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The Superconvergence of Certain Two-Dimensional Cauchy Principal Value Integrals

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Abstract: The composite rectangle (midpoint) rule for the computation of multi-dimensional singular integrals is discussed, and the superconvergence results is obtained. When the local coordinate is coincided with certain priori known coordinates, we get the convergence rate one order higher than the global one. At last, numerical examples are presented to illustrate our theoretical analysis which agree with it very well.

1 Introduction

We consider certain two-dimensional Cauchy principal value integral of the form

$$I(f,t,s) = \fint_{-1}^{1}\fint_{-1}^{1} \frac{f(x,y)}{(x-t)(y-s)}dxdy, \quad (t,s) \in (-1,1)\times(-1,1), \tag{1}$$

where $\fint_{-1}^{1}\fint_{-1}^{1}$ denotes a Cauchy Principle value integral and $(t,s) \in (-1,1)\times(-1,1)$ the singular point.

Cauchy principal value integrals have recently attracted a lot of attention. The main reason for this interest is probably due to the fact that Cauchy principal value integral equations have shown to be an adequate tool in boundary element methods [Yu (1985); Yu (1993); Yu (2002); Yu and Huang (2008); Liu (2007); Liu and Yu (2008); Young, Chen, Chen and Kao (2007); He, Lim, and Lim (2008)] and many engineering problems [Chen and Hong (1999);Zhou, Li, and Yu (2008)] for the modeling of many physical situations, such as acoustics, fluid mechanics, elasticity, fracture mechanics and electromagnetic scattering problems and so on. Numerous work has been devoted in developing efficient quadrature formulae, such as the Gaussian method [Ioakimidis (1985); Hui and Shia (1999);Monehato (1994)], the transformation method [Elliott and Venturino (1997)]; the Newton-Cote methods

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[Linz (1985); Yu (1992); Wu and Yu (1999); Du (2001);Li Wu and Yu (2009);Li zhang and Yu (2010)] and so on.

In this paper, we focus on certain kind of two-dimensional Cauchy principal value integrals which have not been extensively studied. In the paper of [Criscuolo and G. Mastroianni (1987)], the authors have considered product rules of Gauss type for the numerical approximation of two-dimensional Cauchy principal value integrals with respect to generalized smooth Jacobi weight functions. In the reference of [Monegato (1984)], based on the quadrature of one-dimensional Cauchy principal value integral, generalized quadrature rule for two-dimensional Cauchy principal value integrals is presented.

It is the aim of this paper to investigate the superconvergence phenomenon of rectangle rule for it and, in particular, to derive the error estimates. The superconvergence phenomenon of the hypersingular integral is studied in [Wu and Sun (2005); Wu and Sun (2008)] and the superconvergence phenomenon of the Cauchy principal value integral is studied in [Liu, Wu and Yu (2010)]. In this paper, we examine the convergence property of rectangle rule for certain kind of the two-dimensional Cauchy principal value integrals and generalize the above one-dimensional convergence results to cover this new situation. Moreover, we give an error expansion of the corresponding remainder when the density function $f(x,y)$ belong to C^2. As the special function of the error expansion equal zero, we get the superconvergence phenomenon, i.e.,when the singular point coincides with some a priori known point, the convergence rate is higher than what is globally possible.

The rest of this paper is organized as follows. In Sect.2, after introducing some basic formulae of the rectangle rule, we present the main results. In Sect.3, we finished the proof. Finally, several numerical examples are provided to validate our analysis.

2 Main result

Let $-1 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ and $-1 = y_0 < y_1 < \cdots < y_{m-1} < y_m = 1$ be a uniform partition of the area $[-1,1] \times [-1,1]$ with mesh size $h_x = 2/n$ and $h_y = 2/m$. In order to simplify our analysis, we set $h = h_x = h_y = 2/n$, it is not difficult to extend our analysis to the quasi-uniform meshes.

