Natural Boundary Element Method for Stress Field in Rock Surrounding a Roadway with Weak Local Support

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Abstract: Weak local support is a very common phenomenon in roadway support engineering. It is a problem that needs to be studied thoroughly at the theoretical level. So far, the literature on stress field theory of rock surrounding a roadway is largely restricted to analytical solutions of stress for roadways with a uniform support or no support at all. The corresponding stress solution under conditions of local or weak local support has not been provided. Based on a mechanical model of weak local support at the boundary of a circular roadway and the boundary element method on boundary value problems of bi-harmonic equations of a circular exterior domain, the boundary integration formula of an Airy stress function is deduced for roadways with weak local support. Using the boundary surface force of the roadway to calculate the boundary stress function and its normal derivative and substituting them into an integration formula, we derived a concrete expression of the stress function under various levels of support resistance. Furthermore, we have provided the rules of the distribution of stress and strain in the rock surrounding a roadway with different areas of weak support and support resistance. From the integration formula, the solutions for analytical stress can be directly obtained for the rock surrounding a roadway with uniform support or without any support at all. The solutions are exactly the same as those given in the literature.

Keywords: Roadway with weak local support; boundary value problem of bi-harmonic equation; natural boundary element method; boundary integration formula of stress function; stress field.

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1 Introduction

In mining engineering, stress distribution inside rocks, surrounding a roadway, is an important basis from which to study the stability of roadways under a variety of supporting loads. Deep underground roadways of various shapes can be simplified as holes in an infinitely elastic body, usually by plane strain models in order to analyze stress and deformation fields in the surrounding rock. Calculation methods such as analytical, finite element, boundary element and other numerical methods are commonly used. The analytical method [Chen (1994)] usually provides only a stress solution for roadways with a uniform support or no support at all. Roadways in mines often present problems of weak boundary support or weak local support [Li, Wang and Hou (2002); Pang, Guo and Liu (2004); Wang and Feng (2005); Li, Wang and Pan (2007)]. Moreover, not until roadway deformation starts to affect production, will changes be made in beams or columns at positions of weak support. The instability of the surrounding rock at these positions suggests a lack of foresight. Weak support is not only a common phenomenon in engineering, but is equally weak in theoretical studies. Therefore, it is necessary to study the stress field in the rock surrounding roadways, given weak support and analyze the mechanism of its instability. Without loss of generality, our study uses a circular roadway as an example. In theory, the natural boundary element method can be used to solve the stress field in the rock, surrounding a circular roadway, under a variety of uniform and local support conditions [Li (2003)].

The natural boundary element method is a branch of a number of boundary element methods, based on a Green functional method, a complex variable method, or a method using a Fourier series to induce a Dirichlet boundary value problem as a differential equation into Poisson integration formula of the studied area or to induce Neumann boundary value problem of differential equation into a strong singular boundary integral equation [Yu (1993)]. The natural boundary element method is widely used to solve problems of a circular interior and exterior domain and other plane and engineering problems. Zhao et al., (2000), Yu and Du (2003), Liu and Yu (2008) and Peng et al., (2009) have investigated coupling methods between natural boundary element and finite element methods. Based on the natural boundary element method on the boundary value problem of a bi-harmonic equation of a circular exterior domain, a boundary integration formula of the Airy stress function φ in polar coordinates is obtained. This uses surface forces on the roadway boundary to calculate the stress function and its normal derivative, substituting them into the integration formula, so that the specific expression of a stress function under various supporting conditions can be obtained, permitting the analysis of stress and related deformation inside the surrounding rock.
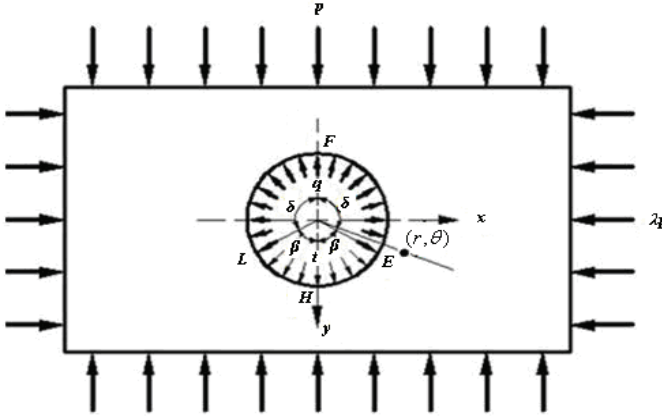
Figure 1: Mechanical model of circular roadway with weak local support

2 Mechanical model of a roadway with weak local support

Fig.1 is a schematic diagram of a roadway under non-uniform boundary support, with an infinite vertical ground stress p and infinite lateral stress λp, where λ is the lateral pressure coefficient. On the roadway boundary within the range of the arc *EFL* and angle $2\delta\,(\pi/2 < \delta < \pi)$ its supporting surface force is q, while the arc *EHL* and angle $2\beta\,(\beta = \pi - \delta)$ supporting surface force is t. The shear modulus and Poisson's ratio of the surrounding rock are denoted by G and μ; x, y is the axis of symmetry through the center of the roadway.

It is obvious that if $(1)\,q = t \neq 0$, i.e., the roadway boundary is supported uniformly; (2) $q = t = 0$, the roadway boundary has no support; (3) when $q > t, 0 < \beta < 90°$, the floor has weak local support and (4) when $t = 0, \beta = 90°$, the floor has no support.

