

Surface Heating Problems of Thermal Propagation in Living Tissue Solved by Differential Transformation Method

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Abstract: The hybrid method, which combines differential transformation and finite difference approximation techniques, is utilized to solve hyperbolic-type heat conduction (bio-heat) problems in one dimension. To capture the thermal behavior in a living tissue subjected to constant or exponential surface heating with the thermal wave model of bio-heat transfer, the relaxation time and the heat wave, which propagates in a direction perpendicular to the skin surface, are considered. The results show that the hybrid method can be used to solve hyperbolic heat conduction problems accurately.

Keywords: differential transformation, bio-heat transfer, relaxation time, thermal wave

1 Introduction

In the analysis of heat conduction problems, the classical Fourier heat conduction law is often adopted under the assumption that the heat transfer speed is sufficiently fast for the boundary or initial conditions to be applied to the objects immediately. This kind of analysis is only suitable for macroscopic heat transfer problems.

When analyzing microscopic heat transfer or very-low-temperature (zero degrees K) problems, the classical Fourier heat conduction law is no longer valid. The hyperbolic model is applied to such problems.

Chen and Liu (1998) applied the differential transformation method to solve boundary value problems. Jang, Chen and Liu (2001) used the two-dimensional differential transformation method to solve partial differential equations. Chen and Ho (1996) applied this method to solve eigenvalue problems and later (1999) proposed the two-dimensional differential transform to solve partial differential equations. Yu and Chen (1998) used the differential transformation method to solve the

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third-order Blasius equation of boundary layer flow and investigated the optimal fin length of a convective-radiative rectangular-profile longitudinal fin under convective boundary conditions and variable thermal conductivity using the method. Chen and Ju (2004) employed the differential transformation method to solve advective-dispersive transport problems. Chen, Lai and Liu (2009) used hybrid differential transformation and the finite difference method to analyze nonlinear micro circular plates. Deng and Liu (2002) conducted an analytical study on bio-heat transfer problems with spatial or transient heating on the skin surface. Rabin and Shitzer (1998) simulated the heat transfer effects of biological tissue after cryosurgery. Torvi and Dale (1994) discussed thermal effects of burnt skin under flash fire conditions. Liu (2008) used the Laplace transform to investigate the thermal behavior in a living tissue subjected to constant, sinusoidal, or step surface heating with the thermal wave model of bio-heat transfer. Liu (2011) employed a dual-phase-lag model to simulate the effect of micro-structural interactions in the fast transient process of heat transport.

In the present study, the differential transformation method is used to solve hyperbolic heat transfer equations in bio-heat transfer problems.

2 Differential transformation method

If $x = x(t)$ is analytic in time domain T , the differential transformation of x at $t = t_0$ can be expressed as:

$$X(k; t_0) = M(k) \left[\frac{d^k}{dt^k} (q(t)x(t)) \right]_{t=t_0}, \quad k \in K \quad (1)$$

where k belongs to the set of non-negative integers denoted as the K domain, $X(k; t_0)$ is the spectrum of $x(t)$ at $t = t_0$, $M(k)(\cdot)$ is the weighting factor, and $q(t)(q(t) \neq 0)$ is regarded as a kernel corresponding to $x(t)$.

If $q(t)$ and $x(t)$ can be expanded using a Taylor series, the inverse transformation of Eq.(1) can be expressed as:

$$x(t) = \frac{1}{q(t)} \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} \frac{X(k; t_0)}{M(k)}, \quad \forall t \in T \quad (2)$$

Generally, the weighting factor $M(k) = \frac{H^k}{k!}$, where H is the time horizon of interest, and kernel $q(t) = 1$. Then, Eq. (1) becomes:

$$X(k) = \frac{H^k}{k!} \left[\frac{d^k x(t)}{dt^k} \right]_{t=0}, \quad k \in K \quad (3)$$

The inverse transformation is:

$$x(t) = \sum_{k=0}^{\infty} \left(\frac{t}{H} \right)^k X(k), \forall t \in T \quad (4)$$

3 Hyperbolic-type bio-heat transfer problems

When the heat transfer effect occurs in a very short time, with a very high temperature gradient, or in an environment with 0 degrees K, the classical heat convection equation is no longer valid. Vernotte (1958) proposed a modified heat flux model for these problems.