We define $f_C(x,y)$ as the piecewise constant interpolation for $f(x,y)$:

$$f_C(x,y) = f(x_{i+0.5}, y_{j+0.5}) \tag{2}$$

then we get

$$\begin{aligned} I_n(f,t,s) &= \fint_{-1}^{1}\fint_{-1}^{1}\frac{f_C(x,y)}{(x-t)(y-s)}dxdy \\ &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} f(x_{i+0.5},y_{j+0.5})\omega_{i,j}(t,s) \\ &= I(f,t,s)-E_n(f,t,s) \end{aligned} \tag{3}$$

where $E_n(f,t,s)$ denotes the error functional and

$$\omega_{i,j}(t,s)=\log\left|\frac{x_{i+1}-t}{x_i-t}\right|\log\left|\frac{y_{j+1}-s}{y_j-s}\right|. \tag{4}$$

To define the constant interpolation polynomials and a linear transformation

$x=\hat{x}_i(\tau):=(\tau+1)(x_{i+1}-x_i)/2+x_i,\quad \tau\in[-1,1],$

$y=\hat{y}_j(\tau):=(\tau+1)(y_{j+1}-y_j)/2+y_j,\quad \tau\in[-1,1]$

from the reference element $[-1,1]$ to the subinterval $[x_i,x_{i+1}]$ and $[y_j,y_{j+1}]$.

Here we present the error estimate for the (composite) rectangle rule with certain two-dimensional Cauchy principal value integrals in the following theorem.

Theorem 1 *Assume $f(x,y)\in C^1[-1,1]\times[-1,1]$. For the rectangle rule $I_n(f,t,s)$ defined as Eq. 3. Assume that $t=x_k+(1+\tau)h/2, s=y_l+(1+\tau)h/2$, there exist a positive constant C, independent of h and t,s, such that*

$$|E_n(f,t,s)|\le C(|\ln h|+|\ln\gamma(\tau)|)^2 h, \tag{5}$$

where

$$\gamma(\tau)=\min_{0\le i\le n}\frac{|t-x_i|}{h}=\frac{1-|\tau|}{2}. \tag{6}$$

Proof: Let $R(x,y)=f(x,y)-f_C(x,y)$, then we have

$$|R(x,y)|\le Ch.$$

As

$$\begin{aligned} &E_n(f,t,s)=\fint_{-1}^{1}\fint_{-1}^{1}\frac{R(x,y)}{(x-t)(y-s)}dxdy \\ &=\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\frac{R(x,y)}{(x-t)(y-s)}dxdy \\ &+\fint_{x_k}^{x_{k+1}}\fint_{y_l}^{y_{l+1}}\frac{R(x,y)}{(x-t)(y-s)}dxdy. \end{aligned} \tag{7}$$

For the first part of Eq. 7,we have

$$
\begin{aligned}
&\left|\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\frac{R(x,y)}{(x-t)(y-s)}dxdy\right| \\
&\leq Ch\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\frac{1}{|(x-t)(y-s)|}dxdy \\
&\leq Ch\sum_{i=0,i\neq k}^{n-1}\int_{x_i}^{x_{i+1}}\frac{1}{|x-t|}dx\sum_{j=0,j\neq l}^{n-1}\int_{y_j}^{y_{j+1}}\frac{1}{|y-s|}dy \\
&\leq C(|\ln h|+|\ln\gamma(\tau)|)^2h.
\end{aligned}
\tag{8}
$$

For the second part of Eq. 7,

$$
\begin{aligned}
&\left|\int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}}\frac{R(x,y)}{(x-t)(y-s)}dxdy\right| \\
&\leq \left|R(t,s)\log\frac{x_{k+1}-t}{t-x_k}\log\frac{y_{l+1}-s}{s-y_l}\right| \\
&+\left|R_x(t,s)h\log\frac{y_{l+1}-s}{s-y_l}\right|+\left|R_y(t,s)h\log\frac{x_{k+1}-t}{t-x_k}\right| \\
&+\left|\int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}}\frac{R_{xy}(\xi_k,\eta_l)}{(x-t)(y-s)}dxdy\right| \\
&\leq Ch|\ln\gamma(\tau)|^2.
\end{aligned}
\tag{9}
$$

Combining Eq. 8 and Eq. 9 together, the proof is completed.

Compared with Riemann integrals, the global convergence rate of the (composite) rectangle rule for integrals Eq. 1 is one order lower for the Riemann integral.

