Opoka and Pietraszkiewicz (2004)

from (7) and (2) we obtain the following RISEs in \mathcal{M} :

$$\begin{aligned}
& N_\alpha^\beta |_\beta + 2A \left(N_\alpha^\lambda N_\lambda^\beta \right) |_\beta - \frac{1}{2} A \left[(1 - \mathbf{v}) N_\beta^\lambda N_\lambda^\beta + \mathbf{v} N_\lambda^\lambda N_\beta^\beta \right] |_\alpha \\
& - D \left\{ \left(b_\alpha^\lambda - \kappa_\alpha^\lambda \right) \left[(1 - \mathbf{v}) \kappa_\lambda^\beta + \mathbf{v} \delta_\lambda^\beta \kappa_\mu^\mu \right] \right\} |_\beta \\
& - \left(b_\alpha^\beta - \kappa_\alpha^\beta \right) \left(D \kappa_\lambda^\lambda |_\beta + m_\beta \right) + 2A \left[(1 + \nu) N_\alpha^\beta p_\beta - \mathbf{v} N_\nu^\nu p_\alpha \right] + p_\alpha = O \left(Eh \frac{\eta \theta^4}{\lambda} \right), \\
& D \kappa_\alpha^\alpha |_\beta + \left(b_\alpha^\beta - \kappa_\alpha^\beta \right) N_\beta^\alpha + p + m^\alpha |_\alpha = O \left(Eh^2 \frac{\eta \theta^2}{\lambda^2} \right),
\end{aligned} \tag{20}$$

$$\begin{aligned}
& \kappa_\alpha^\beta |_\beta - \kappa_\beta^\beta |_\alpha - A(1 + \mathbf{v}) \left[\left(b_\beta^\lambda - \kappa_\beta^\lambda \right) N_\lambda^\beta |_\alpha + \left(b_\alpha^\beta - \kappa_\alpha^\beta \right) N_\lambda^\lambda |_\alpha \right] \\
& - 2A(1 + \mathbf{v}) \left(b_\alpha^\beta - \kappa_\alpha^\beta \right) p_\beta = O \left(\frac{\eta \theta^4}{h\lambda} \right), \\
& AN_\alpha^\alpha |_\beta + \left(b_\alpha^\beta - \frac{1}{2} \kappa_\alpha^\beta \right) \kappa_\beta^\alpha - \left(b_\alpha^\alpha - \frac{1}{2} \kappa_\alpha^\alpha \right) \kappa_\beta^\beta + A(1 + \mathbf{v}) p^\alpha |_\alpha = O \left(\frac{\eta \theta^2}{\lambda^2} \right).
\end{aligned} \tag{21}$$

Note that now (20) and (21) are all quadratic in the independent field variables. Pietraszkiewicz (1989) and Opoka and Pietraszkiewicz (2004) also derived the appropriately reduced four natural intrinsic boundary conditions along $\partial \mathcal{M}_f$ compatible with (20) and (21) (two quadratic and two linear)

$$\begin{aligned}
& [1 + A(N_{VV} - \mathbf{v} N_{\tau\tau})] N_{VV} - D(\sigma_V - \kappa_{VV})(\kappa_{VV} + \mathbf{v} \kappa_{\tau\tau}) + 2D(1 - \mathbf{v})(\boldsymbol{\tau}_\tau + \kappa_{V\tau}) \kappa_{V\tau} \\
& = N_V^* + (\boldsymbol{\tau}_\tau + \kappa_{V\tau}) M_\tau^* + O(Eh\eta\theta^4), \\
& [1 + A(N_{\tau\tau} - \mathbf{v} N_{VV})] N_{V\tau} + 2A(1 + \mathbf{v}) N_{VV} N_{V\tau} + D(\kappa_{VV} + \mathbf{v} \kappa_{\tau\tau})(\boldsymbol{\tau}_\tau + \kappa_{V\tau}) \\
& - 2D(1 - \mathbf{v})(\sigma_\tau - \kappa_{\tau\tau}) \kappa_{V\tau} = N_\tau^* - (\sigma_\tau - \kappa_{\tau\tau}) M_\tau^* + O(Eh\eta\theta^4), \\
& D[\kappa_{VV,V} + \kappa_{\tau\tau,V} + (1 - \mathbf{v}) \kappa'_{V\tau}] + m_V = N^* + M_\tau^* + O\left(Eh \frac{\eta \theta^2}{\lambda}\right), \\
& D(\kappa_{VV} + \mathbf{v} \kappa_{\tau\tau}) = M_V^* + O(Eh^2 \eta \theta^2),
\end{aligned} \tag{22}$$

and four deformational boundary conditions along $\partial \mathcal{M}_d$ (again two quadratic and

two linear)

$$\begin{aligned} \kappa_{\tau\tau} + A(\sigma_{\tau} - \kappa_{\tau\tau})(N_{\tau\tau} - \mathbf{v}N_{v\tau}) &= k_{\tau\tau}^* + O\left(\frac{\eta\theta^4}{h}\right), \\ \kappa_{v\tau} + 2A(1 + \mathbf{v})(\sigma_{\tau} - \kappa_{\tau\tau})N_{v\tau} - A(\boldsymbol{\tau}_{\tau} + \kappa_{v\tau})(N_{v\tau} - \mathbf{v}N_{\tau\tau}) &= k_{v\tau}^* + O\left(\frac{\eta\theta^4}{h}\right), \\ 2A(1 + \mathbf{v})N'_{v\tau} - A(N_{\tau\tau, v} - \mathbf{v}N_{v\tau, v}) + 2A(1 + \mathbf{v})\rho_v N_{v\tau} + A(1 + \mathbf{v})\rho_{\tau}(N_{v\tau} - N_{\tau\tau}) \\ &= k_{n\tau}^* + O\left(\frac{\eta\theta^3}{h}\right), \quad A(N_{\tau\tau} - \mathbf{v}N_{v\tau}) = \gamma_{\tau\tau}^* + O(\eta\theta^2). \end{aligned} \tag{23}$$

In (22) and (23), $\sigma_{\tau} = \boldsymbol{\tau}_{\alpha} b_{\beta}^{\alpha} \boldsymbol{\tau}^{\beta}$ and $\boldsymbol{\tau}_{\tau} = -\mathbf{v}_{\alpha} b_{\beta}^{\alpha} \boldsymbol{\tau}^{\beta}$ are the normal curvature and the geodesic torsion of $\partial\mathcal{M}$, while $\sigma_v = \mathbf{v}_{\alpha} b_{\beta}^{\alpha} \mathbf{v}^{\beta}$ is the normal curvature of the surface curve orthogonal to $\partial\mathcal{M}$ in the outward normal direction. The RISEs (20) and (21) with intrinsic boundary conditions (22) and (23) are valid for unrestricted translations and rotations of the shell material elements.

It follows from the above transformations that the refined intrinsic BVP (20) to (23) can be used if:

1. the surface and boundary forces and couples are given through components in the deformed bases $\bar{\mathbf{a}}_{\alpha}, \bar{\mathbf{n}}$ and $\bar{\mathbf{v}}, \bar{\boldsymbol{\tau}}, \bar{\mathbf{n}}$, respectively;
2. at the shell boundary contour without corners the boundary conditions are prescribed only in terms of the intrinsic fields discussed above;
3. at the shell boundary contour with corners only the deformational boundary conditions are prescribed, or the boundary contour is divided by corner points into an even number of intervals along which alternately either only natural intrinsic or only deformational boundary conditions are prescribed.

These requirements put additional constraints on the range of applications of the RISEs.

6 Some special cases of the refined intrinsic shell equations

Let γ and κ denote the greatest values of the surface strains and bendings at $M \in \mathcal{M}$, respectively, so that the estimates $\gamma_{\alpha\beta} = O(\gamma)$ and $\kappa_{\alpha\beta} = O(\kappa)$ give us upper bounds for the measures. Let also L_{γ} and L_{κ} be the local smallest wave lengths of deformation patterns on \mathcal{M} associated with the surface strains and bendings, respectively, so that the estimates $\gamma_{\alpha}^{\beta}|_{\lambda} = O(\gamma/L_{\gamma})$ and $\kappa_{\alpha}^{\beta}|_{\lambda} = O(\kappa/L_{\kappa})$ can be

used. Let us also remind that the surface curvatures and their spatial derivatives can be estimated as $b_\alpha^\beta = O(1/R)$ and $b_\alpha^\beta|_\lambda = O(1/Rl)$, while components of the surface force and moment vectors by $p_\alpha = O(Eh\gamma/L_\gamma)$, $p = O(Eh\gamma/R)$, and $m_\alpha = O(Eh^3\kappa/L_\kappa)$. Then it is easy to see that all terms in the RISEs can be estimated by some products of six different small parameters: h/R , $(h/l)^2$, γ , $h\kappa$, $(h/L_\gamma)^2$, $(h/L_\kappa)^2$.

Within the accuracy of the first approximation to the strain energy density (12) it has already been assumed that such small terms can be omitted with regard to terms of order unity. However, in different types of shell problems real magnitudes of some of the small terms may be much lower than their upper bounds indicated above. If we are able to predict in advance the type of solution behavior in the whole shell region and propose in advance corresponding proportions between all the small parameters, then it is possible to distinguish in the RISEs the principal terms responsible for the predicted type of shell behavior, and to omit all secondary terms. In this way a number of simplified special versions of the RISEs describing various types of shell behavior can be constructed. We remind here explicitly only three special cases of the RISEs which seem to be most important in applications.

