

A Simple Formula for Complementing FE Analyses in the Estimation of the Effects of Local Conditions in Circular Cylindrical Shells

F. Guarracino¹

Abstract: The design of many engineering problems requires accurate test results and interpretation in order to evaluate the carrying capacity of circular cylindrical shells subjected to various loads including bending. Apparently anomalous values of axial tensile and compressive strains from recent test results have been lately investigated and explained using Finite Element modeling. As a complement to numerical analyses, in the present paper a simple analytical formula for the estimation of the effects of local conditions in tubes testing and design is provided on the basis of an extended Ritz's approach and of the general linear theory of shells. The findings are discussed and validated.

Keywords: Circular cylindrical shells, local effects, extended Ritz's method.

1 Introduction

Several engineering applications require cylindrical shells subjected to various loads including bending and in particular an increasing need for offshore pipelines has been experienced in recent years. In this field the use of the limit state approach provides a comprehensive basis for the calculation of the ultimate conditions for pipes subjected simultaneously to pressure and bending loads. The ultimate state of the pipeline deformation or loading is calculated using a model that describes the characteristic ultimate moment or strain related to the geometry and material properties of the pipe. The design factors are calculated using statistical descriptions of the scatter of test results compared to the mean values together with the statistical descriptions of the variables composing the particular model, e.g. material strength, modulus etc. In the process described above, it is generally assumed that the scatter of tests results from minor and usually random variations in the variables is included in the model. In the case of a pipe, these variations would generally relate to the differences in the geometries of the test pipes from their corresponding

¹ Department of Structural Engineering, University of Naples, Federico II, Italy

nominal values, say for pipe wall thickness or out-of-roundness.

The purpose of the present paper is to provide a simple formula for the evaluation of apparently anomalous results from tests on pipes under bending that have been observed in the past and successively explained by means of Finite Elements analyses (Guarracino (2003); Guarracino, Fraldi and Giordano (2008); Guarracino, Walker and Giordano (2009)).

In order to do so, recourse has been made to the trial function method (Ritz (1909)), which represents one class of techniques which possesses the characteristics for providing synthetic solutions to structural mechanics problems. As known, in the trial function method the unknown solution is approximated by a set of base functions containing constants or functions. The key point has consisted in choosing a trial function family which could provide a satisfactory approximation to the problem at hand and, at the same time, allow the treatment of the problem by means of a computer algebra systems (CAS), that is a software package which allow manipulation of mathematical expressions in symbolic form.

The proposed formula can complement numerical analyses and be useful to highlight the potential influence that these anomalies might have on the process of providing design calculation guidance using the limit state method. Last but not least, the proposed formula also sheds light in a direct and physically intelligible manner on the effects of preventing the natural ovalisation of the cross-section under bending, which takes place on account of the well-known von Kármán effect (von Kármán (1911)).

2 Résumé of some test results

Normally, testing a section of a circular cylindrical shell in purely bending loading is carried out on the basis that the test specimen deforms according to simple bending beam theory. Primarily this implies that while the material remains elastic the application of purely bending moment will induce maximum tensile and compressive strains that are identical in magnitude. A typical test rig for a medium diameter pipe, of about 700mm diameter, is shown in Fig.1. The test rig applies a four-point bending condition with the central section of the test pipe assumed to be subjected to bending action only, with no, or at most very little, shear or axial forces.

The actual implementation of the loading points and supports is shown Fig.2, which also displays the deformed shape of the pipe.

From the limit state point of view, for load-controlled conditions of design the most relevant condition is the maximum bearable moment. Conversely, for displacement-controlled design the most relevant condition is the strain at which the reduction of load-bearing capacity first occurs.

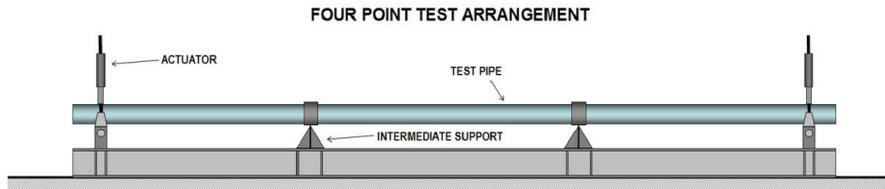


Figure 1: Typical four-point bending test arrangement.



Figure 2: Actual loading points and supports.

Since the pipe is assumed to be an extremely simple structural element, and the simple beam theory is supposed to hold true, it has been common practice to assume that the axial strains have identical values in tension and compression and that the strains can be calculated directly from the curvature or the vertical displacements of the central section of the pipe. The ultimate strain values from tests in which the pipe has been loaded to the point of local buckling have usually been inferred from measurements of the deformations. Only fairly recently it has become a common testing procedure in the industry to attach strain gauges to the specimen to measure axial strains directly.