3.1 Hyperbolic-type heat conduction equation

Consider the following 3-dimensional Fourier heat conduction equation:

$$\vec{q} = -K\nabla T \quad (5)$$

where \vec{q} is the heat flux, K is the conductivity ($W/m \cdot ^\circ C$), and T is the temperature of the system ($^\circ C$). The modified heat flux model (Vernotte 1958) is:

$$\vec{q} + \tau \frac{\partial \vec{q}}{\partial t} = -K\nabla T \quad (6)$$

where $\tau = \frac{\alpha}{V^2}$ is the relaxation time (*sec*), α is the thermal diffusivity (m^2/sec), and V is the heat wave velocity (m/sec). For material with homogeneous properties, the relaxation time is about $10^{-8} \sim 10^{-14}s$. Since the heat wave moves very fast, it is very difficult to observe heat wave phenomena. In contrast, bio-tissue is composed of various materials with different properties; its relaxation is about $20 \sim 30s$.

The bio-heat heat equation can be written as:

$$-\nabla \cdot \vec{q} + W_b C_b (T_b - T) + q_m + q_r = \rho C \frac{\partial T}{\partial t} \quad (7)$$

where ρ , C , and T are the density, specific heat, and temperature of the bio-tissue, respectively. W_b and C_b are the perfusion rate and specific heat of blood, respectively. q_m is the heat generation from metabolism, q_r is the heat source term, and T_b is blood temperature inside capillaries.

Substituting Eq. (6) into Eq. (7) yields:

$$\begin{aligned} \nabla \cdot (K\nabla T) + W_b C_b (T_b - T) + q_m + q_r + \tau \left(-W_b C_b \frac{\partial T}{\partial t} + \frac{\partial q_m}{\partial t} + \frac{\partial q_r}{\partial t} \right) \\ = \rho C \left(\tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} \right) \end{aligned} \quad (8)$$

Eq.(8) is a hyperbolic-type heat conduction equation.

3.2 Hyperbolic-type heat conduction equation in cylindrical coordinates

Consider a 2-dimensional cylindrical coordinate system in r and z directions and let $q_m = \text{constant}$ and $q_r = 0$. From Eq. (8), the hyperbolic-type heat conduction equation can be expressed in cylindrical coordinates as:

$$K \left(\frac{1}{r} \left(\frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial r^2} + \frac{\partial^2 T}{\partial z^2} \right) + W_b C_b (T_b - T) + q_m - \tau W_b C_b \frac{\partial T}{\partial t} = \rho C \left(\tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} \right) \quad (9)$$

Assume that $T_i(r, z, 0)$ is the initial steady temperature. Then, Eq. (9) becomes:

$$K \left(\frac{1}{r} \left(\frac{\partial T_i}{\partial r} \right) + \frac{\partial^2 T_i}{\partial r^2} + \frac{\partial^2 T_i}{\partial z^2} \right) + W_b C_b (T_b - T_i) + q_m = 0 \quad (10)$$

Let $\theta = T - T_i$. From Eq. (9) and Eq. (10):

$$\rho C \tau \frac{\partial^2 \theta}{\partial t^2} + (\rho C + \tau W_b C_b) \frac{\partial \theta}{\partial t} + W_b C_b \theta - K \left(\frac{1}{r} \left(\frac{\partial \theta}{\partial r} \right) + \frac{\partial^2 \theta}{\partial r^2} + \frac{\partial^2 \theta}{\partial z^2} \right) = 0 \quad (11)$$

A differential transformation to time variable t yields:

$$\begin{aligned} & \rho C \tau \frac{(k+1)(k+2)}{H^2} U(k+2) + (\rho C + \tau W_b C_b) \frac{(k+1)}{H} U(k+1) + W_b C_b U(k) \\ & = K \left(\frac{1}{r} \left(\frac{\partial U(k)}{\partial r} \right) + \frac{\partial^2 U(k)}{\partial r^2} + \frac{\partial^2 U(k)}{\partial z^2} \right) \end{aligned} \quad (12)$$

where $U(k) = U(r, z, k)$ is the differential transformation function at θ .