6.1 The almost inextensional bending nonlinear theory of shells

This version of shell theory describes the behavior of shell problems in which the membrane strains are much smaller than the strains associated with bending. In such problems the spatial variability of bendings should also be lower than in the general case of deformation. Thus, if ε is a reference small parameter such that $1 + \varepsilon^2 \simeq 1$, then the six small parameters can be related to ε by $h/R \sim h\kappa \sim h/L_\kappa \sim \varepsilon^2$, $\gamma \sim \varepsilon^4$, $h/l \sim h/L_\gamma \sim \varepsilon$. If we apply these relations to estimate all terms in the RISEs and omit all those terms, whose relative error with regard to the principal terms is ε^2 or less, we obtain the following consistently simplified set of equations of the almost inextensional bending nonlinear theory of shells (less error terms):

$$\begin{aligned}
 N_\alpha^\beta|_\beta - D \left\{ \left(b_\alpha^\lambda - \kappa_\alpha^\lambda \right) \left[(1 - \mathbf{v}) \kappa_\lambda^\beta + \mathbf{v} \delta_\lambda^\beta \kappa_\mu^\mu \right] \right\} |_\beta \\
 - \left(b_\alpha^\beta - \kappa_\alpha^\beta \right) \left(D \kappa_\lambda^\lambda |_\beta + m_\beta \right) + p_\alpha = 0, \\
 D \kappa_\alpha^\alpha |_\beta + \left(b_\alpha^\beta - \kappa_\alpha^\beta \right) N_\beta^\alpha + p + m^\alpha |_\alpha = 0, \\
 \kappa_\alpha^\beta |_\beta - \kappa_\beta^\beta |_\alpha = 0, \quad \left(b_\alpha^\beta - \frac{1}{2} \kappa_\alpha^\beta \right) \kappa_\beta^\alpha - \left(b_\alpha^\alpha - \frac{1}{2} \kappa_\alpha^\alpha \right) \kappa_\beta^\beta = 0.
 \end{aligned} \tag{24}$$

Appropriately reduced four intrinsic natural and deformational boundary conditions are given by Opoka and Pietraszkiewicz (2004).

Note that three compatibility conditions (24)_{3,4} do not depend on $N^{\alpha\beta}$. Thus, for some combinations of the shell geometry and boundary conditions these compati-

bility conditions can be solved independently of the membrane state generated by $N^{\alpha\beta}$, so that such problems may be kinematically determined. If $b_{\alpha\beta} \rightarrow 0$ the relations (24) transform smoothly to the almost inextensional bending theory of plates.

Koiter (1980, Section 6) suggested that the geometrically nonlinear inextensional bending theory of shells should be defined by taking the limit $A \rightarrow 0$ in (24). Simmonds (1979, Section 4.1) obtained such a version of shell theory by non-dimensionalising all the fields, with the non-dimensional stress resultants defined as $\tilde{N}_\beta^\alpha = R^2 D^{-1} N_\beta^\alpha$, and then taking the limit $AD/R^2 \rightarrow 0$. Wempner and Talaslidis (2003) applied an estimation procedure similar to ours, but used a common wave length of deformation patterns defined by $L = \min(l, L_\gamma, L_\kappa)$. Under all the three different procedures mentioned above these authors obtained the almost inextensional bending intrinsic shell equations energetically equivalent to (24). But neither of the above papers discussed the corresponding intrinsic boundary conditions associated with the almost inextensional bending shell equations.

In shell structures designed for strength the almost inextensional bending state should rather be avoided, because it is usually associated with occurrence of larger translations of the base surface. However, some shell structures working in this state are specially designed either to allow maximal flexibility (toroidal compensators, bellows, etc.) or the shell shape is imposed by its function (turbine and compressor blades, ship propellers, etc.).

6.2 The almost membrane nonlinear theory of shells

This version of shell theory describes the behavior of shell problems in which the membrane strains are much larger than the strains associated with bending. In such problems the spatial variability of the membrane strains should also be lower than in the general case of deformation. Thus, the six small parameters can be related to ε by $h/R \sim \gamma \sim h/L_\gamma \sim \varepsilon^2$, $h\kappa \sim \varepsilon^4$, $h/l \sim h/L_\kappa \sim \varepsilon$. Again, applying these relations to estimate all terms in the RISEs (20) and (21), and omitting all small terms, we obtain the following consistently simplified equations of the almost membrane theory of shells (less error terms):

$$\begin{aligned}
 N_\alpha^\beta|_\beta + p_\alpha &= 0, & b_\alpha^\beta N_\beta^\alpha + p &= 0, \\
 \kappa_\alpha^\beta|_\beta - \kappa_\beta^\alpha|_\alpha - A(1 + \mathbf{v}) \left(b_\beta^\lambda N_\lambda^\beta|_\alpha + b_\alpha^\beta N_\lambda^\lambda|_\alpha \right) + A\mathbf{v}b_\beta^\beta N_\lambda^\lambda|_\alpha - 2A(1 + \mathbf{v})b_\alpha^\beta p_\beta &= 0, \\
 AN_\alpha^\alpha|_\beta + b_\alpha^\beta \kappa_\beta^\alpha - b_\beta^\alpha \kappa_\alpha^\alpha + A(1 + \mathbf{v})p^\alpha|_\alpha &= 0.
 \end{aligned}
 \tag{25}$$

Notice that all intrinsic equations (25) are linear with regard to N_α^β and κ_α^β . Appro-

proprately reduced four natural intrinsic and deformational boundary conditions are given by Opoka and Pietraszkiewicz (2004), and they are linear in these variables as well.

The equilibrium equations $(25)_{1,2}$ do not depend on κ_α^β , which indicates that some shell problems of the almost membrane type can be statically determined. However, if $b_\alpha^\beta \rightarrow 0$, the principal term of $(25)_2$ disappears in the limit and this equilibrium equation becomes irrelevant for the almost membrane nonlinear theory of plates. In order to remove such a degenerate behavior, if necessary, in $(25)_2$ we might take into account also terms of higher order smallness, which then replaces this equation by

$$D\kappa_\alpha^\alpha |_\beta^\beta + (b_\alpha^\beta - \kappa_\alpha^\beta) N_\beta^\alpha + p + m^\alpha |_\alpha = 0. \quad (26)$$

As a result, the BVP consisting of six linear PDE $(25)_{1,3,4}$ and the nonlinear one (26) as well as of appropriate four linear boundary conditions might now be transformed smoothly to the almost membrane intrinsic theory of plates. However, such a supplemented BVP is not statically determined any more.

Koiter (1980, Section 6) suggested that the geometrically nonlinear almost membrane theory should be defined by taking the limit $D \rightarrow 0$ in (20) to (23). Simmonds (1979, Section 4.5) derived six equations for the membrane shell theory again by non-dimensionalising all the fields, but with the non-dimensional stress resultants defined as $\tilde{N}_\alpha^\beta = AN_\alpha^\beta$, and then by taking the limit $AD/R^2 \rightarrow 0$. Wempner and Talaslidis (2003) applied an estimation procedure similar to ours, but used a common length L defined above and without correcting degeneration when $b_\alpha^\beta \rightarrow 0$. Unfortunately, each of the three alternative procedures mentioned above lead to the almost membrane intrinsic shell equations slightly different from (25) with or without (26) presented here and energetically inequivalent to each other. Neither of the above authors discussed the intrinsic boundary conditions associated with the almost membrane intrinsic shell equations as well.

6.3 Intrinsic bending nonlinear theory of shells

In terms of N_α^β and κ_α^β the intrinsic bending theory of shells can be characterized by the following range of the bending-to-strain ratio:

$$\max \left(h\kappa \frac{L_\kappa}{L_\gamma}, \frac{h}{R} \frac{L_\kappa}{L_\gamma} \right) \ll \frac{h\kappa}{\gamma} \ll \min \left(\frac{l}{L_\gamma} \frac{R}{h}, \frac{L_\kappa}{L_\gamma} \frac{1}{h\kappa}, \frac{L_\kappa R}{L_\gamma h} \right). \quad (27)$$

Within this range of shell deformation the RISEs (20) and (21) can be reduced to

the following set of six PDE (less error terms):

$$\begin{aligned}
 N_\alpha^\beta|_\beta + p_\alpha &= 0, \quad D\kappa_\alpha^\alpha|_\beta^\beta + (b_\alpha^\beta - \kappa_\alpha^\beta)N_\beta^\alpha + p + m^\alpha|_\alpha = 0, \\
 \kappa_\alpha^\beta|_\beta - \kappa_\beta^\alpha|_\alpha &= 0, \\
 AN_\alpha^\alpha|_\beta^\beta + \left(b_\alpha^\beta - \frac{1}{2}\kappa_\alpha^\beta\right)\kappa_\beta^\alpha - \left(b_\beta^\alpha - \frac{1}{2}\kappa_\beta^\alpha\right)\kappa_\alpha^\beta + A(1 + \mathbf{v})p^\alpha|_\alpha &= 0.
 \end{aligned} \tag{28}$$

The corresponding reduced intrinsic boundary conditions were given first by Pietraszkiewicz (1980a) and modified by Opoka and Pietraszkiewicz (2004). Note that four equations (28)_{1,3} and all boundary conditions are linear while only two equations (28)_{2,4} are quadratic in the intrinsic field variables $N_\alpha^\beta, \kappa_\alpha^\beta$.

The intrinsic bending shell equations in the form equivalent to (28), but without $m^\alpha|_\alpha$ and often also without $A(1 + \mathbf{v})p^\alpha|_\alpha$ was proposed in a number of papers and reproduced in several books. Let us mention here the classical papers by Mushtari (1949), Alumiya (1949), and Koiter (1966). However, neither of these authors discussed the corresponding intrinsic boundary conditions.