Some time ago tests were carried out on 152mm diameter pipe to determine the

minimum curvature to which the tube could be deformed prior to the occurrence of a reduction of the load-bearing capacity (see Ellinas, Walker, Langfield and Vines (1985)). An arrangement similar to that in Fig.1 was used, and strain gauges to measure axial and circumferential strains were attached at intervals of 100mm apart along the central test section. In the design of the test rig it was assumed that a central test section of about 5D would suffice to ensure that the end effects due to the loading conditions would diminish to a negligible level along the major part of that section. Fig.3 shows results of the axial strain values along the top and bottom of the pipe section for two levels of the applied loading. It is evident that the axial strains are fairly uniform along the length of the test section but there are significant differences in the averaged values of the compressive strains compared to the tensile strains.

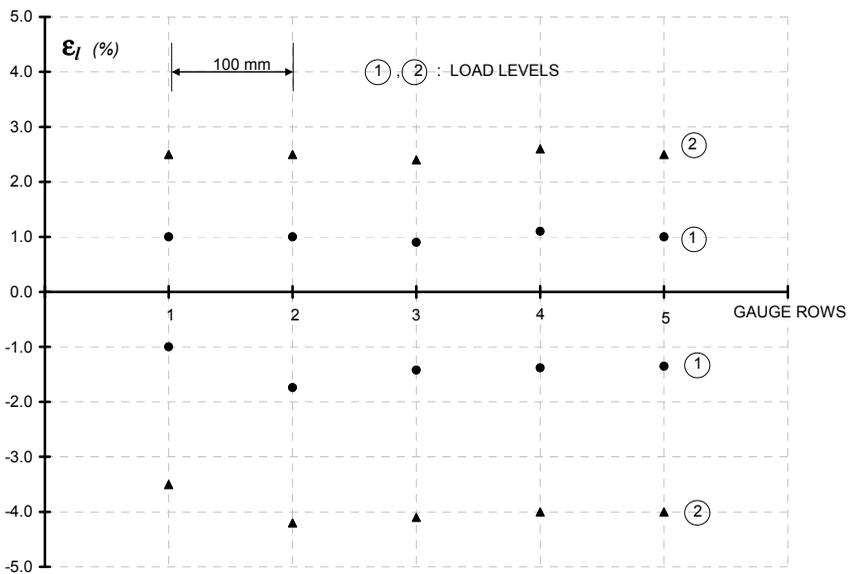


Figure 3: Results from a 152mm diameter pipe bend test (Ellinas, Walker, Langfield and Vines (1985))

At that time the evident anomaly between the measured strains and the expected values vis-à-vis the simple bending theory was not followed up, and even after checking that the strain gauges were correctly positioned and the instrumentation was functioning properly the cause of the anomaly was not further investigated.

Some time later, proving tests were carried out on sections of 609mm diameter pipes containing a thin liner made from a corrosion resistant material (Walker, Holt

and Wilmot (2003)). The purpose of the tests was to determine accurately the level of strain to which the pipe could be bent before the liner buckled locally. The test arrangement of Fig.1 had a loading arm 2m long to create the moment in the central section of the test pipe. The test section was arranged to be 3.5D. The load was applied to the test pipe using straight bars and loose yokes around part of the pipe circumference. A number of axial strain gauges were attached along the top and bottom centre lines of the pipe at intervals from the support points. The values of strain were monitored as the load values were progressively increased. Fig.4 shows the values for the top and bottom gauges averaged along the test sections and plotted against the corresponding value of applied load.

It is evident from Fig.4 that there is a systematic difference between the averaged strains along the top and the bottom of the pipe. At the maximum load level, the averaged axial tensile strains were 1.28 times the corresponding averaged compressive strains.