Similar to Fig.1 and according to the actual support condition of the roadway, we can also establish mechanical models of the roof or sides with weak support.

Using the natural boundary element method, the boundary integration formula of the stress function is derived inside the surrounding rock, given the polar coordinates in Fig.1. Without loss of generality, we denote the roadway radius a as unit one. For the plane problem of a constant body force, the stress function $\varphi(r, \theta)$ is a bi-harmonic function, i.e. $\nabla^4 \varphi(r, \theta) = 0$. According to the complex variable method [Xu (1990)], the stress function can be expressed as

$$\varphi(r, \theta) = Re[\bar{z}\varphi_1(z) + \theta_1(z)] \tag{1}$$

where $z = re^{i\theta}, r \geq 1, \varphi_1(z), \theta_1(z)$ are two analytical functions; for an infinitely large plane problem with multiple holes, we have:

$$\varphi_1(z) = -\frac{1+\mu}{8\pi}(X+iY)\ln z + Bz + \sum_{n=0}^{\infty} a_n z^{-n} \tag{2}$$

$$\theta_1'(z) = \psi_1(z) = \frac{3-\mu}{8\pi}(X-iY)\ln z + (B_1+iC_1)z + \sum_{n=0}^{\infty} b_n z^{-n} \tag{3}$$

where X and Y are principal vectors of the roadway boundary surface force along the x and y directions. In Fig. 1, we have:

$$X = 0 \tag{4}$$

$$Y = -\int_{-\delta}^{\delta} q\cos\alpha d\alpha + \int_{\delta-\frac{\pi}{2}}^{\frac{3\pi}{2}-\delta} t\cos\alpha d\alpha = 2(t-q)\sin\delta \tag{5}$$

a_n, b_n are Laurent series coefficients; without changing the stress state conditions, we can let $a_0 = 0, b_0 = 0$. B, B_1 and C_1 are determined by the principal stresses $\sigma_1, \sigma_2 (\sigma_1 > \sigma_2)$ in infinity and by the angle between the principal stress σ_1 and the x axis [Xu (1990)], where:

$$B = \frac{1}{4}(\sigma_1+\sigma_2) \quad B_1+iC_1 = -\frac{1}{2}(\sigma_1-\sigma_2)e^{-2i\alpha} \tag{6}$$

According to Fig.1, when $\lambda < 1$, then $\sigma_1 = -\lambda p > \sigma_2 = -p$, the angle between σ_1 and the positive x axis, $\alpha = \pi$; substituting them into Eq. (6), we have:

$$B = -\frac{1+\lambda}{4}p, \quad B_1 = -\frac{1-\lambda}{2}p, C_1 = 0 \tag{7}$$

Substituting $X = 0$ into Eq. (3), the integral becomes:

$$\theta_1(z) = -i\frac{3-\mu}{8\pi}Y \cdot z(\ln z - 1) + \sum_{n=1}^{\infty}(-\frac{1}{n}b_{n+1}z^{-n}) + b_1\ln z + \frac{(B_1+iC_1)}{2}z^2 + B' + iC' \tag{8}$$

where B', C' are constants of integration.

Substituting Eqs. (8) and (2) into Eq. (1):

$$\varphi(r,\theta) =$$

$$Re\left[-i\frac{1+\mu}{8\pi}\bar{z}Y\ln z + Br^2 - i\frac{3-\mu}{8\pi}Yz(\ln z - 1) + \sum_{n=1}^{\infty}(a_n\bar{z}z^{-n} - \frac{b_{n+1}}{n}z^{-n})\right]$$

$$+Re\left[b_1\ln z + \frac{(B_1+iC_1)}{2}z^2 + B' + iC'\right] \tag{9}$$

Substituting $z = re^{i\theta}, \bar{z} = re^{-i\theta}, r \geq 1$ into Eq. (9), we obtain:

$$\varphi(r,\theta) = \frac{r\theta\cos\theta}{2\pi}Y + \frac{[(2-2\mu)\ln r - (3-\mu)]r\sin\theta}{8\pi}Y + Br^2 + b_1\ln r + B'$$

$$+ \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} [\frac{a_{|n|-1}}{2}r^2 + (-\frac{b_{|n|+1}}{2|n|})]r^{-|n|}e^{in\theta} + \frac{r^2}{2}(B_1\cos 2\theta - C_1\sin 2\theta), \ r \geq 1$$

$$(10)$$

Various undetermined coefficients in Eq.(10) may be determined by the roadway boundary stress function, its normal derivative and the stress condition at infinity.