Please note that the four equations (28)_{1,3} have a divergent structure with regard to one field variable. This allows one to satisfy these equations within the estimated error by

$$N_\alpha^\beta = \varepsilon_{\alpha\lambda}\varepsilon^{\beta\rho}\left(F|_\rho^\lambda + \delta_\rho^\lambda KF\right) + P_\alpha^\beta, \quad \kappa_\alpha^\beta = W|_\alpha^\beta + \delta_\alpha^\beta KW, \tag{29}$$

provided that the Gaussian curvature K of \mathcal{M} satisfies the relations

$$\frac{L_\gamma}{l}|K|L_\gamma^2 \ll 1, \quad \frac{L_\kappa}{l}|K|L_\kappa^2 \ll 1. \tag{30}$$

In (29), F is the stress function, W is the deformation function, and P_α^β is a particular solution of $P_\alpha^\beta|_\beta + p_\alpha = 0$. Introducing (29) into (28)_{2,4} we obtain

$$\begin{aligned}
 D(W|_\alpha^\alpha + 2KW)|_\beta^\beta + \left(b_\alpha^\beta - W|_\alpha^\beta - \delta_\alpha^\beta KW\right) \left[\varepsilon^{\alpha\lambda}\varepsilon_{\beta\rho}\left(F|_\lambda^\rho + \delta_\lambda^\rho KF\right) + P_\beta^\alpha\right] \\
 + p + m^\alpha|_\alpha &= 0, \\
 A(F|_\alpha^\alpha + 2KF)|_\beta^\beta - \varepsilon^{\alpha\lambda}\varepsilon_{\beta\rho}\left(b_\alpha^\beta - \frac{1}{2}W|_\alpha^\beta - \frac{1}{2}\delta_\alpha^\beta KW\right) (W|_\lambda^\rho + \delta_\lambda^\rho KW) \\
 + A\left[P_\alpha^\alpha|_\beta^\beta - (1 + \mathbf{v})P_\alpha^\beta|_\alpha\right] &= 0.
 \end{aligned} \tag{31}$$

These are the nonlinear intrinsic bending equations for shells of slowly varying curvatures proposed by Rychter (1988) and Pietraszkiewicz (1989). The intrinsic

boundary conditions to be used with (31) are given by Opoka and Pietraszkiewicz (2004).

Under the more restrictive assumption $|K|L^2 \ll 1$, which seem to be used already by Aron (1874) and also proposed by Chien (1944), section 12, we can also omit all terms with K in (31). This leads to the nonlinear intrinsic bending equations of quasi-shallow shells discussed by Koiter (1966). Appropriate intrinsic boundary conditions follow then directly from reduction of those given in Opoka and Pietraszkiewicz (2004).

7 Refined intrinsic shell equations in the rotated base

Let ∇_s be the surface gradient operator at $M \in \mathcal{M}$. Then the surface deformation gradient tensor $\mathbf{F} \in E \otimes T_M \mathcal{M}$ is defined by

$$\mathbf{F} = \nabla_s \chi(\mathbf{r}) = \bar{\mathbf{r}}_{,\alpha} \otimes \mathbf{a}^\alpha, \quad (32)$$

where \otimes is the tensor product and $\chi : \mathcal{M} \rightarrow \bar{\mathcal{M}}$ is the deformation function. Due to the relation $\bar{\mathbf{r}}_{,\alpha} = \bar{\mathbf{a}}_\alpha \in T_{\bar{M}} \bar{\mathcal{M}}$ the field \mathbf{F} maps $d\mathbf{r} \in T_M \mathcal{M}$ into $d\bar{\mathbf{r}} \in T_{\bar{M}} \bar{\mathcal{M}}$, so that $d\bar{\mathbf{r}} = \mathbf{F}d\mathbf{r}$. Since the tangent planes $T_M \mathcal{M}$ and $T_{\bar{M}} \bar{\mathcal{M}}$ lie in the same 3D Euclidean space \mathcal{E} , there is a rotation $\mathbf{R} \in SO(3)$ that takes one plane to the other. This in conjunction with the theorem of Tissot given in do Carmo (1976) and discussion by Pietraszkiewicz, Szwabowicz, and Vallée (2008) justifies the following representation of \mathbf{F} :

$$\mathbf{F} = \mathbf{V}\mathbf{R}, \quad \mathbf{r}_\alpha = \mathbf{R}\mathbf{a}_\alpha = \mathbf{V}^{-1}\bar{\mathbf{a}}_\alpha, \quad (33)$$

where $\mathbf{V} \in T_{\bar{M}} \bar{\mathcal{M}} \otimes T_M \mathcal{M}$ is the left surface stretch tensor, and \mathbf{r}_α are the surface rotated non-holonomic base vectors. The fields \mathbf{R} and \mathbf{V} satisfy the relations

$$\begin{aligned} \mathbf{R} &= \mathbf{r}_\alpha \otimes \mathbf{a}^\alpha + \bar{\mathbf{n}} \otimes \mathbf{n}, \quad \mathbf{R}^T = \mathbf{R}^{-1}, \\ \det \mathbf{R} &= +1, \quad \mathbf{V} = \bar{\mathbf{a}}_\alpha \otimes \mathbf{r}^\alpha, \quad \det \mathbf{V} = \sqrt{\frac{\bar{a}}{a}} > 1. \end{aligned} \quad (34)$$

The relative surface strain measures associated with the base \mathbf{r}_α , $\bar{\mathbf{n}}$ are introduced through the following formulae, see Pietraszkiewicz (1989):

$$\begin{aligned} \boldsymbol{\varepsilon} &= \mathbf{V} - \bar{\mathbf{a}} = (\mathbf{a}_{\beta} + \mathbf{u}_{,\beta} - \mathbf{r}_\beta) \otimes \mathbf{r}^\beta = \boldsymbol{\varepsilon}_\beta \otimes \mathbf{r}^\beta, \quad \boldsymbol{\varepsilon}_\beta = \eta_{\alpha\beta} \mathbf{r}^\alpha, \quad \eta_{\alpha\beta} = \eta_{\beta\alpha}, \\ \boldsymbol{\lambda} &= (\bar{\mathbf{n}}_{,\beta} - \mathbf{R}\mathbf{n}_{,\beta}) \otimes \mathbf{r}^\beta = \mathbf{R}_{,\beta} \mathbf{n} \otimes \mathbf{r}^\beta = \boldsymbol{\lambda}_\beta \otimes \mathbf{r}^\beta, \quad \boldsymbol{\lambda}_\beta = \mu_{\alpha\beta} \mathbf{r}^\alpha, \quad \mu_{\alpha\beta} \neq \mu_{\beta\alpha}. \end{aligned} \quad (35)$$

The components $\eta_{\alpha\beta}$ and $\mu_{\alpha\beta}$ are related to $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ by

$$\begin{aligned} \gamma_{\alpha\beta} &= \eta_{\alpha\beta} + \frac{1}{2}\eta_{\alpha}^{\lambda}\eta_{\lambda\beta}, \\ \kappa_{\alpha\beta} &= \frac{1}{2}\left[\left(\delta_{\alpha}^{\lambda} + \eta_{\alpha}^{\lambda}\right)\mu_{\lambda\beta} + \left(\delta_{\beta}^{\lambda} + \eta_{\beta}^{\lambda}\right)\mu_{\lambda\alpha}\right] - \frac{1}{2}\left(b_{\alpha}^{\lambda}\eta_{\lambda\beta} + b_{\beta}^{\lambda}\eta_{\lambda\alpha}\right). \end{aligned} \tag{36}$$

In the nonlinear theory of thin shells the rotation tensor \mathbf{R} depends upon the surface gradient of the translation field $\nabla_s \mathbf{u}$ by explicit formulae given by Pietraszkiewicz (1977,1980a,b). In order to use the virtual rotations $\delta \mathbf{R}$ as independent field variables in the PVW (3), dependence of \mathbf{R} upon $\nabla_s \mathbf{u}$ can be enforced implicitly through three constraint conditions put on the virtual strains $\delta \eta_{\alpha\beta}$:

$$\varepsilon^{\alpha\beta} \mathbf{r}_{\alpha} \cdot \delta \eta_{\lambda\beta} \mathbf{r}^{\lambda} = 0, \quad \bar{\mathbf{n}} \cdot \delta \eta_{\lambda\beta} \mathbf{r}^{\lambda} = 0. \tag{37}$$

In the interior domain of \mathcal{M} the constraints (37) can be introduced into the surface integral (3) of the PVW with the help of respective Lagrange multipliers S and Q^{β} . It was shown in Pietraszkiewicz (1989) that, in order to express also the boundary terms at each $\partial \mathcal{M}$ explicitly through now independent virtual rotations, it is necessary to introduce into (3) a curvilinear integral over $\partial \mathcal{M}$ with the constraints (37)₁ multiplied by the Lagrange multiplier $B\tau^{\beta}$. Additionally, in (3) the external virtual work performed by the static moments \mathbf{h} and \mathbf{H}^* should be expressed directly in terms of now independent virtual rotations. As a result, the PVW (3) should be modified to the form

$$\begin{aligned} \iint_{\mathcal{M}} \left(\mathbf{N}^{\beta} \cdot \delta \eta_{\lambda\beta} \mathbf{r}^{\lambda} + H^{\alpha\beta} \mathbf{r}_{\alpha} \cdot \delta \mu_{\lambda\beta} \mathbf{r}^{\lambda} \right) dA + \int_{\partial \mathcal{M}} B\tau^{\beta} \bar{\mathbf{n}} \cdot \delta \eta_{\lambda\beta} \mathbf{r}^{\lambda} ds \\ = \iint_{\mathcal{M}} \left(\mathbf{p} \cdot \delta \mathbf{u} + \mathbf{m} \cdot \boldsymbol{\omega} \right) dA + \int_{\partial \mathcal{M}_f} \left(\mathbf{N}^* \cdot \delta \mathbf{u} + \mathbf{M}^* \cdot \boldsymbol{\omega}_{\tau} \right) ds, \end{aligned} \tag{38}$$

where now

$$\begin{aligned} \mathbf{N}^{\beta} &= R^{\alpha\beta} \mathbf{r}_{\alpha} + Q^{\beta} \bar{\mathbf{n}}, \quad \mathbf{M}^{\beta} = \bar{\mathbf{n}} \times H^{\alpha\beta} \mathbf{r}_{\alpha}, \quad R^{\alpha\beta} = S^{\alpha\beta} + \varepsilon^{\alpha\beta} S, \\ \mathbf{M}^* &= \bar{\mathbf{n}} \times \mathbf{H}^*, \quad \mathbf{m} = \bar{\mathbf{n}} \times \mathbf{h}, \end{aligned} \tag{39}$$

$$\begin{aligned} \boldsymbol{\omega} &= \frac{1}{2}(\mathbf{1} \times \mathbf{1}) \cdot (\delta \mathbf{R} \mathbf{R}^T) = \frac{1}{2}(\mathbf{r}^{\alpha} \times \delta \mathbf{r}_{\alpha} + \bar{\mathbf{n}} \times \delta \bar{\mathbf{n}}), \\ \boldsymbol{\omega}_{\tau} &= \frac{1}{2}(\mathbf{1} \times \mathbf{1}) \cdot (\delta \mathbf{R}_{\tau} \mathbf{R}_{\tau}^T) = \frac{1}{2}(\bar{\mathbf{v}} \times \delta \bar{\mathbf{v}} + \bar{\boldsymbol{\tau}} \times \delta \bar{\boldsymbol{\tau}} + \bar{\mathbf{n}} \times \delta \bar{\mathbf{n}}). \end{aligned} \tag{40}$$