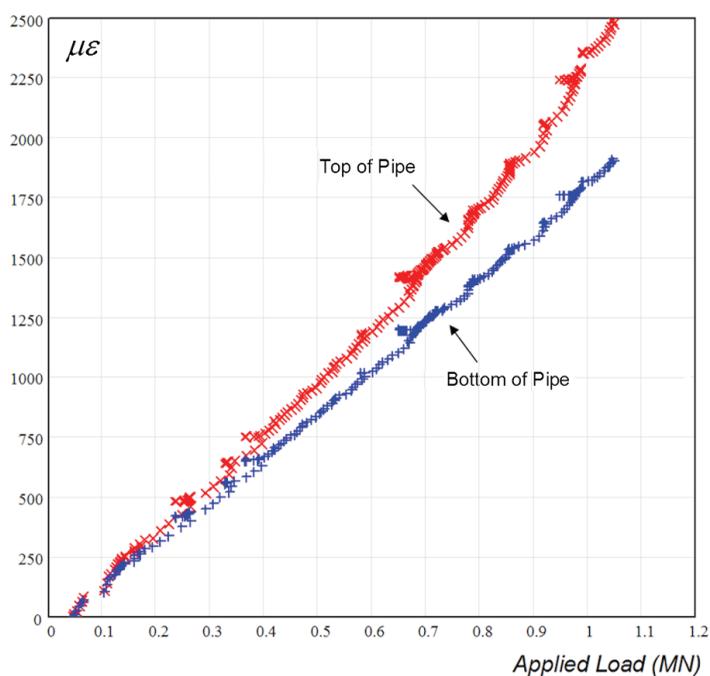


Figure 4: Averaged strain values plotted against corresponding values of applied loading (Walker, Holt and Wilmot (2003): Maximum Ratio of Tensile to Compressive averaged strains = 1.28 (D=609.6mm, t=18.9mm, (D/t=32), X65 material)).

In view of the importance of the results of the tests in providing the allowable levels of strain for the lined pipe an investigation was made with regard to the underlying cause of the anomaly. This is described extensively in Guarracino, Fraldi and Giordano (2008) and in Guarracino, Walker and Giordano (2009) with the aid of several Finite Element models intended to replicate the conditions in bending tests or in pipelines that have changes in cross-section and are subjected to bending, with special attention paid to the constraint arrangements.

The investigation determined that the cause lay in the effect of the imposed ovalisation applied by the saddles at the load points. This result pointed to a proposal for the modification of the loading application in which the loads were applied, not through local stiffening of the pipe wall or saddles, but through the neutral axis of the pipe, as shown in Fig.5.

The test pipe was fitted with strain gauges, as before, and also with gauges to measure the ovality of the pipe. The values of the axial strains measured by the gauges along the test section of the pipe were very uniform. As expected, with the modified loading and support arrangement, the averaged measured values of compressive strains agreed very closely with the corresponding values of the tensile strains.

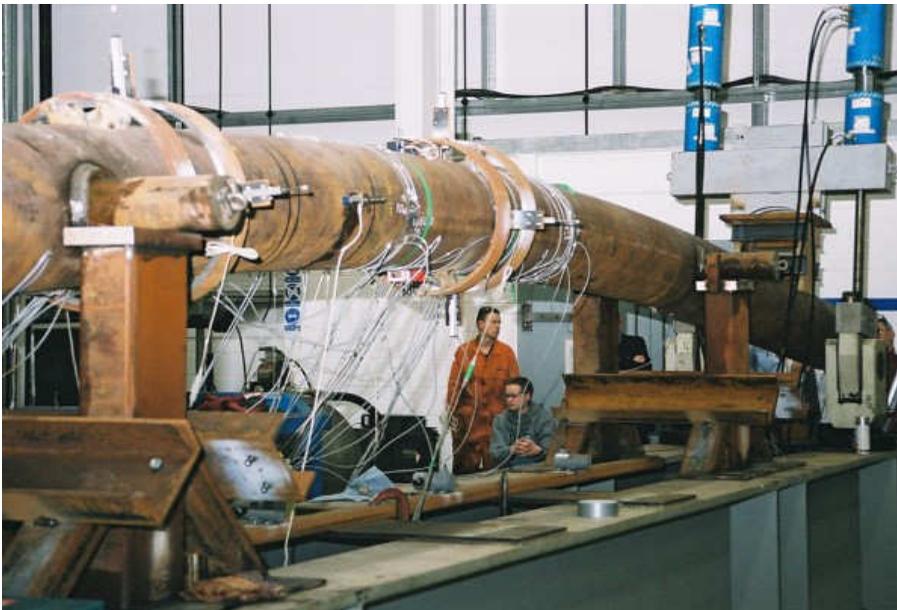


Figure 5: Test arrangement with modified support and load application points.

3 Finite Elements analysis of the effects of testing arrangement

As anticipated in the previous Section, following the observation of the apparent anomaly in the variation of the tensile and compressive strains compared with the values expected on the basis of the simple theory of bending, numerical modeling has been carried out to investigate the root cause of the anomaly.

Essentially, it was found that during the test of a short section of pipe, the practical loading and support arrangements can result in boundary conditions that may impose some degree of ovalisation at the point of load application or, alternatively, decrease the development of the natural ovalisation (von Kármán (1911)). It has generally been assumed that such boundary effects would have a minor consequence on the deformations of the test pipe and would persist for only a short distance along the test length. However, the investigation has shown a hitherto unsuspected mechanism in which the imposition or the prevention of the ovalisation at the loading point of the test pipe will set up an axial strain system that is additional to the usual axial strain caused by simple bending. The conjunction of the two strain systems thus causes a difference between the axial compressive and tensile values.

Fig. 6 shows the Finite Element simulation of a test arranged as in Figs. 1 and 3 on a 609.6mm diameter pipe ($t=18.9\text{mm}$).