Let

$$
\phi_\tau(t,s)=\begin{cases}-\dfrac{1}{2}\displaystyle\int_{-1}^{1}\int_{-1}^{1}\frac{\tau}{(\tau-t)(\xi-s)}d\tau d\xi, & |t|<1,|s|<1,\\ -\dfrac{1}{2}\displaystyle\int_{-1}^{1}\int_{-1}^{1}\frac{\tau}{(\tau-t)(\xi-s)}d\tau d\xi, & |t|>1,|s|>1.\end{cases}
\tag{10}
$$

We also set $J:=(-\infty,-1)\bigcup(-1,1)\bigcup(1,+\infty)$ and the operator $W:C(J)\to C(-1,1)$ be defined by

$$
Wf(\tau):=f(\tau)+\sum_{j=1}^{\infty}f(2j+\tau)+\sum_{j=1}^{\infty}f(-2j+\tau).
\tag{11}
$$

Then we have

$$S_{0x}(\tau) := \phi_x(\tau) + \sum_{j=1}^{\infty} \phi_x(2j+\tau) + \sum_{j=1}^{\infty} \phi_x(-2j+\tau) \tag{12}$$

$$S_{0y}(\tau) := \phi_y(\tau) + \sum_{j=1}^{\infty} \phi_y(2j+\tau) + \sum_{j=1}^{\infty} \phi_y(-2j+\tau). \tag{13}$$

The superconvergence results of constant rectangle rules are given in the following.

Theorem 2 *Let $S_{0x}(\tau), S_{0y}(\tau)$ be defined by Eq. 12 and Eq. 13 respectively. Assume that $t \neq x_i, s \neq y_j$ for any $i, j = 0, 1, \cdots, n$. For the constant rectangle rule Eq. 3, there exists a positive constant C, independent of h and t, s, such that*

$$E_n(f,t,s) = f_y(t,s)S_{0x}(\tau)h + f_x(t,s)S_{0y}(\tau)h + \mathscr{R}_n(s), \tag{14}$$

where $t = x_k + \frac{1+\tau}{2}h, s = y_l + \frac{1+\tau}{2}h \;\; k,l = 0,1,\cdots,n-1, \;\; \tau \in (-1,1)$.

$$|\mathscr{R}_n(s)| \leq C[(|\ln h| + |\ln \gamma(\tau)|)^2 + \eta(t)]h^2. \tag{15}$$

$\gamma(\tau)$ defined as Eq. 6 and $\eta(t) = \max\{\frac{1}{t-a}, \frac{1}{b-t}\}$.

Compared theorem 1 with theorem 2, when the special function $S_{0x}(\tau) = S_{0y}(\tau) = 0$ we get the superconvergence phenomenon. One can see that the superconvergence rate of the (composite) rectangle at certain points is one order higher than their global convergence rate which is the same as the Riemann integral .

3 Proof of Theorem 2

In the following analysis, C will denote a generic positive constant which is independent of h and t, s. Let P_l and Q_l denote the Legendre polynomial [Andrews (2002)] of degree l and the associated Legendre function of the second kind, respectively.

Lemma 1 *Let $f(x,y) \in C^2[-1,1]\times[-1,1]$ and $f_C(x,y)$ be defined as Eq. 2, there holds*

$$f(x,y) - f_C(x,y) = f_x(x,y)(x - x_{i+0.5}) + f_y(x,y)(y - y_{j+0.5}) + \mathscr{R}_{ij}(\alpha_i, \beta_j), \tag{16}$$

where $\alpha_i \in (x_i, x_{i+1}), \beta_j \in (y_j, y_{j+1})$

$$\begin{aligned} \mathscr{R}_{ij}(\alpha_i,\beta_j) &= 0.5 f_{xx}(x,y)(x-x_{i+0.5})^2 + f_{xy}(x,y)(x-x_{i+0.5})(y-y_{j+0.5}) \\ &\quad +0.5 f_{yy}(y-y_{j+0.5})^2 \end{aligned} \tag{17}$$

and

$$|\mathscr{R}_{ij}(\alpha_i,\beta_j)| \le Ch^2. \tag{18}$$

Proof. By taking Taylor expansion for

$$f_C(x,y) = f(x,y) - f_x(x,y)(x-x_{i+0.5}) - f_y(x,y)(y-y_{j+0.5}) - \mathscr{R}_{ij}(\alpha_i,\beta_j), \tag{19}$$

and noting the definition of $f_C(x,y)$, we obtain Eq. 16 and complete the proof.