Let r and z divide into N_1 and N_2 equal parts, with the distance of each part being Δr and Δz . Take the central difference approximation of Eq. (12). Then, the iteration equation of each finite difference grid (r_i, z_j) can be expressed as:

$$\begin{aligned} & \rho C \tau \frac{(k+1)(k+2)}{H^2} U_{i,j}(k+2) + (\rho C + \tau W_b C_b) \frac{(k+1)}{H} U_{i,j}(k+1) + W_b C_b U_{i,j}(k) \\ & = K \left(\frac{1}{r_{i,j}} \left(\frac{U_{i+1,j}(k) - U_{i-1,j}(k)}{2\Delta r} \right) + \frac{U_{i+1,j}(k) - 2U_{i,j}(k) + U_{i-1,j}(k)}{(\Delta r)^2} \right. \\ & \quad \left. + \frac{U_{i,j+1}(k) - 2U_{i,j}(k) + U_{i,j-1}(k)}{(\Delta z)^2} \right) \end{aligned} \quad (13)$$

4 Numerical simulation

In the present study, 1-dimensional bio-heat transfer problems with various boundary conditions are simulated. Human skin is taken as the bio-tissue with $\rho = 1000 \text{ kg/m}^3$ and $C = C_b = 4200 \text{ J/kg}^\circ\text{C}$.

4.1 Case 1: 1-dimensional surface heating with constant temperature

In case 1, the temperature variation perpendicular to the skin surface is simulated (see Fig. 1).

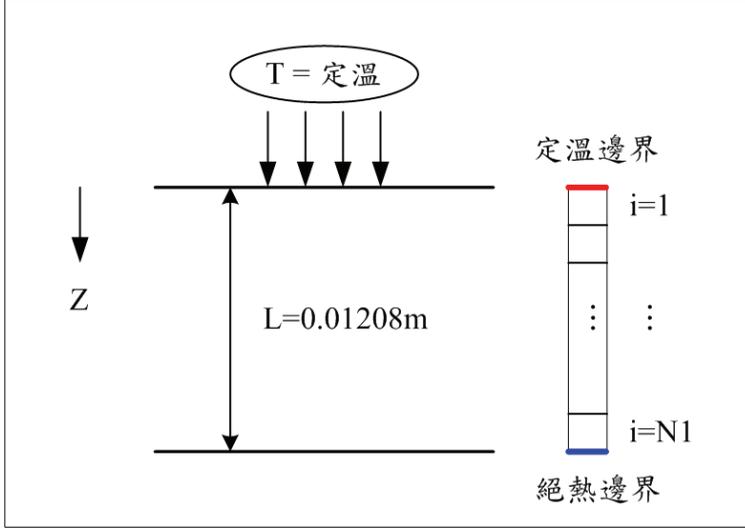


Figure 1: Physical model for case 1.

The governing equation for case 1 can be expressed as:

$$\rho C \tau \frac{\partial^2 \theta}{\partial t^2} + (\rho C + \tau W_b C_b) \frac{\partial \theta}{\partial t} + W_b C_b \theta - K \frac{\partial^2 \theta}{\partial z^2} = 0 \quad (14)$$

When human skin comes into contact with a hot metal, its temperature can be seen as constant. Assuming that there is no heat flux at $z = L$, the initial and boundary conditions are:

$$\theta(z, 0) = 0 \quad (15)$$

$$\frac{\partial \theta(z, 0)}{\partial t} = 0 \quad (16)$$

$$\theta(0, t) = 12 \quad (17)$$

$$\frac{\partial \theta(L, t)}{\partial z} = 0 \quad (18)$$

where $L = 0.01208m$.

Applying the differential transformation to Eq. (15)-(17) yields:

$$U_i(0) = 0 \quad (19)$$

$$U_i(1) = 0 \quad (20)$$

$$U_1(0) = 12, U_1(k) = 0_k \neq 0 \quad (21)$$

$$U_{N1}(k) - U_{N1-1}(k) = 0 \quad (22)$$

In Eq. (14), only heat conduction along the z direction is considered. The differential transformation form of Eq. (14) is:

For $\tau = 0$:

$$U_{i,j}(k+1) = \frac{H}{\rho C(k+1)} \left\{ -W_b C_b U_{i,j}(k) + K \left(\frac{U_{i,j+1}(k) - 2U_{i,j}(k) + U_{i,j-1}(k)}{(\Delta z)^2} \right) \right\} \quad (23)$$

For $\tau \neq 0$:

$$U_{i,j}(k+2) = \frac{H^2}{\rho C \tau (k+1)(k+2)} \left\{ -(\rho C + \tau W_b C_b) \frac{(k+1)}{H} U_{i,j}(k+1) - W_b C_b U_{i,j}(k) \right. \\ \left. + K \left(\frac{U_{i,j+1}(k) - 2U_{i,j}(k) + U_{i,j-1}(k)}{(\Delta z)^2} \right) \right\} \quad (24)$$

4.2 Case 2: 1-dimensional surface heating with exponential function

When human skin comes into contact with a hot metal, the temperature of the mental surface decreases with time (see Fig. 2). The human skin temperature can thus be seen as an exponential function, $e^{\beta t}$.