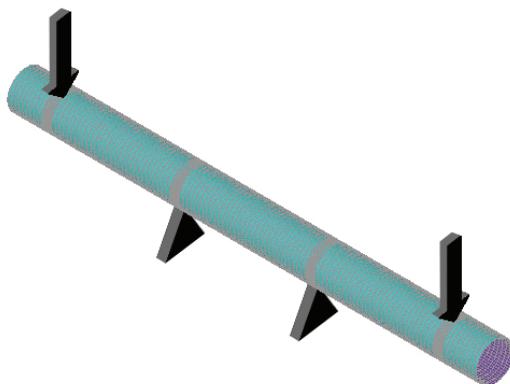


Figure 6: Outline of the Finite Element modelling of a test arranged as in Figs. 1 and 2.

To this purpose the commercially available non-linear Finite Element code, ANSYS® v.11.0 (2007) was employed. The case under consideration was modeled by means of 9000 four-nodes SHELL63 elements for an overall length of 8 m. Both the load-

ing and the support zones were supposed to span over a length of one fourth of the mean radius of the pipe. In this respect a preliminary mesh refinement sensitivity analysis was carried out. The material properties of a generic isotropic high-grade steel, i.e. $E = 2.07 \times 10^5 \text{ N/mm}^2$ and $\nu = 0.3$, were assumed.

As a matter of fact, the Finite Element analysis confirmed that the rigid loading yokes at the supports and at the loading points induce some degree of ovalisation on the pipe and this results in an increased value of the tensile strains at the top of the pipe. This outcome makes it clear that in test design and result analysis special attention must be paid to the effects of loading and constraint arrangement of the pipe being tested.

Fig. 7 shows the values along the pipe axis of the longitudinal strains at the top, at the bottom and at the side of the pipe, as yielded by the Finite Element analysis.

It may be seen that the effects of the loading and of the constraints propagate in a quite complex manner along the full length. It is also evident in the figure that at the mid-span of the model, where the bending moment can be considered constant, the ratio of the tensile axial strain to the corresponding compressive strain is about 1.25 times. Since generally this is the section of pipe that is assumed to be free from the boundary support effects and to have strain levels pertaining to simple bending theory, it can be seen that the analysis confirms the anomaly observed in the tests, see Fig. 4, and also confirms that the presence of the test loading conditions will affect the axial strain levels at which local axial buckling will be initiated in a test pipe.

4 Analytical treatment of the problem by means of an extended Ritz's method

Given the findings from the experimental tests and the Finite Element analyses, it was felt that it would have been useful to have an elementary formula which could be employed to evaluate analytically the propagation of the effect on the membrane strains in the tube wall of the local loading or, alternatively, of the prevented ovalisation at some section.

Evidently, in order to constitute an effective guidance for test design and results interpretation and to complement usefully the Finite Element analyses, such a formula has to be as simple as possible. Conversely, the analysis of circular cylindrical shells under non axis-symmetric loading conditions poses several problems and, notwithstanding the fact that for this reason several simplified shell theories have been proposed in the past (see Calladine (1983)), as a matter of fact very few analytical solutions are available.

With the aim of keeping the model at a level which could be successfully analysed to the purpose, the case of the application of two opposite forces, F , acting at some

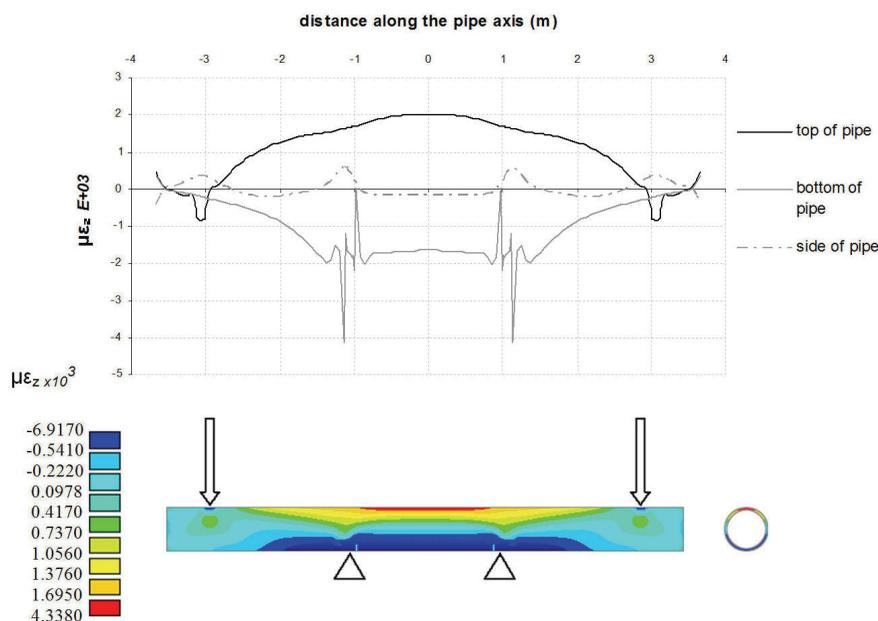


Figure 7: Finite Element simulation of a test arranged as in Figs. 1 and 2: strain values along the pipe axis – Applied Load 1MN.