Here $S^{\alpha\beta}$ and $\varepsilon^{\alpha\beta} S$ are symmetric and skew parts of $R^{\alpha\beta}$, $\mathbf{1}$ is the metric tensor of the 3D Euclidean space, w and w_{τ} are the virtual rotation vectors in the interior

of \mathcal{M} and along $\partial\mathcal{M}$, respectively, and for any 2^{nd} -order tensors \mathbf{C}, \mathbf{D} we have $\mathbf{C} \cdot \mathbf{D} = \text{tr}(\mathbf{C}^T \mathbf{D})$. Please note that all surface couple vectors $\mathbf{M}^\beta, \mathbf{M}^*$ and \mathbf{m} in (15) do not have normal components, that is $\mathbf{M}^\beta \cdot \bar{\mathbf{n}} = \mathbf{M}^* \cdot \bar{\mathbf{n}} = \mathbf{m} \cdot \bar{\mathbf{n}} = 0$. This is the fundamental property of the nonlinear theory of thin shells.

For kinematically admissible virtual deformation the fields $\delta\mathbf{u}$ and ω_τ vanish identically along $\partial\mathcal{M}_d$, and the principle of virtual work (38) leads to the modified local equilibrium equations

$$\mathbf{N}^\beta|_\beta + \mathbf{p} = \mathbf{0}, \quad \mathbf{M}^\beta|_\beta + \bar{\mathbf{a}}_\beta \times \mathbf{N}^\beta + \mathbf{m} = \mathbf{0} \quad \text{at each regular } M \in \mathcal{M}, \quad (41)$$

the five natural boundary conditions

$$\mathbf{N}_v - \mathbf{N}^* = \mathbf{0}, \quad \mathbf{K}_v - \mathbf{M}^* = \mathbf{0} \quad \text{along regular parts of } \partial\mathcal{M}_f, \quad (42)$$

with the corresponding five work-conjugate kinematic boundary conditions

$$\mathbf{u} - \mathbf{u}^* = \mathbf{0}, \quad \mathbf{R}_\tau \mathbf{n} - \mathbf{R}_\tau^* \mathbf{n} = \mathbf{0} \quad \text{along } \partial\mathcal{M}_d. \quad (43)$$

In (42), we have defined $\mathbf{N}_v = \mathbf{N}^\beta \mathbf{v}_\beta$ and $\mathbf{K}_v = \mathbf{M}^\beta \mathbf{v}_\beta + a_\tau \mathbf{B} \mathbf{R}_\tau \mathbf{v}$.

The equilibrium equations formally similar to (41) were first derived by Alamyae (1949,1956) and rederived by Simmonds and Danielson (1970,1972), but their definitions of \mathbf{N}^β and \mathbf{M}^β were somewhat different from (39)_{1,2}. The boundary conditions (42) and (43) were first proposed by Pietraszkiewicz (1989). Several mixed variational principles in terms of the relative surface strains $\eta_{\alpha\beta}$ (or corresponding $S^{\alpha\beta}$) and the finite rotation tensor \mathbf{R} were constructed by Atluri (1984).

As a result of the above transformations, the surface virtual strain energy density can be presented in the alternative form

$$\begin{aligned} \delta\Sigma &= N^{\alpha\beta} \delta\gamma_{\alpha\beta} + M^{\alpha\beta} \delta\kappa_{\alpha\beta} = S^{\alpha\beta} \delta\eta_{\alpha\beta} + H^{\alpha\beta} \delta\mu_{\alpha\beta}, \\ S^{\alpha\beta} &= N^{\alpha\beta} + \frac{1}{2} \left(\eta_\lambda^\alpha N^{\lambda\beta} + \eta_\lambda^\beta N^{\alpha\lambda} \right) - \frac{1}{2} \left[(b_\lambda^\alpha - \mu_\lambda^\alpha) M^{\lambda\beta} + (b_\lambda^\beta - \mu_\lambda^\beta) M^{\alpha\lambda} \right], \\ H^{\alpha\beta} &= (\delta_\lambda^\alpha + \eta_\lambda^\alpha) M^{\lambda\beta}, \end{aligned} \quad (44)$$

where now $S^{\alpha\beta} = S^{\beta\alpha}$, but $H^{\alpha\beta} \neq H^{\beta\alpha}$, in general.

When expressed by components in the rotated basis $\mathbf{r}_\alpha, \bar{\mathbf{n}}$, the vector equilibrium equations (41) lead to the six local scalar equilibrium equations expressed entirely in terms of the surface strain measures $\eta_{\alpha\beta}, \mu_{\alpha\beta}, k_\alpha$, stress measures $R^{\alpha\beta}, H^{\alpha\beta}, Q^\alpha$, and components of \mathbf{p}, \mathbf{m} in the base $\mathbf{r}_\alpha, \bar{\mathbf{n}}$. However, if the shell BVP is to be

formulated in the intrinsic form, these six scalar equilibrium equations should be accompanied by appropriate six scalar compatibility conditions expressed in terms of the same surface strain measures.

It follows from $(\mathbf{R}\mathbf{R}^T)_{,\beta} = \mathbf{0}$ that $\mathbf{R}_{,\beta}\mathbf{R}^T$ is the skew-symmetric tensor expressible through the axial bending vector \mathbf{l}_β by

$$\mathbf{R}_{,\beta}\mathbf{R}^T = \mathbf{l}_\beta \times \mathbf{1}, \quad \mathbf{l}_\beta = \varepsilon^{\alpha\lambda} \mu_{\alpha\beta} \mathbf{r}_\lambda + k_\beta \bar{\mathbf{n}}. \quad (45)$$

The integrability conditions $\varepsilon^{\alpha\beta} \mathbf{u}_{,\alpha\beta} = \mathbf{0}$ and $\varepsilon^{\alpha\beta} \mathbf{R}_{,\alpha\beta} = \mathbf{0}$ lead to

$$\varepsilon^{\alpha\beta} (\mathbf{l}_{\alpha|\beta} + \mathbf{l}_\beta \times \mathbf{r}_\alpha) = \mathbf{0}, \quad \varepsilon^{\alpha\beta} \left(\mathbf{l}_{\alpha|\beta} + \frac{1}{2} \mathbf{l}_\alpha \times \mathbf{l}_\beta \right) = \mathbf{0}. \quad (46)$$

These are vector forms of compatibility conditions for the nonlinear deformation of the shell base surface first proposed by Shkutin (1976). Their component forms relative to the rotated base \mathbf{r}_α , $\bar{\mathbf{n}}$ were derived by Alamyae (1949,1955) and independently by Simmonds and Danielson (1970). Several other equivalent scalar, vector, and tensor forms of 2D compatibility conditions at the shell base surface follow directly from the 3D compatibility conditions derived by Pietraszkiewicz and Badur (1983) for the non-linear continuum mechanics, if the Kirchhoff-Love kinematic constraints are imposed.

Let us introduce the decompositions

$$\begin{aligned} \mu_{\alpha\beta} &= \rho_{\alpha\beta} + \varepsilon_{\alpha\beta} \rho, & \rho_{\alpha\beta} &= \frac{1}{2} (\mu_{\alpha\beta} + \mu_{\beta\alpha}), & \rho &= \frac{1}{2} \varepsilon^{\alpha\beta} \mu_{\alpha\beta}, \\ H^{\alpha\beta} &= G^{\alpha\beta} + \varepsilon^{\alpha\beta} G, & G^{\alpha\beta} &= \frac{1}{2} (H^{\alpha\beta} + H^{\beta\alpha}), & G &= \frac{1}{2} \varepsilon_{\alpha\beta} H^{\alpha\beta}. \end{aligned} \quad (47)$$

For the symmetric surface fields $S^{\alpha\beta}$, $G^{\alpha\beta}$, $\eta_{\alpha\beta}$, $\rho_{\alpha\beta}$ we obtain the constitutive equations analogous to those given in (14) and (15). With the error of these constitutive equations the six scalar equilibrium equations and the six scalar compatibility conditions, following from (41) and (46) in the base \mathbf{r}_α , $\bar{\mathbf{n}}$, can be consistently simplified, each into the set of six equations. Then three of each set can be solved either for k_λ , ρ or for Q^β , S , respectively, which can then be eliminated from the remaining shell equations using additional estimates $G = O(Eh^2\eta\theta^2)$ and $\rho = O(\eta\theta/\lambda)$. As a result, this reduction process leads to two sets of six PDE for twelve symmetric surface fields related by the constitutive equations. Using the constitutive equations one can eliminate any six of the fields to obtain a definite system of six PDE for the remaining six surface measures. However, elimination of $S^{\alpha\beta}$ and $\rho_{\alpha\beta}$ would introduce greater errors which in critical cases may lead to some loss of accuracy

of the solution. Thus, eliminating $G^{\alpha\beta}$ and $\eta_{\alpha\beta}$, we obtain