Assuming that there is no heat flux at $z = L$, the governing equation and initial and boundary conditions are:

$$\rho C \tau \frac{\partial^2 \theta}{\partial t^2} + (\rho C + \tau W_b C_b) \frac{\partial \theta}{\partial t} + W_b C_b \theta - K \frac{\partial^2 \theta}{\partial z^2} = 0 \quad (25)$$

$$\theta(z, 0) = 0 \quad (26)$$

$$\frac{\partial \theta(z, 0)}{\partial t} = 0 \quad (27)$$

$$\theta(0, t) = 100 \cdot e^{\beta t} \quad (28)$$

$$\frac{\partial \theta(L, t)}{\partial z} = 0 \quad (29)$$

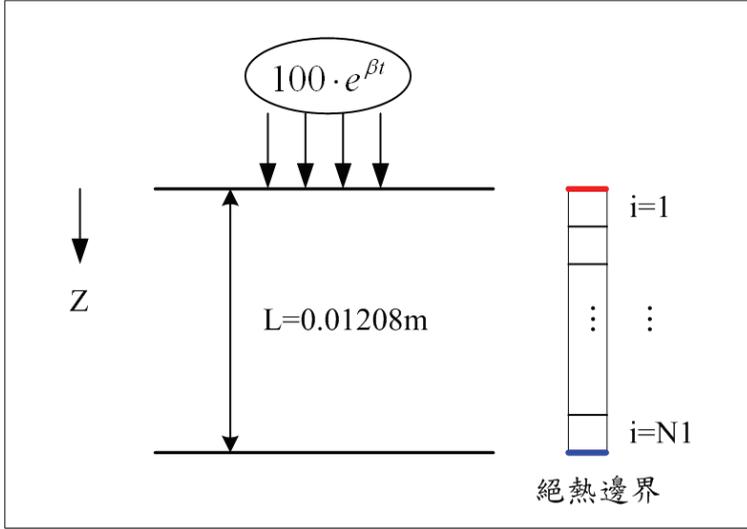


Figure 2: Physical model for case 2.

Applying the differential transformation to Eq. (26)-(29) yields:

$$U_i(0) = 0 \quad (30)$$

$$U_i(1) = 0 \quad (31)$$

$$U_1(k) = 100 \frac{(\beta H)^k}{k!} e^{\beta \cdot n \cdot H}, \quad n : \text{time step} \quad (32)$$

$$U_{N1}(k) - U_{N1-1}(k) = 0 \quad (33)$$

Here, $t' = \frac{t}{t_{\max}}$, where t_{\max} is the time required for θ to decrease to 0. That is, when $\beta = -0.01$ and -0.02 , $t_{\max} \approx 1000s$, and when $\beta = -0.1$, $t_{\max} \approx 100s$.

5 Results and discussion

For case 1:

Fig. 3 shows the temperature variation solved by a sixth-order differential transformation with a time step of $0.01s$ at $z = 0.01m$ for 3 grid sizes for case 1 with parameters set to $\tau = 0s$, $K = 0.2^W/m \cdot ^\circ C$, $W_b = 0.5^k g/m^3 \cdot s$. It can be observed that $m = 101$ is the best grid number for $\tau = 0s$.

Fig. 4 shows the temperature variation solved by a seventh-order differential transformation with a time step of $0.1s$ at $z = 0.01m$ for 3 grid sizes for case 1 with

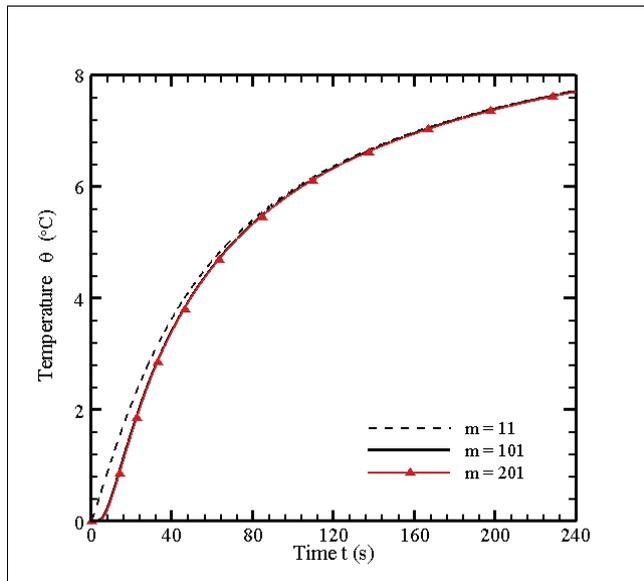


Figure 3: Grid independence ($\tau = 0s, z = 0.01m, K = 0.2^W/m \cdot ^\circ C, W_b = 0.5^k g/m^3 \cdot s$).