diameter of an infinitely long tube has been considered. Even if this is not strictly the case of the loading condition of Figs.1 and 6, for which the Finite Element results are shown in Fig.7, nevertheless it can be employed to estimate the effects of the loading transmitted by the yokes and by the supports through the consequential mean amount of the ovalisation imposed on the cross section. The result can be added to the solution from pure bending and a first assessment of overall strains and stresses is thus made possible by hand calculation.

For the solution of the problem, reference is made to an extended Ritz’s approach (see, for example, Courant and Hilbert (1953), Guarracino and Walker (1999)).

As a matter of fact, the Ritz’s method has been extensively used by structural engineers well through the middle of the twentieth century until it has progressively lost ground to its more versatile localised form, i.e. the Finite Element Method. Nevertheless, many formulae of primary practical importance have been found by this mean, which still form the basis of the understanding of a large number of mechanical problems.

Basically, the Ritz method leads to a solution which satisfies approximately the con-

ditions of equilibrium and the static boundary conditions. With a known material law, the chosen approximate displacements must satisfy the kinematic conditions and displacement boundary conditions, that is they must be kinematically admissible. Here reference is made to the extended version of this method, in which the boundary conditions to be satisfied by the trial functions are only partially fulfilled.

A difficulty of the Ritz's method certainly consists in the extensive calculations required, but the appearance of computer algebra systems (CAS), that are software programs which allow manipulation of mathematical expressions in symbolic form, has now made possible the treatment of many problems abandoned in the past.

It is also well known that the complexity of the governing equations of the general linear theory of thin shells motivated the development of a wide range of approximate theories associated with simplifications of these relationships and several viable shell theories are available for circular cylindrical shells. However, versions of these theories that correspond to the same level of approximation generally differ from each other in small ways. Therefore the strain-displacement relationships adopted in the present work are those from the general linear theory of shells and simplifications are made a posteriori by neglecting some higher order terms in the Ritz's expression of the total potential energy.

The advantage of the proposed procedure lies in the extreme simplicity of its final expression, which can give a meaningful physical insight into the parameters which govern the problem at hand and can complement three-dimensional and computationally expensive Finite Elements analyses. In fact the solution, even if approximate, provides the possibility of carrying out a synthetic and comprehensive analysis of the tube state of strain.

Essentially, the development follows steps similar to Timoshenko's method (see Timoshenko and Woinowsky-Kreiger (1959)) in deriving the solution to the same problem in the case of inextensional deformation, with the difference that in the current treatment the extensional deformation of the mid-surface of the shell is allowed. Therefore, since the technique is a classical one, only the particular assumptions at the basis of the present formulation are highlighted in what follows.

A circular cylindrical shell is taken into consideration. The cylindrical coordinate system has the x -axis coincident with the axis of the cylinder. The azimuth, θ , and the radial distance, r , are measured in the plane normal to the x -axis. In such a reference system u , v , and w are the corresponding displacements of a generic point. As said before, the membrane and bending components of strain are assumed to be those in the general linear theory of shells (see Calladine (1983)), i.e.

$$\begin{aligned} \bar{\epsilon}_{xx} &= u_{,x} & \bar{\epsilon}_{\theta\theta} &= (v_{,\theta} - w)/R & \bar{\epsilon}_{x\theta} &= u_{,\theta}/R + v_{,x} \\ \chi_{xx} &= -w_{,xx} & \chi_{\theta\theta} &= -(v_{,\theta} + w_{,\theta\theta})/R^2 & \chi_{x\theta} &= -(v_{,x} + w_{,x\theta})/R \end{aligned} \quad (1)$$

where R is the radius of the middle surface of the shell and comma "," indicates partial differentiation.

It is worth pointing out explicitly that recourse to the Vlasov's semi-membrane theory of cylindrical shells (see Vlasov (1964)), very often employed in the analysis of thin-walled cylindrical tubes, would not capture the essential features of the problem since imposing

$$\bar{\epsilon}_{\theta\theta} = 0, \quad \bar{\epsilon}_{x\theta} = 0, \quad \chi_{xx} = 0, \quad \chi_{x\theta} = 0 \tag{2}$$

leads to a model consisting of innumerable sets of transverse elementary curvilinear strips connected by hinged bonds. Each such strip works in bending in the plane of the shell cross section and internal forces are transmitted from strip to strip by means of rods. The latter can transmit only in-plane normal and shear forces, thus neglecting the flexural stiffness along the longitudinal fibers of the cylinder.