Lemma 2 *Assume* $(t,s) \in (x_k,x_{k+1}) \times (y_l,y_{l+1})$ *and let* $a_i = 2(t-x_i)/h-1, b_j = 2(s-x_j)/h-1, 0 \le i,j \le n-1$. *Then, we have*

$$\phi_x(a_i,b_j) = \begin{cases} -\dfrac{1}{h}\displaystyle\fint_{x_i}^{x_{i+1}}\fint_{y_j}^{y_{j+1}}\frac{x-x_{i+0.5}}{(x-t)(y-s)}dxdy, & i=k, j=l, \\ -\dfrac{1}{h}\displaystyle\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\frac{x-x_{i+0.5}}{(x-t)(y-s)}dxdy, & i \ne k, j \ne l \end{cases} \tag{20}$$

and

$$\phi_y(a_i,b_j) = \begin{cases} -\dfrac{1}{h}\displaystyle\fint_{x_i}^{x_{i+1}}\fint_{y_j}^{y_{j+1}}\frac{y-y_{j+0.5}}{(x-t)(y-s)}dxdy, & i=k, j=l, \\ -\dfrac{1}{h}\displaystyle\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\frac{y-y_{j+0.5}}{(x-t)(y-s)}dxdy, & i \ne k, j \ne l. \end{cases} \tag{21}$$

Proof: By the definition of Eq. 1 and $i=k, j=l$, we have:

$$\begin{aligned}
&\fint_{x_i}^{x_{i+1}}\fint_{y_j}^{y_{j+1}}\frac{x-x_{i+0.5}}{(x-t)(y-s)}dxdy \\
&= \fint_{x_i}^{x_{i+1}}\frac{x-x_{i+0.5}}{x-t}dx \fint_{y_j}^{y_{j+1}}\frac{1}{y-s}dy \\
&= \lim_{\varepsilon\to 0}\left\{\left(\int_{x_i}^{t-\varepsilon} + \int_{t+\varepsilon}^{x_{i+1}}\right)\frac{x-x_{i+0.5}}{x-t}dx\right\}\lim_{\varepsilon\to 0}\left\{\left(\int_{y_j}^{s-\varepsilon} + \int_{s+\varepsilon}^{y_{j+1}}\right)\frac{1}{y-s}dy\right\} \\
&= \frac{h}{2}\lim_{\varepsilon\to 0}\left\{\left(\int_{-1}^{a_i-\frac{2\varepsilon}{h}} + \int_{a_i+\frac{2\varepsilon}{h}}^{1}\right)\frac{\tau}{\tau-a_i}d\tau\right\}\lim_{\varepsilon\to 0}\left\{\left(\int_{-1}^{b_l-\frac{2\varepsilon}{h}} + \int_{b_j+\frac{2\varepsilon}{h}}^{1}\right)\frac{1}{\xi-b_j}d\xi\right\} \\
&= \frac{h}{2}\fint_{-1}^{1}\fint_{-1}^{1}\frac{\tau}{(\tau-a_i)(\xi-b_j)}d\tau d\xi \\
&= -h\phi_x(a_i,b_j).
\end{aligned}$$

The first identity in Eq. 21 is then verified. The second identity can be obtained by applying the approach to the correspondent Riemann integral.
Define

$$\mathscr{H}^{kl}(x,y) = f(x,y) - f_C(t,s) - f_x(t,s)(x - x_{k+0.5}) - f_y(t,s)(y - y_{l+0.5}). \tag{22}$$

Lemma 3 *Under the assumption of Theorem 2, there holds that*

$$\left| \fint_{x_k}^{x_{k+1}} \fint_{y_l}^{y_{l+1}} \frac{\mathscr{H}^{kl}(x,y)}{(x-t)(y-s)} dxdy \right| \le C \left| \ln \gamma(\tau) \right|^2 h^2. \tag{23}$$

Proof: By the definition of Cauchy principal integral, we have

$$\begin{aligned} \fint_a^b \fint_c^d \frac{f(x,y)}{(x-t)(y-s)} dxdy &= f(t,s) \log \frac{b-t}{t-a} \log \frac{d-s}{s-c} \\ &+ f_x(t,s)(b-a) \log \frac{d-s}{s-c} \\ &+ f_y(t,s)(d-c) \log \frac{b-t}{t-a} \\ &+ \int_a^b \int_c^d \frac{R(x,y)}{(x-t)(y-s)} dxdy, \end{aligned} \tag{24}$$

then follow Eq. 22, we have

$$\begin{aligned} &\fint_{x_k}^{x_{k+1}} \fint_{y_l}^{y_{l+1}} \frac{\mathscr{H}^{kl}(x,y)}{(x-t)(y-s)} dxdy \\ &= \mathscr{H}^{kl}(t,s) \log \frac{x_{k+1}-t}{t-x_k} \log \frac{y_{l+1}-s}{s-y_l} \\ &+ \mathscr{H}_x^{kl}(t,s) h \log \frac{y_{l+1}-s}{s-y_l} + \mathscr{H}_y^{kl}(t,s) h \log \frac{x_{k+1}-t}{t-x_k} \\ &+ \int_{x_k}^{x_{k+1}} \int_{y_l}^{y_{l+1}} \frac{R(x,y)}{(x-t)(y-s)} dxdy. \end{aligned} \tag{25}$$