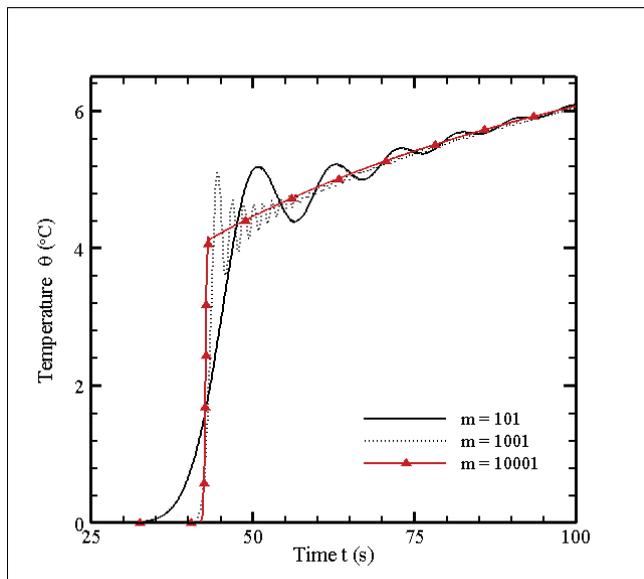


Figure 4: Grid independence ($\tau = 20s, z = 0.01m, K = 0.2^W/m \cdot ^\circ C, W_b = 0.5^k g/m^3 \cdot s$).

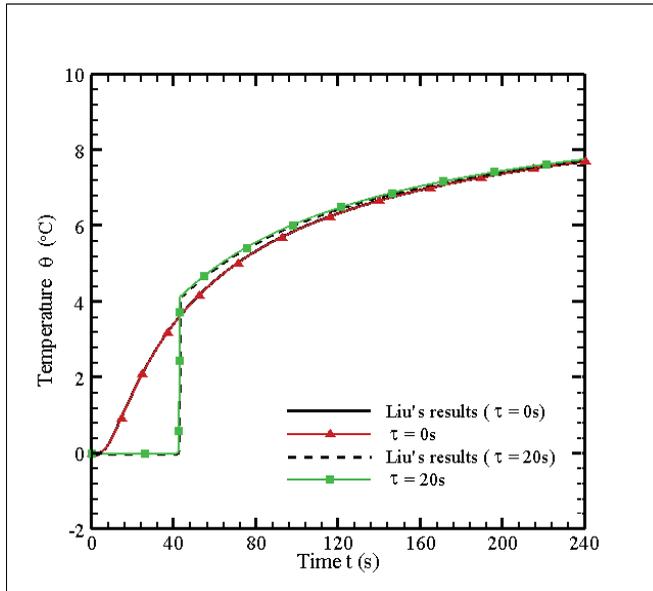


Figure 5: Simulation results compared to those reported by Liu (2008) with $K = 0.2 \text{ W/m} \cdot \text{°C}$, $W_b = 0.5 \text{ kg/m}^3 \cdot \text{s}$, $z = 0.00208 \text{ m}$.

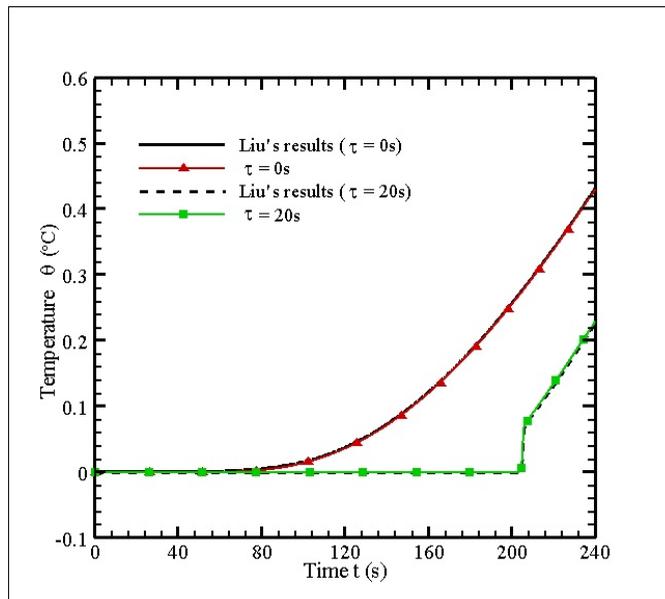


Figure 6: Simulation results compared to those reported by Liu (2008) with $K = 0.2 \text{ W/m} \cdot \text{°C}$, $W_b = 0.5 \text{ kg/m}^3 \cdot \text{s}$, $z = 0.01 \text{ m}$.