In order to evaluate the deformation induced by two opposite forces, F , acting along a vertical diameter at the section of symmetry $x = 0$, the components of displacement varying along the length of the cylinder for $x \in [0, +\infty[$ are taken in the form

$$\begin{aligned} u &= C_1 R \alpha e^{-\alpha_1 x} \sum_{n=1}^N \frac{1}{n} [A_n \sin(n\phi) + B_n \cos(n\phi)] [M_1 \sin(\alpha_2 x) + N_1 \cos(\alpha_2 x)] \\ v &= C_2 e^{-\alpha_1 x} \sum_{n=1}^N [A_n \cos(n\phi) - B_n \sin(n\phi)] [M_2 \sin(\alpha_2 x) + N_2 \cos(\alpha_2 x)] \\ w &= C_3 e^{-\alpha_1 x} \sum_{n=1}^N n [A_n \sin(n\phi) + B_n \cos(n\phi)] [M_3 \sin(\alpha_2 x) + N_3 \cos(\alpha_2 x)] \end{aligned} \tag{3}$$

where α_1, \dots, N_3 are constants that must be calculated, for the case of loading at hand.

This is a noticeable differentiation of the present approach from the classical Timoshenko formulation, which assumes the change of curvature in the direction of the generatrix to be equal to zero.

It must also be pointed out that the derivatives of the displacement field (3) result discontinuous with respect to the section of symmetry $x = 0$ and, therefore, the displacement field cannot be considered kinematically admissible over the whole length of the tube, that is for $x \in]-\infty, +\infty[$. However, the extended Ritz's method provides a solution that fulfills the displacement conditions approximately, as well as the conditions of equilibrium and the static boundary conditions. Actually, the resulting total potential energy of the system, Π , can be interpreted in a somewhat weak form with the consequence that in the present case the Ritz's approximation

does not necessarily predicts the stiffness of the system to be higher than the actual one.

As anticipated, the assumption (3) implies a considerable computational effort at a symbolic level to define the functional Π . In fact, the integration over the whole cylinder of the strain energy density, i.e.

$$\varphi = E / [2(1 - \nu^2)] \left\{ [\bar{\epsilon}_{xx} + \bar{\epsilon}_{\theta\theta} + z(\chi_{xx} + \chi_{\theta\theta})]^2 + 2(1 - \nu) \left[(\bar{\epsilon}_{x\theta} + 2\chi_{x\theta}z)^2 / 4 - (\bar{\epsilon}_{xx} + \chi_{xx}z)(\bar{\epsilon}_{\theta\theta} + \chi_{\theta\theta}z) \right] \right\} \tag{4}$$

is required. E and ν stand for the Young's modulus and the Poisson's ratio, respectively, and z is the distance from the middle surface. In Eq.(4) the strain components variation across the shell thickness is approximated to the first order according to the Kirchhoff's assumptions, which stipulate the error of the linear theory of the order of t/R , t being the thickness of the shell.

On account of what has been said above, the whole operation has been performed with the aid of the symbolic system MATHEMATICA® (Wolfram (1999)).

Following Ritz's approach, equations for calculating the constants α_1, \dots, N_3 have been then obtained by imposing the total potential energy to be a stationary value for a value of the upper bound of summation, N , equal to 4. The latter has been found adequate to a physically satisfactory representation of the solution. The equations have been successively solved and the results have been expanded in series, trigonometrically fit and simplified in order to obtain a practical expression.

The end result of this lengthy procedure can be summarised in the following formula, which provides the top and bottom membrane strain along the cylinder axis on account of the deformation induced by two opposite forces, F , acting along the vertical diameter at the mid-span

$$\bar{\epsilon}_{xx} = - \frac{2(1 - \nu^2)F}{ERt} e^{-\frac{\beta x}{2}} \cos \beta x \tag{5}$$

β is given by

$$\beta = \frac{\pi^2 \sqrt[4]{(1 - \nu^2)}}{4\sqrt{R^3/t}} \tag{6}$$

An important finding is that the expression of what can be considered the natural half-wavelength of the problem, i.e. $\lambda = \pi/\beta$, results proportional to the term $\sqrt{R^3/t}$, whereas in the case of circular shells subject to axial symmetric loading, it is proportional to \sqrt{Rt} (see Timoshenko and Woinowsky-Kreiger (1959)).

Therefore it is confirmed that the effect of the loading represented by two opposite forces, F , acting along a vertical diameter, cannot be seen as a rapidly decaying classic local perturbation. This is visualised in Fig.8, where the function $\cos \beta x$, which governs Eq.(5) when β is given by Eq.(6), and characterises a solution for an axisymmetric loading when β is given by

$$\beta = \frac{\sqrt[4]{3(1 - \nu^2)}}{\sqrt{Rt}} \tag{7}$$

is shown.