$$\begin{aligned}
& S_{\alpha}^{\beta}|_{\beta} + A \left[(1 + \mathbf{v}) S_{\alpha}^{\lambda} - \mathbf{v} \delta_{\alpha}^{\lambda} S_{\kappa}^{\kappa} \right] |_{\beta} S_{\lambda}^{\beta} - \frac{1}{2} A \left[(1 + \mathbf{v}) S_{\lambda}^{\beta} S_{\beta}^{\lambda} - \mathbf{v} S_{\lambda}^{\lambda} S_{\beta}^{\beta} \right] |_{\alpha} \\
& - \frac{1}{2} D (1 - \mathbf{v}) \left(b_{\alpha}^{\lambda} \rho_{\lambda}^{\beta} - b_{\lambda}^{\beta} \rho_{\alpha}^{\lambda} \right) |_{\beta} - D \left(b_{\alpha}^{\beta} - \rho_{\alpha}^{\beta} \right) \rho_{\lambda}^{\lambda} |_{\beta} \\
& + \hat{p}_{\alpha} - \left(b_{\alpha}^{\beta} - \rho_{\alpha}^{\beta} \right) \hat{m}_{\beta} = O \left(Eh \frac{\eta \theta^4}{\lambda} \right), \\
& D \rho_{\alpha}^{\alpha} |_{\beta}^{\beta} + \left(b_{\alpha}^{\beta} - \rho_{\alpha}^{\beta} \right) S_{\beta}^{\alpha} + p + \hat{m}^{\alpha} |_{\alpha} = O \left(Eh^2 \frac{\eta \theta^2}{\lambda^2} \right), \\
& \rho_{\alpha}^{\beta} |_{\beta} - \rho_{\beta}^{\beta, \alpha} + \frac{1}{2} A (1 + \mathbf{v}) \left[\left(b_{\alpha}^{\lambda} - \rho_{\alpha}^{\lambda} \right) S_{\lambda}^{\beta} - \left(b_{\lambda}^{\beta} - \rho_{\lambda}^{\beta} \right) S_{\alpha}^{\lambda} \right] |_{\beta} \\
& - A \left(b_{\alpha}^{\beta} - \rho_{\alpha}^{\beta} \right) S_{\lambda}^{\lambda, \alpha} - A (1 + \mathbf{v}) \left(b_{\alpha}^{\beta} - \rho_{\alpha}^{\beta} \right) \hat{p}_{\beta} = O \left(\frac{\eta \theta^4}{h \lambda} \right), \\
& A S_{\alpha}^{\alpha} |_{\beta}^{\beta} + \left(b_{\alpha}^{\beta} - \frac{1}{2} \rho_{\alpha}^{\beta} \right) \rho_{\beta}^{\alpha} - \left(b_{\alpha}^{\alpha} - \frac{1}{2} \rho_{\alpha}^{\alpha} \right) \rho_{\beta}^{\beta} + A (1 + \mathbf{v}) \hat{p}^{\alpha} |_{\alpha} = O \left(\frac{\eta \theta^2}{\lambda^2} \right).
\end{aligned} \tag{48}$$

The set of refined intrinsic shell equations (48) given in this form by Pietraszkiewicz (2001a,b) is written only in terms of the intrinsic surface components relative to the rotated base \mathbf{r}_{α} , $\bar{\mathbf{n}}$. Within the indicated errors, the equations (48) are energetically equivalent to alternative forms of the refined intrinsic shell equations proposed by Danielson (1970); Koiter and Simmonds (1973); and Pietraszkiewicz (1977,1980a,1989). However,

- the system (48) of PDE is expressed through the surface fields $S^{\alpha\beta}$, $\rho_{\alpha\beta}$ appearing naturally in the nonlinear theory of thin shells and needing no special modifications;
- when linearised (48) reduce to the shell equations of the “best” formulation of the linear shell theory according to Budiansky and Sanders (1963), see also Koiter 1960);
- the system (48) follows from six sets of six equations which obey the static-geometric analogy in the nonlinear range of deformation, see Alamyae (1955), Pietraszkiewicz (1989,2001a);
- corresponding sets of natural intrinsic and deformational boundary conditions were provided by Pietraszkiewicz (1989,2001a).

From (48), similarly as in section 6, it is possible to construct many reduced sets of intrinsic shell equations valid under additional assumptions about orders of curvature and variability of the middle surface, stretching-to-bending ratio, variability of stretching and bending deformations, etc. Some of these special cases were discussed by Pietraszkiewicz (1989,2001a). However, in our computer age it seems more appropriate to apply direct numerical methods to the complete system (48), which would allow one to analyze numerically all possible cases of the non-linear behavior of thin elastic shells.

8 Position of the base surface of deformed shell

When for a definite geometry of \mathcal{M} given by the position vector $\mathbf{r}(\theta^\alpha)$ one solves an appropriate set of the RISEs described in sections 5-7, the fields $N^{\alpha\beta}$ (and therefore $\gamma_{\alpha\beta}$) and $\kappa_{\alpha\beta}$ become known functions of the coordinates θ^α in some domain \mathcal{U} . Thus, our next problem is to determine the position vector $\bar{\mathbf{r}}$ of $\bar{\mathcal{M}}$ from known $\mathbf{r}(\theta^\alpha)$ and two fields $\gamma_{\alpha\beta}(\theta^\alpha)$ and $\kappa_{\alpha\beta}(\theta^\alpha)$ satisfying the compatibility conditions (21) within the prescribed accuracy.

In the linear theory of thin shells the problem of determining the translation field (and thus the position of base surface of the deformed shell in space) from known linearised surface strains and bendings was discussed already by Lur'e (1940). The explicit curvilinear surface integrals are given, for example, in the books by Chernykh (1964) and Gol'denveiser (1976).

In differential geometry of a surface immersed in the 3D Euclidean space \mathcal{E} the fundamental theorem first given by Bonnet (1867) and reformulated in many papers and books (see for example Ciarlet and Larssoneur, 2002, and references given there) states that two fundamental forms $\bar{a}_{\alpha\beta}d\theta^\alpha d\theta^\beta$ and $\bar{b}_{\alpha\beta}d\theta^\alpha d\theta^\beta$ of the surface $\bar{\mathcal{M}}$ determine locally its position in space up to a rigid-body motion. This solves the problem of existence of such a surface. But in the nonlinear theory of shells such a statement for $\bar{\mathcal{M}}$ is not satisfactory, because in engineering one usually needs to know the position of $\bar{\mathcal{M}}$ in space *uniquely*, which should usually be described by the position vector $\bar{\mathbf{r}}$. This goal can be achieved only by formulating an appropriate system of PDE and solving it for a unique set of boundary and/or initial conditions fixing the surface in the ambient space. Only recently Pietraszkiewicz and Szwabowicz (2007) worked out two such procedures the results of which are briefly described in two subsections 8.1 and 8.2 given below.

8.1 Position of $\bar{\mathcal{M}}$ through vector ODE

When $\bar{a}_{\alpha\beta}$ and $\bar{b}_{\alpha\beta}$ of $\bar{\mathcal{M}}$ are calculated from known $a_{\alpha\beta}$, $b_{\alpha\beta}$ and $\gamma_{\alpha\beta}$, $\kappa_{\alpha\beta}$, by the Gauss and Weingarten relations we obtain the following system of four PDE

for the base vectors $\bar{\mathbf{a}}_\alpha, \bar{\mathbf{n}}$:

$$\bar{\mathbf{a}}_{\alpha,\beta} = \bar{\Gamma}_{\alpha\beta}^\lambda \bar{\mathbf{a}}_\lambda + \bar{b}_{\alpha\beta} \bar{\mathbf{n}}, \quad \bar{\mathbf{n}}_{,\beta} = -\bar{b}_\beta^\lambda \bar{\mathbf{a}}_\lambda, \quad (49)$$

where $\bar{\Gamma}_{\alpha\beta}^\lambda$ are the Christoffel symbols of the second kind computed from the metric components $\bar{a}_{\alpha\beta}$ alone.

Let us introduce the column vector $\bar{\mathbf{X}}$ and two square 3×3 matrices \bar{A}_α defined by

$$\bar{\mathbf{X}} = \begin{bmatrix} \bar{\mathbf{a}}_1 \\ \bar{\mathbf{a}}_2 \\ \bar{\mathbf{n}} \end{bmatrix}, \quad \bar{A}_\alpha = \begin{bmatrix} \bar{\Gamma}_{1\alpha}^1 & \bar{\Gamma}_{1\alpha}^2 & \bar{b}_{1\alpha} \\ \bar{\Gamma}_{2\alpha}^1 & \bar{\Gamma}_{2\alpha}^2 & \bar{b}_{2\alpha} \\ -\bar{b}_\alpha^1 & -\bar{b}_\alpha^2 & 0 \end{bmatrix}, \quad (50)$$

which allow one to express the relations (49) in the form of the total system of PDE

$$\bar{\mathbf{X}}_{,\alpha} = \bar{A}_\alpha \bar{\mathbf{X}}. \quad (51)$$

Integrability conditions of (51) are just the Gauss-Mainardi-Codazzi (GMC) equations of $\bar{\mathcal{M}}$.

Applying the Frobenius-Dieudonné theorem, see for example Maurin (1980), it has been proved that instead of solving the system (51) of PDE one can cover the domain \mathcal{U} with a dense set of paths C leaving radially from an arbitrarily chosen initial point p_0 labeled by $\theta_0^\alpha \in \mathcal{U}$, and then compute a particular solution $\bar{\mathbf{X}}(\theta^\alpha)$ corresponding to the initial condition $\bar{\mathbf{X}}(\theta_0^\alpha) = \bar{\mathbf{X}}_0$ of the initial value problem for the system of ODE

$$\frac{d\bar{\mathbf{X}}}{ds} = \bar{A}^C \bar{\mathbf{X}}, \quad \bar{A}^C = \bar{A}_\alpha \frac{d\theta^\alpha}{ds} \quad (52)$$

along each of the paths C . The solution to (52) may be obtained, for example, by the method of successive approximations in the form of infinite series, see for example Gantmakher (1960),

$$\bar{\mathbf{X}} = \sum_{i=1}^{\infty} \hat{\mathbf{Y}}_i, \quad \hat{\mathbf{Y}}_0 = \bar{\mathbf{X}}_0, \dots, \quad \hat{\mathbf{Y}}_i = \int_{p_0}^p \bar{A}^C(s) \hat{\mathbf{Y}}_{i-1}(s) ds, \dots \quad (53)$$

Having determined $\bar{\mathbf{X}}$, the bases $\bar{\mathbf{a}}_\alpha$ become known and the position vector of $\bar{\mathcal{M}}$ is calculated by the quadrature

$$\bar{\mathbf{r}}(\theta^\alpha) = \bar{\mathbf{r}}_0 + \int_{p_0}^p \bar{\mathbf{a}}_\alpha d\theta^\alpha, \quad (54)$$

where $\bar{\mathbf{r}}_0$ is the initial value of $\bar{\mathbf{r}}(\theta^\alpha)$ at some arbitrarily chosen point $p_0 \in \mathcal{U}$.