parameters set to $\tau = 20s$, $K = 0.2^W/m \cdot ^\circ C$, $W_b = 0.5^k g/m^3 \cdot s$. For insufficient grid sizes ($m < 10001$), some numerical instability influences the simulation accuracy.

The simulation results (Fig.5-6) show good agreement with those reported by Liu (2008). When $\tau = 0s$, the temperature distributions increase smoothly. When $\tau = 20s$, the heat waves move with a finite velocity, which can be expressed as $V = \sqrt{\alpha/\tau}$, the time heat wave required to reach $z=0.00208m$ is $t = 0.00208/V = 0.00208/\sqrt{0.2/(1000 \times 4200 \times 20)} = 42.627s$, and

$$t = 0.01/V = 0.01/\sqrt{0.2/(1000 \times 4200 \times 20)} = 204.939s$$

for $z = 0.01m$.

The blood flow rate plays a very important role in bio-heat transfer. Fig. 7 shows that the blood temperature is higher on the skin surface than it is inside capillaries. The blood flow takes the heat away proportionally.

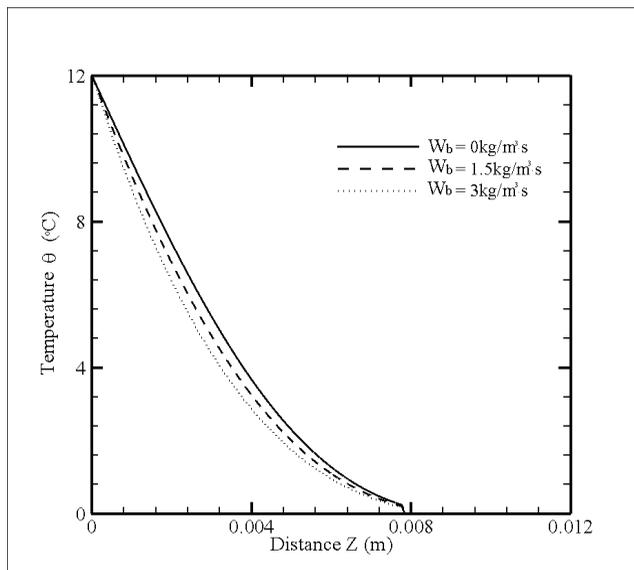


Figure 7: Temperature distribution for various blood flow rates W_b with $\tau = 20s$, $t = 160s$, $K = 0.2^W/m \cdot ^\circ C$.

For case 2:

Fig. 8-9 show the results of grid independence. The appropriate grid sizes are 101 for $\tau = 0s$ and 3001 for $\tau = 20s$.

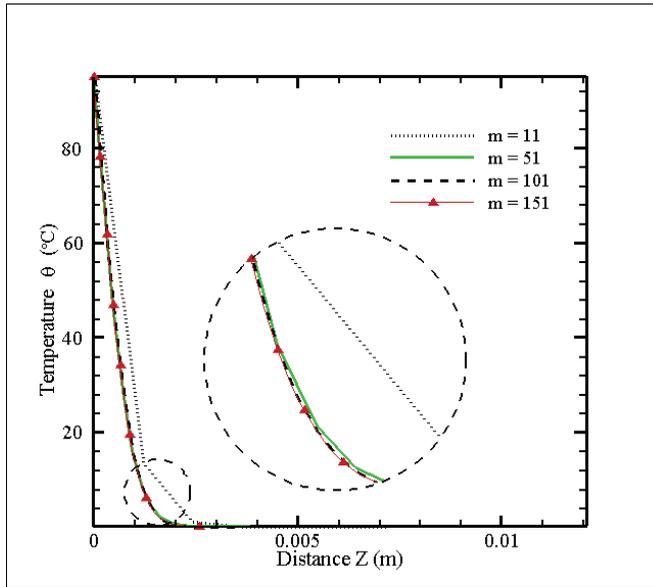


Figure 8: Grid independence ($\tau = 0s$, $t' = 0.02$, $\beta = -0.01$, $K = 0.2^W/m \cdot ^\circ C$, $W_b = 0.5^k g/m^3 \cdot s$).