Now let we estimate Eq. 25 term by term. For the first term of Eq. 25 we have

$$\left| \mathscr{H}^{kl}(t,s) \log \frac{x_{k+1}-t}{t-x_k} \log \frac{y_{l+1}-s}{s-y_l} \right| \le Ch^2 \left(\ln |\gamma(\tau)| \right)^2. \tag{26}$$

For the second term, we have

$$\left| \mathscr{H}_x^{kl}(t,s) h \log \frac{y_{l+1}-s}{s-y_l} \right| \le Ch^2 |\ln \gamma(\tau)| \tag{27}$$

and

$$\left|\mathscr{H}_y^{kl}(t,s)h\log\frac{x_{k+1}-t}{t-x_k}\right| \le Ch^2|\ln\gamma(\tau)|. \tag{28}$$

As for the last term of Eq. 25, we have

$$\left|\int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}}\frac{R(x,y)}{(x-t)(y-s)}dxdy\right| \le Ch^2. \tag{29}$$

Combining Eq. 26, Eq. 27, Eq. 28 and Eq. 29 leads to Eq. 23 and the proof is completed.

Lemma 4 *For $\tau\in(-1,1)$, we have*

$$\left|\sum_{i=k}^{\infty}\phi_x(2i+\tau)+\sum_{i=n-k+1}^{\infty}\phi_x(-2i+\tau)\right| \le Ch\eta(t) \tag{30}$$

$$\left|\sum_{i=l}^{\infty}\phi_y(2i+\tau)+\sum_{i=n-l+1}^{\infty}\phi_y(-2i+\tau)\right| \le Ch\eta(s). \tag{31}$$

Proof: By the definition of $\phi_\tau(t,s)$,we have

$$\phi_x(\tau)=\left(2+\tau\log\left|\frac{1-\tau}{1+\tau}\right|\right)\log\left|\frac{1-\xi}{1+\xi}\right| = 4Q_1(\tau)Q_0(\xi), \tag{32}$$

which means

$$\phi_x(t) = -\frac{1}{2}\int_{-1}^{1}\frac{1-\tau^2}{(t-\tau)^2}d\tau\int_{-1}^{1}\frac{1}{s-\xi}d\xi = -4Q_1(t)Q_0(s). \tag{33}$$

Noting that $t=x_k+\frac{\tau+1}{2}h=a+(k+\frac{\tau+1}{2})h$, we have $2(t-a)/h=\tau+2k+1$ and

$$\begin{aligned}\left|\sum_{i=k}^{\infty}\phi_x(2i+\tau)\right| &\le C\sum_{i=k}^{\infty}\int_{-1}^{1}\frac{dt}{|2i+\tau-t|^2}\\ &= C\int_{\tau+2k+1}^{\infty}\frac{dx}{x^2}=\frac{C}{(\tau+2k+1)}=\frac{Ch}{t-a}.\end{aligned} \tag{34}$$

On the other hand, since $b=a+nh$, we have $2(b-t)/h=2(n-k)-1-\tau$ and

$$\begin{aligned}\left|\sum_{i=n-k+1}^{\infty}\phi_x(\tau-2i)\right| &\le C\sum_{i=n-k+1}^{\infty}\int_{-1}^{1}\frac{dt}{|2i-\tau+t|^2}\\ &= C\int_{2(n-k)-1-\tau}^{\infty}\frac{dx}{x^2}=\frac{C}{2(n-k)-1-\tau}=\frac{Ch}{b-t}.\end{aligned} \tag{35}$$

The proof for Eq. 31 is similar. And the proof of Lemma 4 is completed.