From Eq. (10) we can obtain the stress function $\varphi_0(\theta)$ and its normal derivative $\varphi_n(\theta)$ for the roadway boundary.

$$\varphi_0(\theta) = \varphi(1,\theta) = \frac{\theta\cos\theta}{2\pi}Y - \frac{(3-\mu)\sin\theta}{8\pi}Y + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} [\frac{a_{|n|-1}}{2} + (-\frac{b_{|n|+1}}{2|n|})]e^{in\theta}$$

$$+ B + B' + \frac{1}{2}(B_1\cos 2\theta - C_1\sin 2\theta)$$

$$(11)$$

$$\varphi_n(\theta) = \frac{\partial\varphi}{\partial n} = -\frac{\partial\varphi}{\partial r}\bigg|_{r=1} = -\frac{\theta\cos\theta}{2\pi}Y + \frac{1}{8\pi}(1+\mu)Y\sin\theta$$

$$+ \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} [\frac{(|n|-2)}{2}a_{|n|-1} - \frac{b_{|n|+1}}{2}]\cdot e^{in\theta} - (2B+b_1) - (B_1\cos 2\theta - C_1\sin 2\theta)$$

$$(12)$$

Let

$$\varphi_0(\theta) = \frac{\theta\cos\theta}{2\pi}Y + \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \tag{13}$$

$$\varphi_n(\theta) = -\frac{\theta\cos\theta}{2\pi}Y + \sum_{n=-\infty}^{\infty} d_n e^{in\theta} \tag{14}$$

Comparing the coefficients on the right side of Eqs. (11) and (13) and of Eqs. (12) and (14), respectively, we simplify and have

$$
\begin{cases}
B' = c_0 - B \\
a_{|n|-1} = c_1 - d_1 + \frac{Y}{4\pi i}, & n = 1 \\
a_{|n|-1} = c_{-1} - d_{-1} - \frac{Y}{4\pi i}, & n = -1 \\
a_{|n|-1} = 2c_2 - B_1 - d_2 + \frac{C_1}{i}, & n = 2 \\
a_{|n|-1} = 2c_{-2} - B_1 - d_{-2} - \frac{C_1}{i}, & n = -2 \\
a_{|n|-1} = |n|c_n - d_n, & n \neq 0, \pm 1, \pm 2
\end{cases}
\tag{15}
$$

$$
\begin{cases}
b_{|n|+1} = -2B - d_0, & n = 0 \\
b_{|n|+1} = -c_1 - d_1 - \frac{(1-\mu)Y}{8\pi i}, & n = 1 \\
b_{|n|+1} = -c_{-1} - d_{-1} + \frac{(1-\mu)Y}{8\pi i}, & n = -1 \\
b_{|n|+1} = -B_1 - 2d_2 + \frac{C_1}{i}, & n = 2 \\
b_{|n|+1} = -B_1 - 2d_{-2} - \frac{C_1}{i}, & n = -2, \\
b_{|n|+1} = |n| \cdot [(|n|-2)c_n - d_n], & n \neq 0, \pm 1, \pm 2
\end{cases}
\tag{16}
$$

Substituting a_n, b_n into Eq. (10), where parts of the series become:

$$
\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left[\frac{a_{|n|-1}}{2} r^2 + \left(-\frac{b_{|n|+1}}{2|n|} \right) \right] r^{-|n|} e^{in\theta} =
$$

$$
\sum_{n=-\infty}^{\infty} \left\{ \frac{1}{2}(r^2-1)|n|c_n + c_n - \frac{r^2-1}{2}d_n \right\} r^{-|n|} e^{in\theta}
$$

$$
- c_0 - \frac{1-r^2}{2}d_0 + \frac{Y}{4\pi} r^{-1} \sin\theta \left[\frac{1-\mu}{2} + r^2 \right] + \frac{1}{2}(B_1 \cos 2\theta - C_1 \sin 2\theta)(r^{-2} - 2),
$$

$$
r \geq 1
\tag{17}
$$

The part of the series on the right side of Eq.(17) can be rewritten as the convolution form:

$$
\sum_{-\infty}^{\infty} \left\{ \frac{1}{2}(r^2-1)|n|c_n + c_n - \frac{1}{2}(r^2-1)d_n \right\} r^{-|n|} e^{in\theta}
$$

$$
= \frac{1}{4\pi}(r^2-1) \left[\sum_{n=-\infty}^{\infty} |n| r^{-|n|} e^{in\theta} \right] * \left[\sum_{n=-\infty}^{\infty} c_n e^{in\theta} \right] + \frac{1}{2\pi} \left[\sum_{n=-\infty}^{\infty} r^{-|n|} e^{in\theta} \right]
$$

$$
\times \left[\sum_{n=-\infty}^{\infty} c_n e^{in\theta} \right] - \frac{(r^2-1)}{4\pi} \left[\sum_{n=-\infty}^{\infty} r^{-|n|} e^{in\theta} \right] \right\} * \left[\sum_{n=-\infty}^{\infty} d_n e^{in\theta} \right]
\tag{18}
$$

According to the basic formula of generalized functions [Yu (1993)], when $r > 1$

$$\sum_{n=-\infty}^{\infty} r^{-|n|} e^{in\theta} = \frac{r^2 - 1}{1 + r^2 - 2r\cos\theta} \tag{19}$$

$$\sum_{n=-\infty}^{\infty} |n| \, r^{-|n|} e^{in\theta} = \frac{2r^3 \cos\theta - 4r^2 + 2r\cos\theta}{(1 + r^2 - 2r\cos\theta)^2} \tag{20}$$