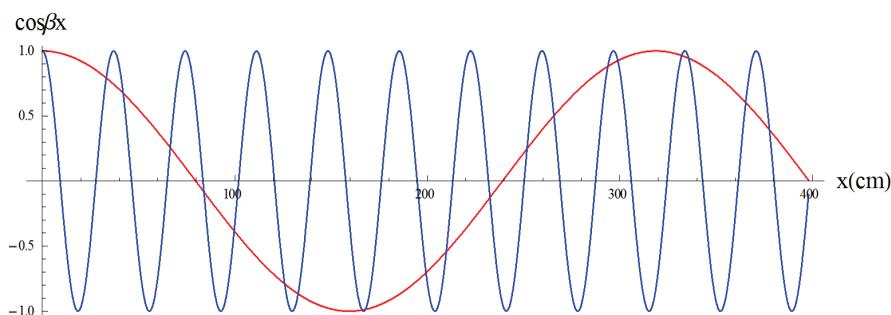


Figure 8: Plot along the tube axis of the periodic function governing the present solution (red) and that for an axisymmetric loading (blue) for values pertaining to the specimen tested in Fig.4.

Finally, it is worth noticing that the top and bottom mid-surface strain is inversely proportional to both the radius of the tube, R , and to its thickness, t .

5 Validation of the proposed formula

In order to validate the proposed formula, Eq.(5), Finite Element modeling has been applied to determine the intensity and the length over which the effects of the loading propagate and hence influence the membrane axial strains. In order to do so, a very simple modeling was applied to pipes with different diameters and wall thicknesses. The modeling consisted in applying a vertical load of 1MN exactly on a vertical support. In this manner no bending moment was expected along the axis of the pipe and all the strains were due to the imposed ovalisation of the loaded section. As seen before, these strains are essentially similar to those which, in the case of bending tests, add to the bending strains and can cause a significant alteration in the symmetry with respect to the neutral axis of the section. Again,

it is shown that this effect cannot be seen as a classic local perturbation, since it persists for a greater length along the test pipe than has hitherto been assumed.