Lemma 5 *Under the assumption of Theorem 2,we have*

$$\left|\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\frac{\mathscr{R}_{ij}(x,y)}{(x-t)(y-s)}dxdy\right| \leq C(|\ln h|+|\ln\gamma(\tau)|)^2h^2. \tag{36}$$

Proof: By the definition of Eq. 17, we have

$$\begin{aligned}
&\left|\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\frac{\mathscr{R}_{ij}(x,y)}{(x-t)(y-s)}dxdy\right| \\
&\leq Ch^2\left|\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\frac{1}{|(x-t)(y-s)|}dxdy\right| \\
&\leq Ch^2\sum_{i=0,i\neq k}^{n-1}\int_{x_i}^{x_{i+1}}\frac{1}{|x-t|}dx\sum_{j=0,j\neq l}^{n-1}\int_{y_j}^{y_{j+1}}\frac{1}{|y-s|}dy \\
&\leq C(|\ln h|+|\ln\gamma(\tau)|)^2h^2.
\end{aligned} \tag{37}$$

3.1 Proof of Theorem 2

Proof: By Lemma 1, we have

$$\begin{aligned}
&\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\frac{f(x,y)-f_C(x,y)}{(x-t)(y-s)}dxdy \\
&=\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\frac{f_x(x,y)(x-x_{i+0.5})}{(x-t)(y-s)}dxdy \\
&+\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\frac{f_y(x,y)(y-y_{j+0.5})}{(x-t)(y-s)}dxdy \\
&+\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\frac{\mathscr{R}_{ij}(\alpha_i,\beta_j)}{(x-t)(y-s)}dxdy.
\end{aligned} \tag{38}$$

Following the equation of $\mathscr{H}^{kl}(x,y)$, we have

$$\begin{aligned}
\fint_{x_k}^{x_{k+1}}\fint_{y_l}^{y_{l+1}}\frac{f(x,y)-f_C(x,y)}{(x-t)(y-s)}dxdy = \quad & f_x(t,s)\fint_{x_k}^{x_{k+1}}\fint_{y_l}^{y_{l+1}}\frac{(x-x_{i+0.5})}{(x-t)(y-s)}dxdy \\
&+f_y(t,s)\fint_{x_k}^{x_{k+1}}\fint_{y_l}^{y_{l+1}}\frac{(y-y_{j+0.5})}{(x-t)(y-s)}dxdy \\
&+\fint_{x_k}^{x_{k+1}}\fint_{y_l}^{y_{l+1}}\frac{\mathscr{H}^{kl}(x,y)}{(x-t)(y-s)}dxdy
\end{aligned} \tag{39}$$

and by Taylor expansion of $f_x(x,y)$ and $f_y(x,y)$, we have

$$f_x(x,y) = f_x(t,s) + f_{xx}(t,s)(x-t) + f_{xy}(t,s)(y-s)$$

$$f_y(x,y) = f_y(t,s) + f_{yx}(t,s)(x-t) + f_{yy}(t,s)(y-s).$$

Putting Eq. 38, Eq. 39 together yields

$$\begin{aligned} \int_a^b\!\!\!\!\!\!\!\!-\;\int_c^d\!\!\!\!\!\!\!\!-\; \frac{f(x,y)-f_C(x,y)}{(x-t)(y-s)}dxdy &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} \frac{f(x,y)-f_C(x,y)}{(x-t)(y-s)}dxdy \\ &= f_y(t,s)S_{0x}(\tau)h + f_x(t,s)S_{0y}(\tau)h + \mathscr{R}_n(s), \end{aligned} \tag{40}$$

where

$$\mathscr{R}_n(s) = \mathscr{R}_n^{(1)}(s) + \mathscr{R}_n^{(2)}(s) + \mathscr{R}_n^{(3)}(s) + \mathscr{R}_n^{(4)}(s) + \mathscr{R}_n^{(5)}(s)$$

and

$$\begin{aligned} \mathscr{R}_n^{(1)}(s) = &\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} \frac{f_{xy}(t,s)(x-x_{i+0.5})}{y-s}dxdy \\ &+\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} \frac{f_{xx}(t,s)(x-x_{i+0.5})}{x-t}dxdy, \end{aligned}$$

$$\begin{aligned} \mathscr{R}_n^{(2)}(s) = &\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} \frac{f_{yx}(t,s)(y-y_{j+0.5})}{y-s}dxdy \\ &+\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} \frac{f_{yy}(t,s)(y-y_{j+0.5})}{x-t}dxdy, \end{aligned}$$

$$\mathscr{R}_n^{(3)}(s) = \sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} \frac{\mathscr{R}_{ij}(\alpha_i,\beta_j)}{(x-t)(y-s)}dxdy,$$

$$\mathscr{R}_n^{(4)}(s) = \int_{x_k}^{x_{k+1}}\!\!\!\!\!\!\!\!\!\!-\;\;\int_{y_l}^{y_{l+1}}\!\!\!\!\!\!\!\!\!\!-\;\; \frac{\mathscr{H}^{ij}(x,y)}{(x-t)(y-s)}dxdy,$$

$$\begin{aligned} \mathscr{R}_n^{(5)}(s) &= f_x(t,s)h\left[\sum_{i=k}^{\infty}\phi_x(2i+\tau) + \sum_{i=n-k+1}^{\infty}\phi_x(-2i+\tau)\right] \\ &+f_y(t,s)h\left[\sum_{i=l}^{\infty}\phi_y(2i+\tau) + \sum_{i=n-l+1}^{\infty}\phi_y(-2i+\tau)\right]. \end{aligned}$$