Substituting Eqs. (19) and (20) into Eq. (18) and then using Eqs. (13) and (14), we have

$$\sum_{-\infty}^{\infty} \left\{ \frac{1}{2}(r^2 - 1)\,|n|\,c_n + c_n - \frac{1}{2}(r^2 - 1)d_n \right\} r^{-|n|} e^{in\theta}$$

$$= \int_0^{2\pi} \left\{ \frac{(r^2 - 1)^2[r\cos(\theta - \theta') - 1]}{2\pi[1 + r^2 - 2r\cos(\theta - \theta')]^2} \left[\varphi_0(\theta') - \frac{\theta'\cos\theta'}{2\pi}Y \right] \right. \tag{21}$$

$$\left. - \frac{(r^2 - 1)^2}{4\pi[1 + r^2 - 2r\cos(\theta - \theta')]} \left[\varphi_n(\theta') + \frac{\theta'\cos\theta'}{2\pi}Y \right] \right\} d\theta' = f(r, \theta) + f_1(r, \theta)$$

where

$$f(r, \theta) = \int_0^{2\pi} \left\{ \frac{(r^2 - 1)^2[r\cos(\theta - \theta') - 1]}{2\pi[1 + r^2 - 2r\cos(\theta - \theta')]^2} \varphi_0(\theta') \right.$$

$$\left. - \frac{(r^2 - 1)^2}{4\pi[1 + r^2 - 2r\cos(\theta - \theta')]} \varphi_n(\theta') \right\} d\theta' \tag{22}$$

$$f_1(r, \theta) = -\int_0^{2\pi} \frac{(r^2 - 1)^3}{4\pi[1 + r^2 - 2r\cos(\theta - \theta')]^2} \frac{\theta'\cos\theta'}{2\pi} Y\, d\theta' \tag{23}$$

Substituting Eq. (21) into Eq. (17) and then the results into Eq. (10), the boundary integration formula of the stress function in the rock surrounding the roadway with weak local support is obtained as follows:

$$\varphi(r, \theta) =$$

$$\frac{r\theta\cos\theta}{2\pi}Y + \frac{[(2 - 2\mu)\ln r - (3 - \mu)]}{8\pi} r\sin\theta \cdot Y + \frac{r^{-1}\sin\theta}{4\pi}Y\left[\frac{1 - \mu}{2} + r^2\right] + f(r, \theta)$$

$$+ f_1(r, \theta) + \left(\frac{B_1}{2}\cos 2\theta - \frac{C_1}{2}\sin 2\theta\right)(r^{-2} + r^2 - 2) + (2B + d_0)\left(\frac{r^2 - 1}{2} - \ln r\right),$$

$$\quad r \geq 1$$

$$\tag{24}$$

where B, B_1, C_1 and $f(r, \theta), f_1(r, \theta)$ are respectively determined by the earlier mentioned Eqs. (6), (22) and (23) and for d_0, determined by Eq. (14), we obtain:

$$d_0 = \frac{1}{2\pi} \int_0^{2\pi} [\varphi_n(\theta) + \frac{\theta \cos \theta}{2\pi} Y] d\theta \qquad (25)$$

3 Calculation of the boundary stress function $\varphi_0(\theta)$ and its normal derivative $\varphi_n(\theta)$

In order to calculate $f(r, \theta)$, $f_1(r, \theta)$ in the Eq.(24) by Eqs. (22) and (23), we first need to determine the boundary stress function $\varphi_0(\theta)$ and its normal derivative $\varphi_n(\theta)$ according to the known surface force $\bar{X}, \bar{Y}$ on the roadway. Based on the method of solving the boundary stress function in mechanics of elasticity [Xu (1990)], in Fig. 2, a base point A is selected on the roadway boundary, namely

$$\varphi_A = 0, \quad \left(\frac{\partial \varphi}{\partial x} \right)_A = 0, \left(\frac{\partial \varphi}{\partial y} \right)_A = 0 \qquad (26)$$

Then for an arbitrary point B on the boundary, we have

$$\varphi_0(\theta) = \varphi_B = -\int_A^B (x - x_B) \bar{Y} ds - \int_A^B (y_B - y) \bar{X} ds \qquad (27)$$

and

$$\left(\frac{\partial \varphi}{\partial x} \right)_B = \int_A^B \bar{Y} ds \qquad (28)$$

$$\left(\frac{\partial \varphi}{\partial y} \right)_B = -\int_A^B \bar{X} ds \qquad (29)$$

$$\varphi_n(\theta) = \left(\frac{\partial \varphi}{\partial n} \right)_B = \left(\frac{\partial \varphi}{\partial x} \right)_B (-\cos \theta) + \left(\frac{\partial \varphi}{\partial y} \right)_B (-\sin \theta) \qquad (30)$$

For the separate range $2\delta \, (\pi/2 < \delta < \pi)$ with the supporting force q in Fig.1, the stress function and its normal derivative on the boundary are calculated with Eqs.(27)~(30) as

$$\varphi_0(\theta) = \varphi_B = \begin{cases} -q(1 - \cos \theta), & 0 \le \theta \le \delta - \frac{\pi}{2} \\ -q[\sin(\delta - \theta) - \cos \theta], & \delta - \frac{\pi}{2} \le \theta \le \frac{3}{2}\pi - \delta \\ -q[1 + (2 \sin \delta - 1) \cos \theta], & \frac{3}{2}\pi - \delta \le \theta \le 2\pi \end{cases} \qquad (31)$$
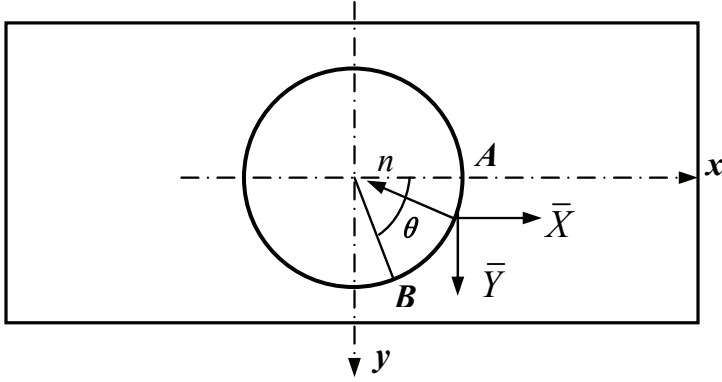
Figure 2: Schematic diagram of calculating the boundary stress function