The solution presented above depends on arbitrarily chosen initial values of the column vector $\bar{\mathbf{X}}_0$ and the position vector $\bar{\mathbf{r}}_0$. These quantities fix uniquely the position of $\bar{\mathcal{M}}$ in the space \mathcal{E} .

8.2 Position of $\overline{\mathcal{M}}$ through the surface deformation gradient

If $\chi : \mathcal{M} \rightarrow \overline{\mathcal{M}}$ is the deformation function of the shell base surface, then the surface deformation gradient tensor $\mathbf{F} \in E \otimes T_M \mathcal{M}$ is defined in (32). Since $\bar{\mathbf{r}}_{,\alpha} = \bar{\mathbf{a}}_\alpha \in T_{\overline{\mathcal{M}}} \overline{\mathcal{M}} \subset \mathcal{E}$, partial derivatives of \mathbf{F} are

$$\mathbf{F}_{,\alpha} = \mathbf{F}\mathbf{A}_\alpha, \quad \mathbf{A}_\alpha = (\bar{\Gamma}_{\lambda\alpha}^\kappa - \Gamma_{\lambda\alpha}^\kappa) \mathbf{a}_\kappa \otimes \mathbf{a}^\lambda + b_{\alpha\kappa}^\kappa \mathbf{a}_\kappa \otimes \mathbf{n} + \bar{b}_{\lambda\alpha} \frac{1}{\sqrt{\bar{a}}} (\mathbf{a}_1 \times \mathbf{a}_2) \otimes \mathbf{a}^\lambda. \quad (55)$$

For known geometry of \mathcal{M} the tensors \mathbf{A}_α are known as well, so that (55)₁ constitute a total system of PDE for \mathbf{F} , which integrability is assured by the GMC equations of $\overline{\mathcal{M}}$. The solution to the system (55)₁ can again be constructed with the help of Frobenius-Dieudonné theorem by choosing arbitrarily two points p_0, p with coordinates $\theta_0^\alpha, \theta^\alpha \in \mathcal{U}$, so that paths C drawn between such points cover the entire domain \mathcal{U} . In a local chart any path may be specified by two equations $\theta^\alpha|_C = \theta^\alpha(s)$, where s is the arc length chosen so that $s(\theta_0^\alpha) = s_0$. The system (55)₁, when restricted to any C , reduces to an ODE of the form

$$\frac{d\mathbf{F}}{ds} = \mathbf{F}\mathbf{A}, \quad \mathbf{A} = \mathbf{A}_\alpha \frac{d\theta^\alpha}{ds}. \quad (56)$$

The general solution to (56) can again be given by the method of successive approximation in the form

$$\mathbf{F} = \mathbf{F}_0 \mathbf{F}_s, \quad \mathbf{F}_0 = \mathbf{F}(s_0), \quad \mathbf{F}_s = \sum_{i=1}^{\infty} \mathbf{H}_i, \quad (57)$$

$$\mathbf{H}_0(s) = \mathbf{1}, \quad \mathbf{H}_i(s) = \int_{p_0}^p \mathbf{H}_{i-1}(t) \mathbf{A}(t) dt, \quad i \geq 1.$$

When $\mathbf{F}(\theta^\alpha)$ is known the position vector of $\overline{\mathcal{M}}$ follows from the quadrature

$$\bar{\mathbf{r}} = \bar{\mathbf{r}}_0 + \int_{p_0}^p \mathbf{F}\mathbf{a}_\alpha d\theta^\alpha, \quad \bar{\mathbf{r}}_0 = \bar{\mathbf{r}}(p_0). \quad (58)$$

Within the nonlinear theory of dislocations in thin elastic shells Zubov (1989,1997) used the spatial deformation gradient $\mathbf{G} \in E \otimes E$ evaluated on \mathcal{M} , which was defined by Pietraszkiewicz (1977) as

$$\mathbf{G} = \nabla \chi(\mathbf{r} + \boldsymbol{\zeta}\mathbf{n})|_{\boldsymbol{\zeta}=0} = \bar{\mathbf{a}}_\alpha \otimes \mathbf{a}^\alpha + \bar{\mathbf{n}} \otimes \mathbf{n}, \quad \bar{\mathbf{a}}_\alpha = \mathbf{G}\mathbf{a}_\alpha, \quad \bar{\mathbf{n}} = \mathbf{G}\mathbf{n}. \quad (59)$$

Thus, if the tensor \mathbf{G} is found the position vector $\bar{\mathbf{r}}$ can also be found by a quadrature similar to (58). However, the tensor \mathbf{G} contains the excessive term $\bar{\mathbf{n}} \otimes \mathbf{n}$ as compared with \mathbf{F} . Within the nonlinear theory of thin shells additional care should be taken to separate this excessive part of \mathbf{G} from the important one.

8.3 Position of $\overline{\mathcal{M}}$ using the right polar decomposition of deformation gradient

For an arbitrary deformation χ of the shell base surface the solutions (53) and (57) may be extremely complex and hardly applicable. Pietraszkiewicz, Szwabowicz, and Vallée (2008) developed an alternative approach to determination of the position vector $\bar{\mathbf{r}}$ which is based on the right or left polar decompositions of the surface deformation gradient \mathbf{F} .

The left polar decomposition (33)₁ was used in section 7 to formulate the RISEs in the rotated surface basis. In the present subsection, in order to show an alternative approach we apply, for definiteness, the right polar decomposition

$$\begin{aligned} \mathbf{F} &= \mathbf{R}\mathbf{U}, \quad \mathbf{R} = \bar{\mathbf{a}}_\alpha \otimes \mathbf{s}^\alpha + \bar{\mathbf{n}} \otimes \mathbf{n}, \quad \mathbf{U} = \mathbf{s}_\alpha \otimes \mathbf{a}^\alpha, \\ \mathbf{s}_\alpha &= \mathbf{U}\mathbf{a}_\alpha = \mathbf{R}^T \bar{\mathbf{a}}_\alpha, \quad \det \mathbf{R} = +1, \quad \det \mathbf{U} = \sqrt{\frac{\bar{a}}{a}} > 1, \end{aligned} \quad (60)$$

where $\mathbf{R}^T \mathbf{V} \mathbf{R} = \mathbf{U} \in T_M \mathcal{M} \otimes T_M \mathcal{M}$ is the right surface stretch tensor, and \mathbf{s}_α are the surface stretched non-holonomic base vectors.

The relative surface strain measures associated with $\mathbf{s}_\alpha, \mathbf{n}$ are defined by, see Pietraszkiewicz (1989),

$$\boldsymbol{\eta} = \mathbf{U} - \mathbf{a} = \eta_{\alpha\beta} \mathbf{a}^\alpha \otimes \mathbf{a}^\beta, \quad \boldsymbol{\mu} = (\mathbf{R}^T \bar{\mathbf{n}}_{,\beta} - \mathbf{n}_{,\beta}) \otimes \mathbf{a}^\beta = \mu_{\alpha\beta} \mathbf{a}^\alpha \otimes \mathbf{a}^\beta, \quad (61)$$

where the components $\eta_{\alpha\beta}$ and $\mu_{\alpha\beta}$ satisfy the relations (35) and (36).

Now the problem of finding $\bar{\mathbf{r}}$ can be solved in three steps.

First, from known $\boldsymbol{\gamma} = \gamma_{\alpha\beta} \mathbf{a}^\alpha \otimes \mathbf{a}^\beta$ the right stretch tensor can explicitly be calculated by

$$\mathbf{U} = \frac{[1 + \sqrt{1 + 2 \operatorname{tr}(\boldsymbol{\gamma}) + 4 \det(\boldsymbol{\gamma})}] \mathbf{a} + 2\boldsymbol{\gamma}}{\sqrt{2 [1 + \operatorname{tr}(\boldsymbol{\gamma}) + \sqrt{1 + 2 \operatorname{tr}(\boldsymbol{\gamma}) + 4 \det(\boldsymbol{\gamma})}]}}. \quad (62)$$

Then, having known $\eta_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ it was proved that \mathbf{R} should satisfy the following total system of two PDE:

$$\begin{aligned} \mathbf{R}_{,\alpha} &= \mathbf{R} \times \mathbf{k}_\alpha, \quad (U^{-1})_\beta^\alpha = \sqrt{\frac{\bar{a}}{a}} \varepsilon^{\alpha\lambda} \varepsilon_{\beta\mu} (\delta_\lambda^\mu + \eta_\lambda^\mu), \\ \mathbf{k}_\alpha &= \varepsilon^{\kappa\rho} [b_{\kappa\alpha} - (U^{-1})_\kappa^\lambda (b_{\lambda\alpha} - \kappa_{\lambda\alpha})] \mathbf{a}_\rho - \sqrt{\frac{\bar{a}}{a}} \varepsilon^{\kappa\rho} (\delta_\alpha^\lambda + \eta_\alpha^\lambda) \eta_{\lambda\kappa|\rho} \mathbf{n}. \end{aligned} \quad (63)$$