For the first term of $\mathscr{R}_n^1(s)$ we have

$$\left|\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\frac{f_{xy}(t,s)(x-x_{i+0.5})}{y-s}dxdy\right|$$
$$\leq C\left|\sum_{i=0,i\neq k}^{n-1}\int_{x_i}^{x_{i+1}}(x-x_{i+0.5})dx\right|\sum_{j=0,j\neq l}^{n-1}\int_{y_j}^{y_{j+1}}\frac{1}{|y-s|}dy$$
$$\leq C(|\ln h|+|\ln\gamma(\tau)|)h^2.$$

For the second term of $\mathscr{R}_n^1(s)$ we have

$$\left|\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\frac{f_{xx}(t,s)(x-x_{i+0.5})}{x-t}dxdy\right|$$
$$\leq Ch^2\left|\sum_{i=0,i\neq k}^{n-1}f_{xx}(t,s)\right|\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\frac{1}{|x-t|}dx$$
$$\leq C(|\ln h|+|\ln\gamma(\tau)|)h^2,$$

which means

$$|\mathscr{R}_n^1(s)|\leq C(|\ln h|+|\ln\gamma(\tau)|)h^2.$$

Similarly, we also have

$$|\mathscr{R}_n^2(s)|\leq C(|\ln h|+|\ln\gamma(\tau)|)h^2.$$

By Lemma Eq. 3 and Lemma Eq. 5, we have

$$|\mathscr{R}_n(s)|\leq C[(|\ln h|+|\ln\gamma(\tau)|)^2+\eta(t)]h^2.$$

The proof is complete. □

For the one-dimensional Cauchy principal integral we get the superconvergence point is $\pm\frac{2}{3}$. As we have proved above, the superconvergence point of certain two-dimensional is $(\pm\frac{2}{3},\pm\frac{2}{3})$. In fact, it is not difficult to extend our results to arbitrary (multi-dimensional) Cauchy Principal value integral and the superconvergence point is $(\pm\frac{2}{3},\cdots,\pm\frac{2}{3})$.

4 Numerical analysis

Example 1 *Consider the Cauchy Principal integral*

$$\int_{-1}^{1}\int_{-1}^{1}\frac{x^3y^6}{(x-t)(y-s)}dxdy,(t,s)\in(-1,1)\times(-1,1) \tag{41}$$

with the exact solution

$$(t^3\log\frac{1-t}{1+t}+2t^2+\frac{2}{3})(2s^5+\frac{2}{3}s^3+\frac{2}{5}s+s^6\log\frac{1-s}{1+s})$$

Table 1: Errors of the rectangle rule with $t=x_{n/4}+(\tau+1)h/2$ and $s=y_{n/4}+(\tau+1)h/2$

n	$(0,0)$	$(\frac{2}{3},\frac{2}{3})$	$(-\frac{2}{3},-\frac{2}{3})$	$(\frac{2}{3},-\frac{2}{3})$	$(-\frac{2}{3},\frac{2}{3})$
64	-8.2852e-003	-4.7026e-004	3.2205e-004	3.0914e-005	-2.1174e-004
128	-4.4420e-003	-1.3073e-004	8.2596e-005	7.3969e-006	-5.9789e-005
256	-2.2993e-003	-3.4453e-005	2.0915e-005	1.7994e-006	-1.5882e-005
512	-1.1697e-003	-8.8430e-006	5.2625e-006	4.4305e-007	-4.0923e-006
1024	-5.8990e-004	-2.2400e-006	1.3199e-006	1.0987e-007	-1.0386e-006