$$\varphi_n(\theta) = \frac{\partial \varphi}{\partial n}\Big|_B = \begin{cases} q(1-\cos\theta), & 0 \leq \theta \leq \delta - \frac{\pi}{2} \\ q[\sin(\delta-\theta)-\cos\theta], & \delta - \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}-\delta \\ q[1+(2\sin\delta - 1)\cos\theta], & \frac{3\pi}{2}-\delta \leq \theta \leq 2\pi \end{cases} \tag{32}$$

Substituting Eqs. (31) and (32) into Eq. (22), its integration result is recorded as $f^q(r,\theta)$, i.e.:

$$f^q(r,\theta) =$$

$$\frac{q(r^2-1)^3}{4\pi}\Big\{ \int_0^{\delta-\frac{\pi}{2}} \frac{(\cos\theta'-1)}{[1+r^2-2r\cos(\theta-\theta')]^2}d\theta' + \int_{\delta-\frac{\pi}{2}}^{\frac{3\pi}{2}-\delta} \frac{(\cos\theta'-\sin(\delta-\theta'))}{[1+r^2-2r\cos(\theta-\theta')]^2}d\theta'$$

$$-\int_{\frac{3\pi}{2}-\delta}^{2\pi} \frac{[1+(2\sin\delta-1)\cos\theta']}{[1+r^2-2r\cos(\theta-\theta')]^2}d\theta'\Big\}$$

$$\tag{33}$$

Similarly, for the separately remaining range $2\pi - 2\delta$ with support resistance t, we obtain:

$$\varphi_0(\theta) = \varphi_B = \begin{cases} 0, & 0 \leq \theta \leq \delta - \frac{\pi}{2} \\ -t[1+\sin(\theta-\delta)], & \delta - \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}-\delta \\ 2t\sin\delta\cos\theta, & \frac{3}{2}\pi-\delta \leq \theta \leq 2\pi \end{cases} \tag{34}$$

$$\varphi_n(\theta) = \frac{\partial \varphi}{\partial n}\Big|_B = \begin{cases} 0, & 0 \le \theta \le \delta - \frac{\pi}{2} \\ t[1 + \sin(\theta - \delta)], & \delta - \frac{\pi}{2} \le \theta \le \frac{3\pi}{2} - \delta \\ -2t \sin \delta \cos \theta, & \frac{3\pi}{2} - \delta \le \theta \le 2\pi \end{cases} \tag{35}$$

Substituting Eqs.(34) and (35) into Eq. (22), the result of the integration is recorded as $f^t(r, \theta)$, i.e.:

$$f^t(r, \theta) = \frac{t(r^2 - 1)^3}{4\pi}$$

$$\{ \int_{\delta - \frac{\pi}{2}}^{\frac{3\pi}{2} - \delta} \frac{-[1 + \sin(\theta' - \delta)]}{[1 + r^2 - 2r\cos(\theta - \theta')]^2} d\theta' + \int_{\frac{3\pi}{2} - \delta}^{2\pi} \frac{2\sin\delta\cos\theta'}{[1 + r^2 - 2r\cos(\theta - \theta')]^2} d\theta' \} \tag{36}$$

Thus,

$$f(r, \theta) = f^q(r, \theta) + f^t(r, \theta) \tag{37}$$

Substituting $Y = (2t - 2q)\sin\delta$ into Eq. (23), we have

$$f_1(r, \theta) = -\int_0^{2\pi} \frac{(r^2 - 1)^3}{4\pi[1 + r^2 - 2r\cos(\theta - \theta')]^2} \frac{\theta'\cos\theta'}{2\pi}(-2q + 2t)\sin\delta d\theta' \tag{38}$$

Substituting $\varphi_n(\theta)$ in Eq.(25) by Eqs. (32), (35) and noting that $Y = (2t - 2q)\sin\delta$, we obtain by integration:

$$d_0 = \frac{\delta}{\pi}q + \frac{\pi - \delta}{\pi}t \tag{39}$$

Substituting Eqs. (37), (38) and (39) into Eq. (24), the stress function $\varphi(r, \theta)$ at any point inside the surrounding rock can be obtained and by the following formulas

$$\sigma_r = \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}, \quad \sigma_\theta = \frac{\partial^2 \varphi}{\partial r^2}, \quad \tau_{r\theta} = -\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \varphi}{\partial \theta}) \tag{40}$$

then the related stress analysis can be carried out. In the following analyses, we take a roadway boundary with uniform support as an example to illustrate that the derivation of Eq. (24) is correct.