Instead of solving the system (63) directly, we may again convert the problem into equivalent infinite sets of initial value problems along curves covering densely the

entire domain \mathcal{U} of coordinates θ^α . Since the integrability conditions $\varepsilon^{\alpha\beta} \mathbf{R}_{,\alpha\beta} = \mathbf{0}$ of (63) are equivalent to the compatibility conditions (48)_{3,4}, which are supposed to be approximately satisfied, by the Frobenius-Dieudonné theorem for any initial value $\mathbf{R}_0 = \mathbf{R}(\theta_0^\alpha)$ prescribed at some point $p_0 \in \mathcal{U}$ there exists a unique solution $\mathbf{R}(\theta^\alpha)$ satisfying this initial value, and all such solutions depend continuously on \mathbf{R}_0 . Thus, we can cover the domain \mathcal{U} with a dense set of paths leaving radially from the initial point p_0 and solve the initial value problem for the system of ODE

$$\frac{d\mathbf{R}}{ds} = \mathbf{R}\mathbf{K}, \quad \mathbf{K} = \mathbf{1} \times \mathbf{k}, \quad \mathbf{k} = \mathbf{k}_\alpha \frac{d\theta^\alpha}{ds}. \tag{64}$$

Solution to the initial value problem (64) may again be obtained with the use of any of the well-known techniques, numerical techniques inclusive. In particular, applying the method of successive approximations, see Maurin (1980), the general solution of (64) can be presented in the form

$$\mathbf{R} = \mathbf{R}_0 \mathbf{R}_s, \quad \mathbf{R}_s = \sum_{i=1}^{\infty} \mathbf{O}_i, \quad \mathbf{O}_0(s) = \mathbf{1}, \quad \mathbf{O}_i(s) = \int_{s_0}^s \mathbf{O}_{i-1}(t) \mathbf{K}(t) dt, \quad i \geq 1. \tag{65}$$

One may point out a number of special cases when equation (64) has the solution in the closed form. In particular, when $\mathbf{k} = k(s)\mathbf{i}$, i.e. when \mathbf{k} has a constant direction along C , then $d\mathbf{i}/ds = \mathbf{0}$ and the tensors \mathbf{O}_i satisfy the conditions $\mathbf{O}_i \mathbf{O}_j = \mathbf{O}_j \mathbf{O}_i$ for any i, j . Then the solution (65) can be presented in the exponential form

$$\mathbf{R}(s) = \exp \left(\mathbf{1} \times \mathbf{i} \int_{s_0}^s k(t) ds \right). \tag{66}$$

A still simpler solution may be obtained if \mathbf{k} itself is constant along C , i.e. when $d\mathbf{k}/ds = \mathbf{0}$. Then from (66) it follows that

$$\mathbf{R}(s) = \exp(s\mathbf{1} \times \mathbf{k}). \tag{67}$$

Moreover, the tensor equation (64) is identical with the one describing spherical motion of a rigid body about a fixed point, where s is time and \mathbf{k} is the angular velocity vector in the spatial representation, see for example Goldstein, Poole, and Safko (2002); Lurie (2001); Heard (2005). In analytical mechanics many ingenious analytical and numerical methods of integration of the equation (64) have been devised for various special classes of the function $\mathbf{k}(s)$. A number of such closed-form solutions was summarized, for example, by Gorr, Kudryashova, and Stepanova (1978). Thus, the results already known in analytical mechanics of rigid-body motion may be of great help when analyzing problems discussed here for thin elastic shells.

9 Some related problems

The notions associated with the intrinsic nonlinear theory of thin shells can also be helpful in analyzing some related problems of nonlinear shell theory and of differential geometry. Four such special related problems are briefly discussed below.

9.1 Position of the surface in space

Pietraszkiewicz and Vallée (2007) proposed a new method of unique determination of position of a surface \mathcal{M} in the 3D Euclidean space \mathcal{E} from known components $a_{\alpha\beta}$ and $b_{\alpha\beta}$ of two fundamental forms of the surface. The idea of analysis came from shell theory and has been summarized in subsection 8.3 of the present paper.

We introduced the second-order tensor $\mathbf{F} = \mathbf{a}_\alpha \otimes \mathbf{i}^\alpha$ which brings the Cartesian plane Ox^1x^2 with the base \mathbf{i}_α into the tangent plane $T_M\mathcal{M}$ with the base \mathbf{a}_α attached to the surface point $M \in \mathcal{M}$. Since both planes lie in the space \mathcal{E} , we introduced a rotation $\mathbf{R} \in SO(3)$ that takes one plane to the other. Then using the Tissot theorem and discussion given in subsection 8.3, we introduced the right polar decomposition of \mathbf{F} :

$$\mathbf{F} = \mathbf{R}\mathbf{U}, \quad \mathbf{R}^T = \mathbf{R}^{-1}, \quad \det \mathbf{R} = +1, \quad \mathbf{U}^T = \mathbf{U}. \quad (68)$$

Notice that the tensors \mathbf{R} and \mathbf{U} here are not the same as in subsection 8.3, although some analogy between them may be noted.

First, we proved that components $U_{\alpha\beta}$ of $\mathbf{U} = U_{\alpha\beta}\mathbf{i}^\alpha \otimes \mathbf{i}^\beta$ in (68) can be found explicitly by pure algebra,

$$U_{\alpha\beta} = \frac{a_{\alpha\beta} + \sqrt{a}\delta_{\alpha\beta}}{\sqrt{\text{tr}(a_{\lambda\mu}) + 2\sqrt{a}}}. \quad (69)$$

Then, differentiating the identity $\mathbf{R}^T\mathbf{R} = \mathbf{1}$ along the coordinate lines we obtained the skew-symmetric tensors $\mathbf{R}^T\mathbf{R}_{,\beta}$ expressible through their axial vectors \mathbf{k}_β according to

$$\begin{aligned} \mathbf{R}^T\mathbf{R}_{,\beta} &= \mathbf{k}_\beta \times \mathbf{1} = \mathbf{1} \times \mathbf{k}_\beta, \quad \mathbf{k}_\beta = \varepsilon^{\kappa\rho} b_{\rho\beta} \mathbf{s}_\kappa + c_\beta \mathbf{n}, \\ \mathbf{s}_\alpha &= U_{\alpha\lambda} \mathbf{i}^\lambda, \quad c_\beta = \frac{1}{2} \varepsilon^{\kappa\rho} \left(\Gamma_{\rho,\beta\kappa} - U_{\rho\lambda} \delta^{\lambda\mu} U_{\mu\kappa,\beta} \right). \end{aligned} \quad (70)$$

The integrability conditions of $\mathbf{R}_{,\alpha} = \mathbf{R}(\mathbf{k}_\alpha \times \mathbf{1})$ are

$$\varepsilon^{\alpha\beta} \left(\mathbf{k}_{\alpha,\beta} - \frac{1}{2} \mathbf{k}_\alpha \times \mathbf{k}_\beta \right) = \mathbf{0}, \quad (71)$$

which in components along $\mathbf{s}_\kappa, \mathbf{i}_3$ take the form

$$\varepsilon^{\alpha\beta} \varepsilon^{\kappa\rho} b_{\rho\alpha|\beta} \mathbf{s}_\kappa + \varepsilon^{\alpha\beta} \left(c_{\alpha|\beta} - \frac{1}{2} \varepsilon^{\kappa\rho} b_{\kappa\alpha} b_{\rho\beta} \right) \mathbf{i}_3 = \mathbf{0}. \tag{72}$$

The relations (71) and (72) proposed by Pietraszkiewicz and Vallée (2007) are equivalent to the GMC equations written in the component form by Vallée and Fortuné (1996) and rederived by Ciarlet, Gratie, and Mardare (2007, 2008).

Applying the same integration method as in subsection 8.3, we can find \mathbf{R} and then the following explicit formula for position of the surface in space \mathcal{E} can be derived:

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{R}_0 \left[\int_0^{\theta^1} \mathbf{R}_1(\xi, 0) \mathbf{U}(\xi, 0) \mathbf{i}_1 d\xi + \mathbf{R}_1 \int_0^{\theta^2} \mathbf{R}_2(\theta^1, \eta) \mathbf{U}(\theta^1, \eta) \mathbf{i}_2 d\eta \right], \tag{73}$$

where \mathbf{r}_0 and \mathbf{R}_0 are constants assumed to be known at M_0 .

Efficiency of the method was illustrated by Pietraszkiewicz and Vallée (2007) on the simple example of establishing the unique position of the surface parameterized by coordinates θ^α such that components of two fundamental forms are given by

$$(a_{\alpha\beta}) = \begin{pmatrix} \frac{1}{\lambda^2} & 0 \\ 0 & \frac{1}{\lambda^2} \end{pmatrix}, \quad (b_{\alpha\beta}) = \begin{pmatrix} \frac{1}{\lambda^2} & 0 \\ 0 & \frac{1}{\lambda^2} \end{pmatrix}, \quad a = \frac{1}{\lambda^4}, \tag{74}$$

$$\lambda = \frac{1}{2} [1 + (\theta^1)^2 + (\theta^2)^2] > 0, \quad \lambda_{,1} = \theta^1, \quad \lambda_{,2} = \theta^2.$$

From the analysis performed according to this method we obtained the sphere parameterized by stereographic projection, see Fig. 1.

9.2 Buckling of the axially compressed circular cylinder

Opoka and Pietraszkiewicz (2009b) presented extensive numerical results on bifurcation buckling analysis of the axially compressed circular cylinder. The analysis was based on two-dimensionally exact intrinsic equilibrium equations (7) which were modified with the help of compatibility conditions (2), see Opoka and Pietraszkiewicz (2009a). Since in this case the boundary conditions were given in terms of translations and their derivatives, the very complex BVP and the corresponding homogeneous shell buckling problem (SBP) were generated automatically in the computer memory by two program packages set up in the symbolic language of MATHEMATICA. The SBP was generated without using any additional approximation following from errors of the constitutive equations (14) and (15). Such an approach allowed us to always account for those a few supposedly small terms which may be critical in finding the correct buckling load of this shell structure very sensitive to imperfections.