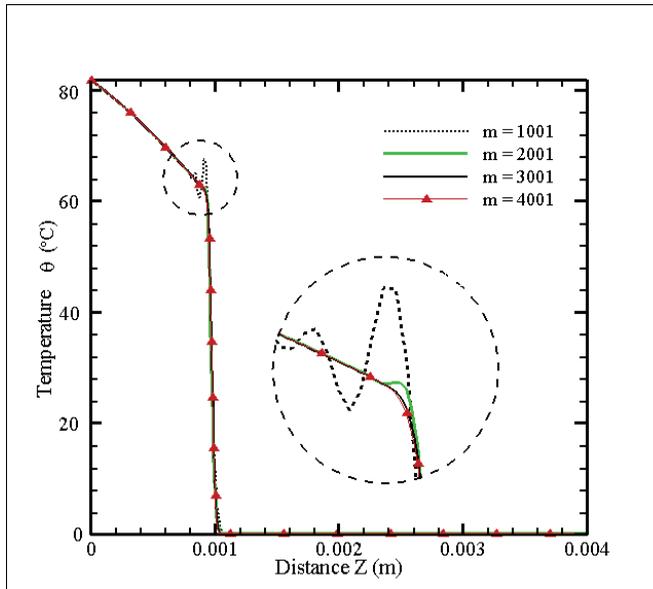


Figure 9: Grid independence ($\tau = 20s$, $t' = 0.02$, $\beta = -0.01$, $K = 0.2^W/m \cdot ^\circ C$, $W_b = 0.5^k g/m^3 \cdot s$).

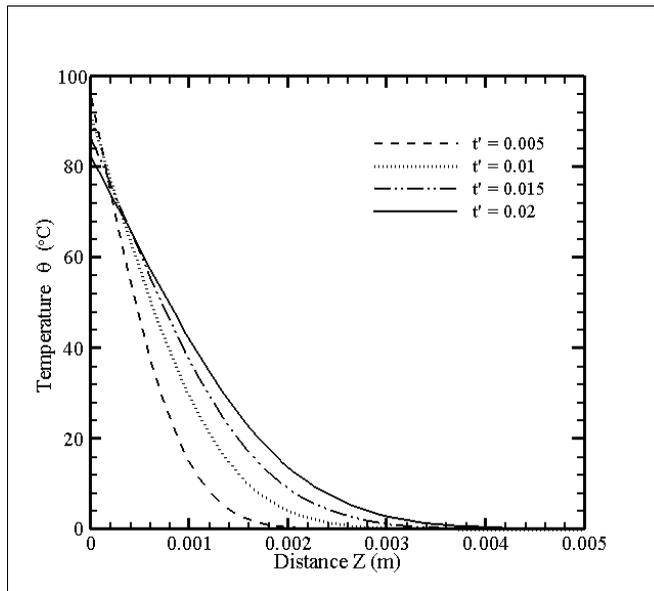


Figure 10: Temperature distributions for various values of t' ($\tau = 0s$, $\beta = -0.01$, $K = 0.2^W/m \cdot ^\circ C$, $W_b = 0.5^k g/m^3 \cdot s$).

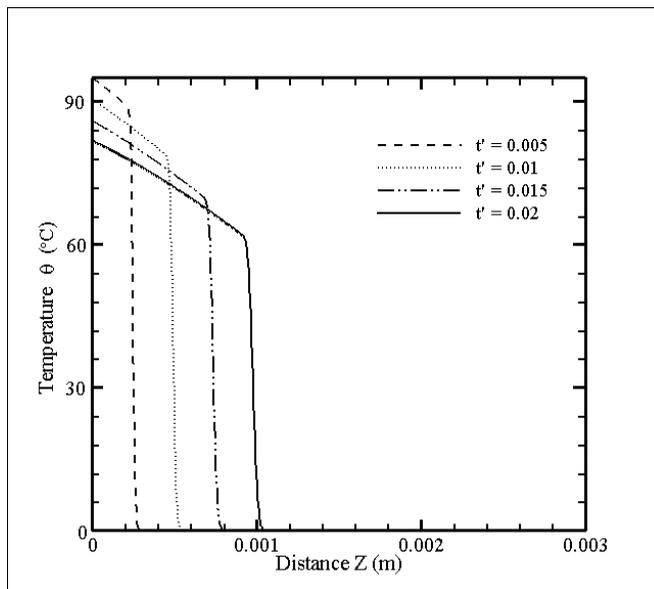


Figure 11: Temperature distributions for various values of t' ($\tau = 20s$, $\beta = -0.01$, $K = 0.2^W/m \cdot ^\circ C$, $W_b = 0.5^k g/m^3 \cdot s$).