4 Stress function and stress formula for roadway boundary with uniform support

When the roadway boundary is supported uniformly, i.e. when $t = q$ in Fig.1, then, by Eqs. (33), (36) and (37), we have:

$$f(r,\theta) = \frac{q(r^2-1)^3}{4\pi} \int_0^{2\pi} \frac{(\cos\theta'-1)}{[1+r^2-2r\cos(\theta-\theta')]^2} d\theta' \tag{41}$$

By integration, Eq. (41) can be expressed as

$$f(r,\theta) = q[r\cos\theta - \frac{(r^2+1)}{2}] \tag{42}$$

Then from Eq. (39), we obtain:

$$d_0 = \frac{\delta}{\pi}q + \frac{\pi-\delta}{\pi}q = q \tag{43}$$

From Eqs.(38) and (5), we respectively obtain $f_1(r,\theta) = 0$ and $Y = 0$. Substituting $B, B_1, C_1, f(r,\theta), f_1(r,\theta)$ and d_0 into Eq. (24), the stress function is obtained for a roadway with uniform support as:

$$\varphi(r,\theta) = -\frac{p-\lambda p}{4}(r^{-2}+r^2-2)\cos 2\theta + \frac{p+\lambda p}{2}(\ln r - \frac{r^2-1}{2})$$
$$+ qr\cos\theta - q(1+\ln r), \quad r \geq 1 \tag{44}$$

Hence, we can obtain the corresponding stress formula for a roadway with uniform support.

$$\begin{cases} \sigma_r = -\frac{p+\lambda p}{2}(1-\frac{1}{r^2}) + \frac{p-\lambda p}{2}(1-\frac{4}{r^2}+\frac{3}{r^4})\cos 2\theta - \frac{q}{r^2} \\ \sigma_\theta = -\frac{p+\lambda p}{2}(1+\frac{1}{r^2}) - \frac{p-\lambda p}{2}(1+\frac{3}{r^4})\cos 2\theta + \frac{q}{r^2} \\ \tau_{r\theta} = -\frac{p-\lambda p}{2}(1+\frac{2}{r^2}-\frac{3}{r^4})\sin 2\theta \end{cases} \quad (r \geq 1) \tag{45}$$

During this derivation, we assumed the radius a of the roadway to be unit one; in general, when the radius is a, we replace r with r/a on the right side of Eq. (45) as

$$\begin{cases} \sigma_r = -\frac{p+\lambda p}{2}(1-\frac{a^2}{r^2}) + \frac{p-\lambda p}{2}(1-\frac{4a^2}{r^2}+\frac{3a^4}{r^4})\cos 2\theta - \frac{a^2 q}{r^2} \\ \sigma_\theta = -\frac{p+\lambda p}{2}(1+\frac{a^2}{r^2}) - \frac{p-\lambda p}{2}(1+\frac{3a^4}{r^4})\cos 2\theta + \frac{q}{r^2}a^2 \\ \tau_{r\theta} = -\frac{p-\lambda p}{2}(1+\frac{2a^2}{r^2}-\frac{3a^4}{r^4})\sin 2\theta \end{cases} \tag{46}$$

Eqs. (45) and (46) are exactly the same as the analytical solution given by the general theory of rock mechanics [Gao and Zhang (2000)]. Particularly, if $q = 0$ in Eqs. (45) and (46), the analytical stress solution of the surrounding rock can be directly obtained for the excavated roadway without any boundary support.

5 Stress and strain distribution in rock surrounding a roadway with weak local support

With ground stress p, the support resistance q and t and the support angle δ or β, the lateral pressure coefficient λ, given Eqs. (24) and (40), we can calculate the stress value of any point (r, θ) inside the surrounding rock in Fig. 1.

We established four groups of support resistance:

(1) q =8 MPa, t =0 MPa: corresponding to the local no-support floor (especially when $\beta = \pi/2$ corresponding to the entire no- support floor);

(2) q =8 MPa, t =3 MPa : floor with relatively weak local support:

(3) q =3 MPa, t =8 MPa: weak support within a wide range of the roadway; only the floor has strong local support;

(4) q =3 MPa, t =3 MPa: the roadway is all around weakly supported

Fig. 3 shows the variation of the normal radial stress σ_r with angle θ at r=2a in the surrounding rock, when p =25 MPa, $\lambda = 0.8$ and the angle of weak support range is $\beta = \pi/2$ and $\beta = \pi/3$, respectively.

From Fig.3, we can see that the stress release in the surrounding rock of the floor is most evident and the two release peaks are symmetric for the floor without support (t =0). Given a roof with strong local support but a floor with weak support ($q = 8$MPa, t =3 MPa), the release of stress in the floor is more obvious than that of the roadway with all round but weak support ($q = t$ =3 MPa). The radial compressive stress increases with the support resistance near the floor and the stress in the surrounding rock gradually approaches the stress field of the roadway with its all round uniform support.

Fig. 4 compares the effect of different weak support angles β on radial stress when the support resistance is kept at q =8 MPa, t =0 MPa. Changes within the range of weak support have a large effect on the stress in the side of the roadway, while for the roof and floor the effect is relatively small.