Table 2: Errors of the rectangle rule with $t=x_{n/4}+(\tau+1)h/2$ and $s=1-(\tau+1)h/2$

n	$(0,0)$	$(\frac{2}{3},\frac{2}{3})$	$(-\frac{2}{3},-\frac{2}{3})$	$(\frac{2}{3},-\frac{2}{3})$	$(-\frac{2}{3},\frac{2}{3})$
64	-1.2770e-001	-2.3418e-002	-7.1350e-002	-6.7546e-002	-2.4717e-002
128	-6.0481e-002	-1.0670e-002	-3.5797e-002	-3.4671e-002	-1.1169e-002
256	-2.7848e-002	-5.0225e-003	-1.7870e-002	-1.7545e-002	-5.1918e-003
512	-1.2584e-002	-2.4239e-003	-8.9124e-003	-8.8207e-003	-2.4772e-003
1024	-5.5957e-003	-1.1883e-003	-4.4467e-003	-4.4212e-003	-1.2044e-003

We adopt the uniform meshes to examine the convergence rate of the rectangle rule with the dynamic point with $t=x_{[n/4]}+(1+\tau)h/2, s=y_{n/4}+(\tau+1)h/2$, $t=x_{n/4}+(\tau+1)h/2, s=b-(\tau+1)h/2$ and $t=b-(\tau+1)h/2, s=b-(\tau+1)h/2$. From the table 1, we know that when the local coordinate of singular point is $(\pm\frac{2}{3},\pm\frac{2}{3})$, the quadrature reach the convergence rate of $O(h^2)$ as for the non-supersingular point the convergence rate is $O(h)$ which agree with our theorematically analysis. For the case of $t=x_{n/4}+(\tau+1)h/2, s=1-(\tau+1)h/2$ and $t=1-(\tau+1)h/2, s=1-(\tau+1)h/2$, table 2 and table 3 show that there are no superconvergence phenomenon for the mid-rectangle rue which coincide with our theoretically analysis.

Table 3: Errors of the rectangle rule with $t = 1-(\tau+1)h/2$ and $s = 1-(\tau+1)h/2$

n	$(0,0)$	$(\frac{2}{3},\frac{2}{3})$	$(-\frac{2}{3},-\frac{2}{3})$	$(\frac{2}{3},-\frac{2}{3})$	$(-\frac{2}{3},\frac{2}{3})$
64	4.3668e-001	4.4585e-002	3.2042e-001	1.2932e-001	1.1029e-001
128	3.1022e-001	3.2652e-002	1.9793e-001	9.4580e-002	7.2841e-002
256	1.9973e-001	2.0936e-002	1.1727e-001	6.1636e-002	4.5156e-002
512	1.2156e-001	1.2686e-002	6.7640e-002	3.7782e-002	2.6955e-002
1024	7.1408e-002	7.4519e-003	3.8292e-002	2.2314e-002	1.5683e-002

Example 2 *Consider the Cauchy Principal integral*

$$\fint_{-1}^{1}\fint_{-1}^{1}\fint_{-1}^{1}\frac{x^3y^3z^3}{(x-t)(y-s)(z-u)}\,dxdydz, (t,s,u)\in(-1,1)\times(-1,1)\times(-1,1) \quad (42)$$

with the exact solution

$$(t^3\log\frac{1-t}{1+t}+2t^2+\frac{2}{3})(s^3\log\frac{1-s}{1+s}+2s^2+\frac{2}{3})(u^3\log\frac{1-u}{1+u}+2u^2+\frac{2}{3})$$

Table 4: Errors of the rectangle rule with $t = x_{n/4}+(\tau+1)h/2$ and $s = y_{n/4}+(\tau+1)h/2$,$u = z_{n/4}+(\tau+1)h/2$

n	$(0,0,0)$	$(-\frac{2}{3},-\frac{2}{3},-\frac{2}{3})$	$(\frac{2}{3},\frac{2}{3},\frac{2}{3})$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$(-\frac{2}{3},\frac{2}{3},\frac{2}{3})$
16	1.4257e-001	2.2695e-002	-8.7573e-003	7.6978e-002	1.4155e-002
32	8.6387e-002	7.5571e-003	-2.2204e-003	4.5511e-002	4.5654e-003
64	4.7313e-002	2.1587e-003	-5.5816e-004	2.4381e-002	1.2873e-003
128	2.4726e-002	5.7526e-004	-1.3987e-004	1.2565e-002	3.4117e-004
256	1.2635e-002	1.4837e-004	-3.5004e-005	6.3706e-003	8.7782e-005

Table 4 describes the case when the singular point coincide with the local coordinate $(\pm\frac{2}{3},\pm\frac{2}{3},\pm\frac{2}{3})$, the convergence rate can reach $O(h^2)$ which is one order higher than the global one, while for the local coordinate takes non-superconvergence point the convergence rate is only $O(h)$ which agree with our theoretically analysis.

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