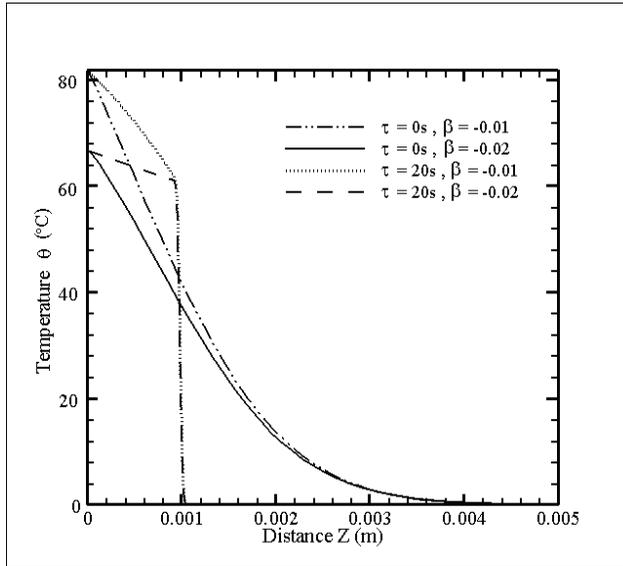


Figure 12: Temperature distribution for various values of β and τ ($t' = 0.02$, $K = 0.2^W/m \cdot ^\circ C$, $W_b = 0.5^k/m^3 \cdot s$).

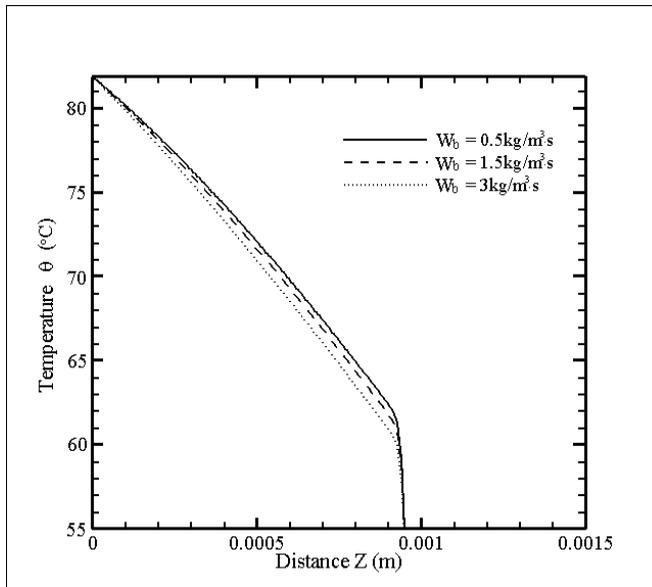


Figure 13: Temperature distribution for various values of blood flow rate W_b ($\tau = 20s$, $t' = 0.02$, $\beta = -0.01$, $K = 0.2^W/m \cdot ^\circ C$).

Fig. 10-11 show that with increasing value of t' , the temperature on the skin surface decreases, but heat wave penetrates deeper into the skin. It is worth mentioning that for $\tau = 20s$, $\beta = -0.01$, $K = 0.2W/m \cdot ^\circ C$, and $W_b = 0.5^k g/m^3 \cdot s$, the heat wave propagates deeper with increasing time and the time required to reach $z = 0.001m$ is $t' = 0.001/\sqrt{0.2/(1000 \times 4200 \times 20)}/1000 = 0.0205s$.

Fig. 12 shows that the temperature changes more apparently with larger β . For $\tau = 0$, the velocity of heat wave propagation is infinity, and for $\tau \neq 0$, the heat wave propagates with a velocity of $V = \sqrt{\alpha/\tau}$.

Fig. 13 shows that an increase in the blood flow rate decreases the temperature but it does not affect the penetration depth.

6 Conclusion

Differential transformation combined with the finite difference method was used to simulate 1-dimensional thermal propagation for living tissue for surface heating problems. The simulation results show good accuracy. Due to the relaxation time effect, the heat transfer propagates in the form of a heat wave, whose velocity can be expressed as $V = \sqrt{\alpha/\tau} = \sqrt{K/\rho C \tau}$. The thermal conductivity, K , is proportional to V^2 .

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