Fig. 5 shows the variation in radial strain with the angle θ in the surrounding rock corresponding to different levels of support resistance when the support angle is $\delta = \pi/2$ and the radius r is 2a. Obviously, for the roadway boundary with weak local support, the roof and floor display an asymmetric distribution of stress in the sides of the roadway, with little compressive strain, so that the surrounding rock is easy to heave at places with weak support. For the floor without support, the radial compressive strain is smallest in the floor. Under these conditions, it is easy to have the phenomenon of floor heave. Conversely, if the roof has weak local support (q =3 MPa, t =8 MPa), the compressive strain is small in the rock surrounding the roof and the surrounding rock can easily collapse. With an increase in the weak
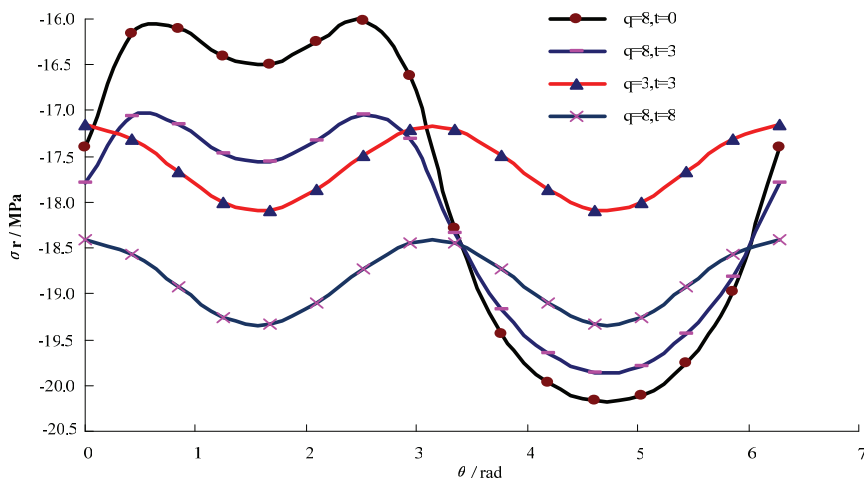
(a) $\beta = \pi / 2$

(b) $\beta = \pi / 3$
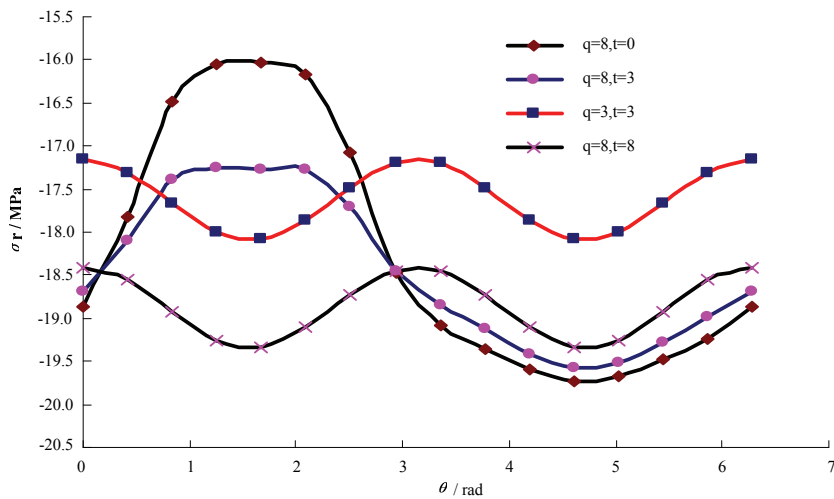
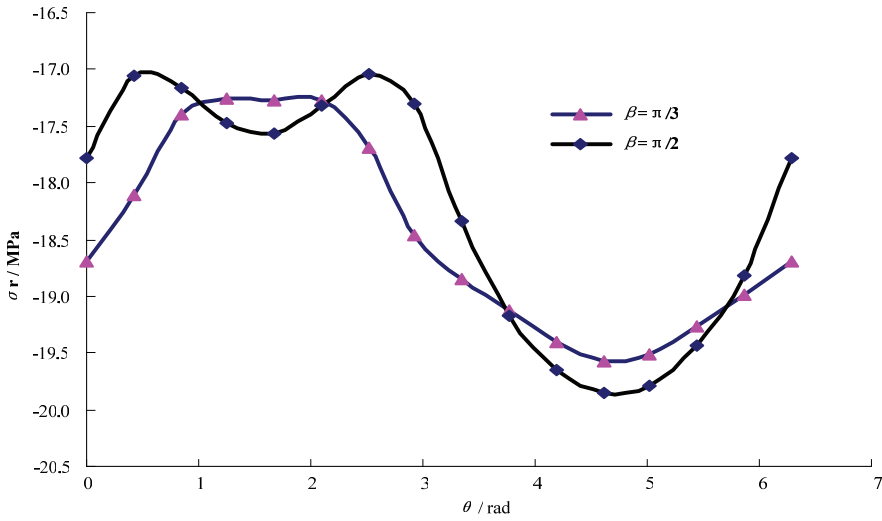Figure 3: Stress σ_r as a function of θ at $r=2a$