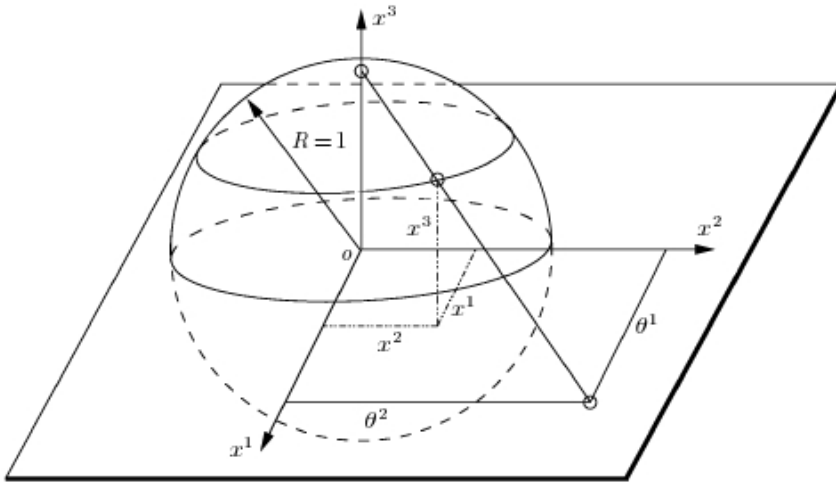


Figure 1: Sphere parameterized by stereographic projection

The numerical analysis of the weighted buckling load ρ was performed for the cylinders with length-to-diameter ratio l in the range $(0.05, 60)$, with eight sets of work-conjugate boundary conditions analogous to those used in the literature and partly summarized in the book by Yamaki (1984), and additionally with six sets of boundary conditions not previously discussed in the literature. In the analysis, $\rho = 1$ corresponds to the classical value of the buckling load of the cylinder given by Lorenz (1911).

The results partly presented in Fig. 2 and Fig. 3 allowed us to formulate several important conclusions, for example:

- omission of small terms of the order of error of constitutive equations in the non-linear tangential equilibrium equations and compatibility conditions leads to the overestimated buckling load for long cylinders with clamped boundaries (the curve $C1_S$ in Fig. 2);
- for some relaxed boundary conditions the buckling load decreases for short cylinders with decrease of the cylinder length, which does not agree with the results by Yamaki (1984), but confirms similar conclusion of Danielson and Simmonds (1969), see curves $C4$ and $S4$ in Fig. 3;
- the results obtained with six additional sets of boundary conditions revealed existence of several new cases, in which by relaxing one kinematic boundary

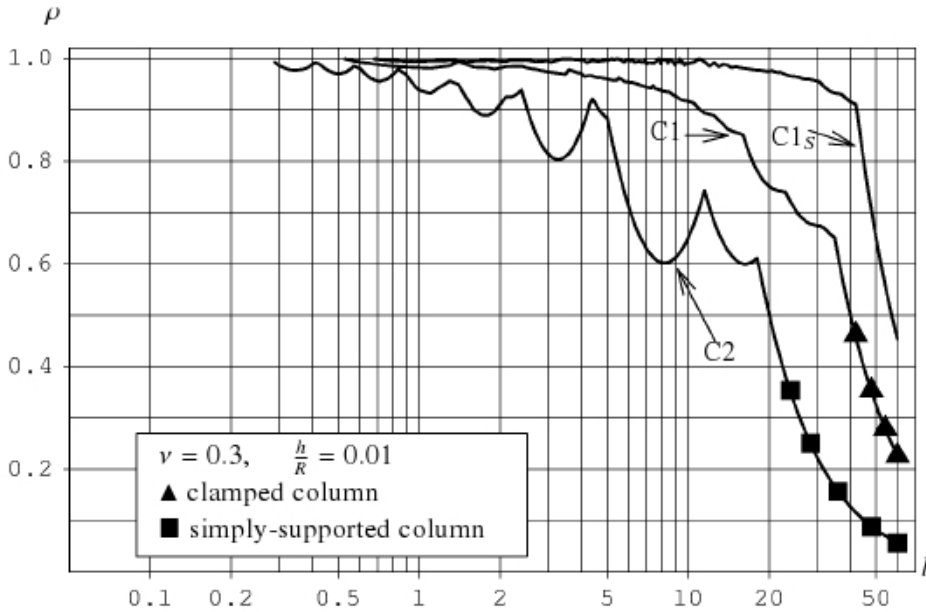


Figure 2: The buckling load of axially compressed perfect cylinder for boundary conditions C1 and C2

condition the buckling load falls down to about one half of the classical value in a wide range of the cylinder length-to-diameter ratios.

For many other results and conclusions we refer to Opoka and Pietraszkiewicz (2009b).

9.3 Position of deformed base surface from its metric components and the height function

Szwabowicz (1999) proposed an alternative BVP for the nonlinear theory of thin elastic shells, which was expressed in terms of three surface strains and the distance (height) function of the deformed base surface from some arbitrarily fixed plane as basic independent field variables. Then Szwabowicz and Pietraszkiewicz (2004) solved the following problem: given the strain tensor of a base surface of a thin shell and the height function find position of the deformed base surface in space. Two alternative procedures supplying the solution were developed. The first one follows from ideas developed by Darboux (1894), whereas the second one is based on the polar decomposition theorem and techniques discussed in section 7 and subsection

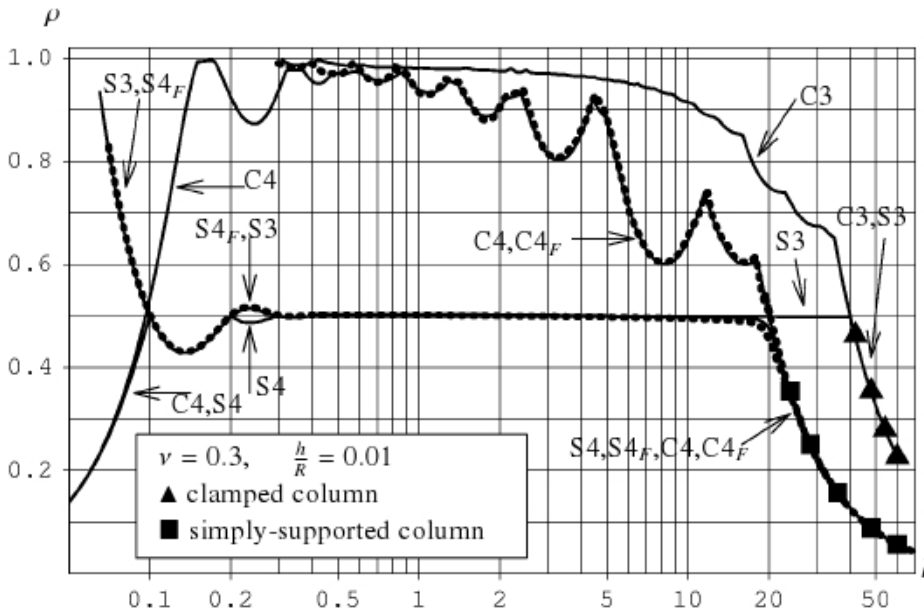


Figure 3: The buckling load of axially compressed perfect cylinder for boundary conditions S3, S4, C3 and C4

8.3. These procedures are purely kinematic, valid for arbitrary surface geometry and for unrestricted surface strains, translations and rotations. The results of that paper suggest that this approach to the non-linear problems of thin shells may be an attractive alternative to other BVPs developed in the literature.

9.4 Flexible shells

Thin elastic shells are sometimes specifically designed to allow maximum displacements realizable within small elastic strains. Four types of such flexible shells are presented in Fig. 1 of Axelrad (1984). Adequate description of such deformation states in thin shells can be given by the nonlinear flexible shell theory (FST). The FST is characterized by the local shell shape and the stress state which vary with one surface coordinate more intensively than with the other. To properly describe this specific case of shell deformation it is convenient to take the coordinates θ^α to be the orthogonal lines of curvatures of the shell midsurface for which $a_{12} = b_{12} = 0$.

In the intrinsic formulation the FST is characterized by the nonlinear shell equa-

tions in the rotated basis which are much simpler than those given in (48). This specialized theory is sometimes called the semi-momentless or semi-membrane shell theory, because it represents adequately the stress states free from bending in, for example, the cross sections $\theta^1 = \text{const}$, which may exhibit a substantial bending in the orthogonal cross sections $\theta^2 = \text{const}$. The survey of this simple but versatile version of the nonlinear intrinsic shell theory is given in the papers by Axelrad (1980,1984); Axelrad and Emmerling (1987); and in the book by Axelrad (1987).

10 Conclusions

We have reviewed some achievements in the intrinsic formulation of the geometrically nonlinear theory of thin elastic shells, foundations of which were laid down by Chien (1944). Three groups of problems have been reviewed: a) several consistent intrinsic BVPs, b) determination of position in space of the deformed base surface from known surface strain measures, and c) four special problems somewhat related to the intrinsic formulation of shell theory.

In 1940's the intrinsic formulation of thin shell equations was supposed to be promising as an alternative to the extremely complex nonlinear shell BVP formulated in terms of translations, when rotations of material elements are unrestricted. However, incredible advances in computer technology during the last 66 years combined with development of powerful numerical methods, and the finite element method in particular, has made the intrinsic approach to formulation of shell equations less and less attractive and finally obsolete in the numerical analyses of shell structures. Nowadays only a few theoretically inclined shell specialists contribute sometimes to this field. As a result, there are still some areas here which are worth to explore. Let me mention two of them.

While the ranges of applicability of the refined intrinsic BVPs (20) - (23), (48) and some of their simplified cases discussed in section 6 are well documented, there is virtually no 2D numerical applications of these BVPs to analyze stress and strain fields in highly nonlinear problems of thin shells.

Three general methods of determination of position of the deformed base surface from the surface strain measures have been discussed in section 8. But again, numerical algorithms and computer codes based on these methods are still not available. Particularly interesting for numerical applications in thin shells might be the method presented in subsection 8.3. It seems that one could use here the algorithms and numerical codes available in the mechanics of rigid-body motion to develop numerical programs for large-rotation analyses of thin elastic shells.

Acknowledgement: This paper contains the full text of the W.-Z. Chien Award

Lecture to be presented at ICCES' 11, April 18-21, in Nanjing (PR China). I would like to thank Professor S.N. Atluri very much indeed for proposing my name to the Award Committee of ICCES for the 2011 ICCES / Wei-Zhang Chien Award.

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