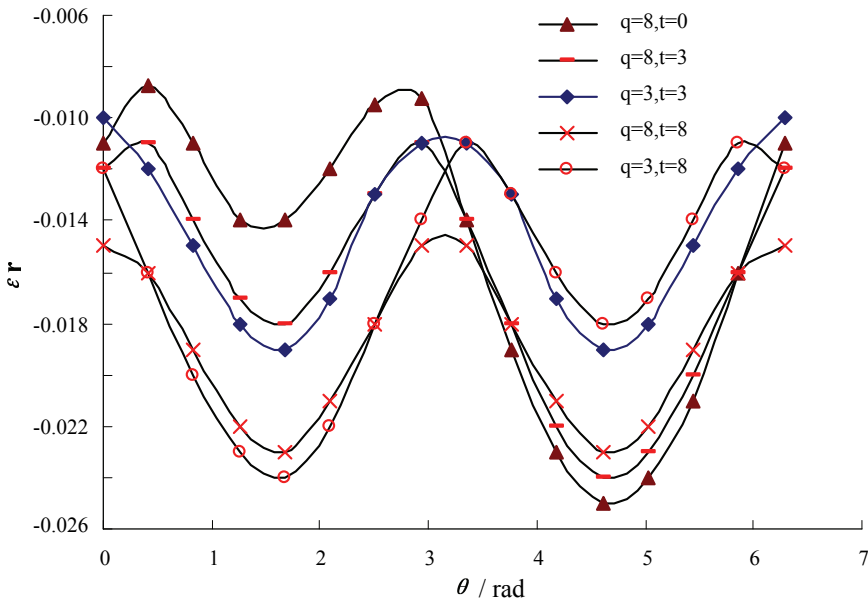Figure 4: Radical normal stress as a function of θ at two different support angles β
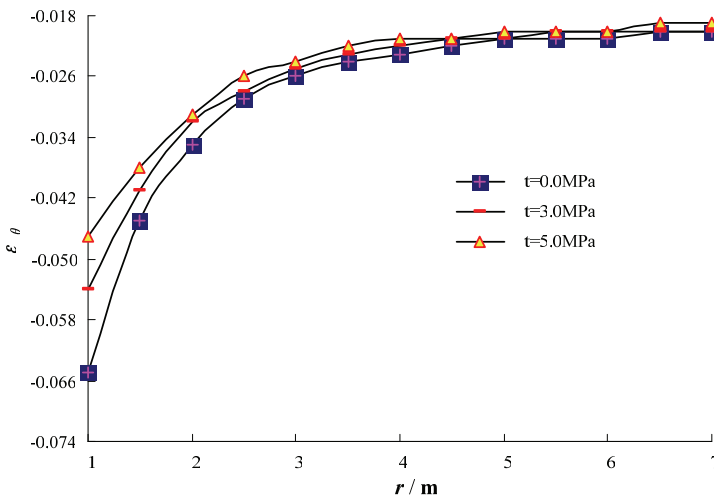

Figure 5: ε_r as a function of θ at $\delta = \pi/2$, r=2a given different support resistance

support resistance, its corresponding radial strain increases, improving the stability of the surrounding rock.

Fig. 6 shows the changes in radial and circumferential strain as a function of depth (radius) of the surrounding rock when p =20 MPa, q =3 MPa, $\delta = \pi/2$ and $\theta = \pi/3$.
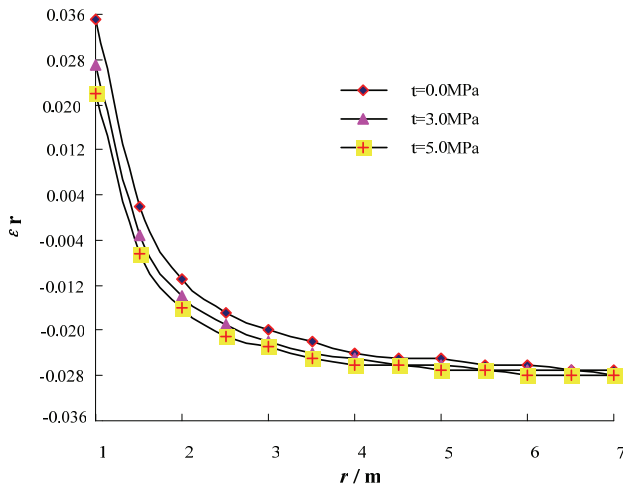
(a) $\varepsilon_r - r$ Curve

(b) $\varepsilon_\theta - r$ Curve

Figure 6: Strains at $\theta = \pi/3$ as a function of radius r given different support resistance

Fig. 6(a) shows that as the radius increases, the radial strain changes from tensile strain on the roadway boundary to compressive strain as it approaches infinity asymptotically. The surrounding rock corresponding to the tensile strain can easily heave out from the roadway boundary, which is extremely detrimental to the stability of the surrounding rock.

According to Fig. 6(b), as the radius increases, the circumferential compressive strain decreases and approaches infinity asymptotically. Under the given support resistance, the roadway boundary will further converge. Therefore, the support conditions of the roadway boundary should be further modified in order to improve the stability of the surrounding rock.

6 Conclusions

(1) Based on a complex variable function method, a complex Fourier series method and by a natural boundary reduction, we derived the boundary integration formula of the stress function in the rock surrounding a circular roadway with weak local support. Using surface force on the roadway boundary to calculate the stress function and its normal derivative, and substituting them into the integration formula, the specific expression of stress functions under various support conditions have been obtained.

(2) We have analyzed rules of the distribution of the stress and deformation fields in the surrounding rock within a range of weak levels of support and different levels of weak support resistance. We have pointed out that the surrounding rock is easy to heave from positions with weak local support. Changes within the range of weak support have a large effect on the stress in the sides of the roadway, while this effect is relatively small for the roof and floor. (3)We have developed the analytical solution of stress in the rock surrounding a roadway with uniform or no support after excavation. The solutions are exactly the same as the answers given in various sources of the literature.

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