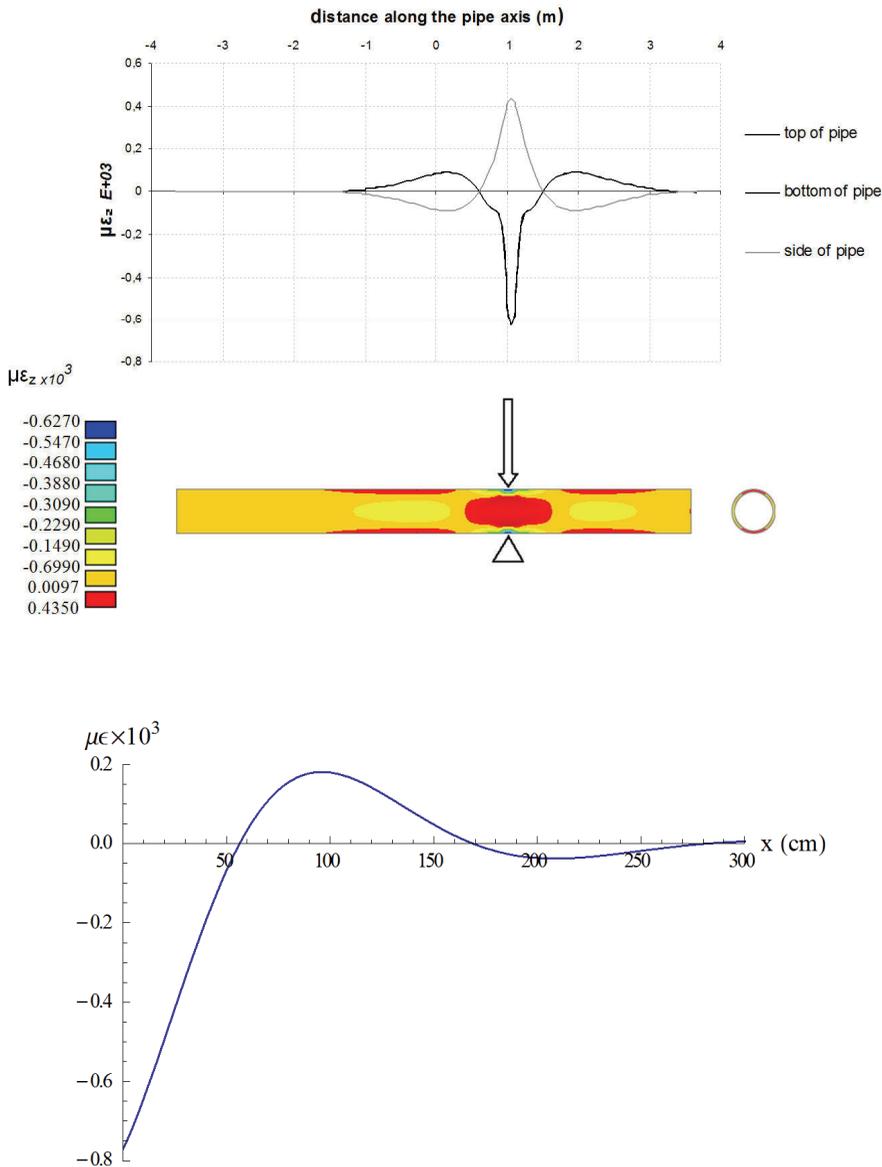


Figure 9: FEM vs. proposed formula results for a pipe with $D/t=16$ (diameter 609.6mm, wall thickness 37.8mm) and applied load of 1MN: mid-surface top and bottom strain values along the pipe axis

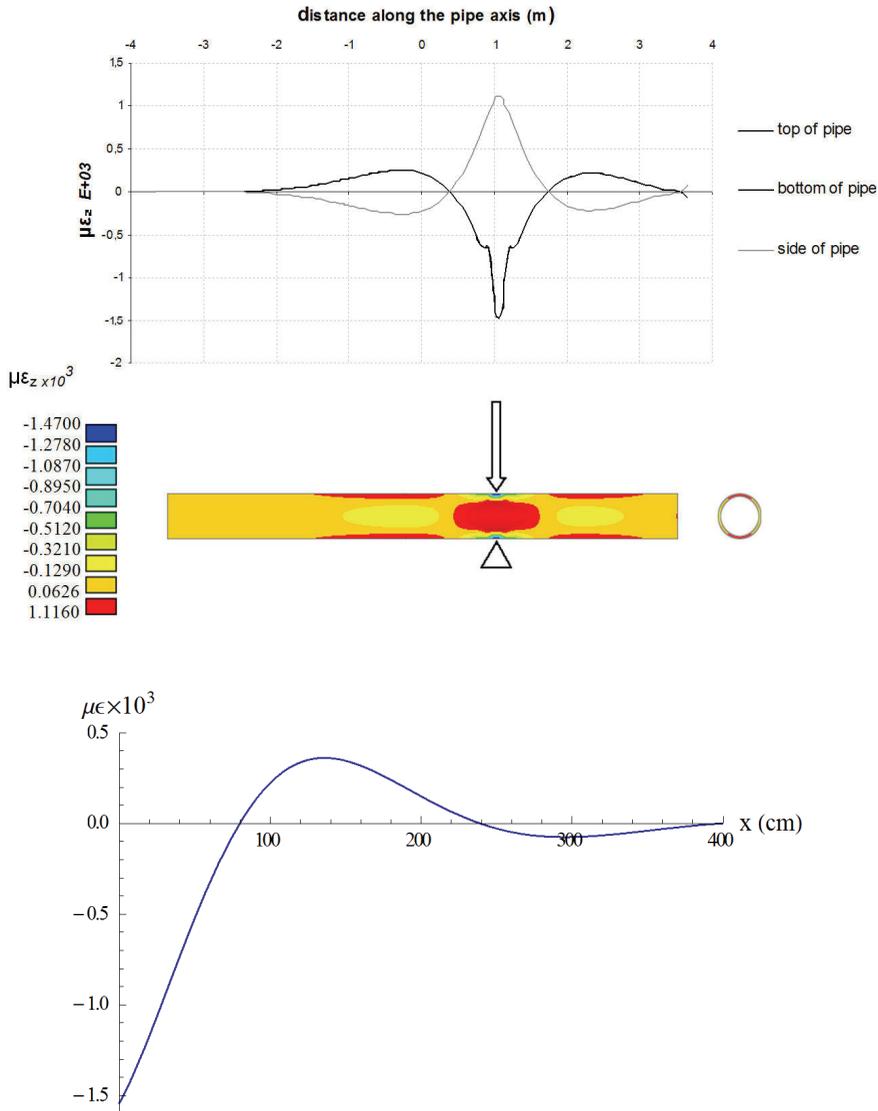


Figure 10: FEM vs. proposed formula results for a pipe with $D/t=32$ (diameter 609.6mm, wall thickness 18.9mm) and applied load of 1MN: mid-surface top and bottom strain values along the pipe axis

Figs. 9-11 show a very good agreement between the results from FE analyses and those from the proposed formula for three different cases, corresponding to D/t ratios of 16, 32 and 64. In fact, both the value of the strain and the propagation of the effect are fundamentally captured by the presented formula.

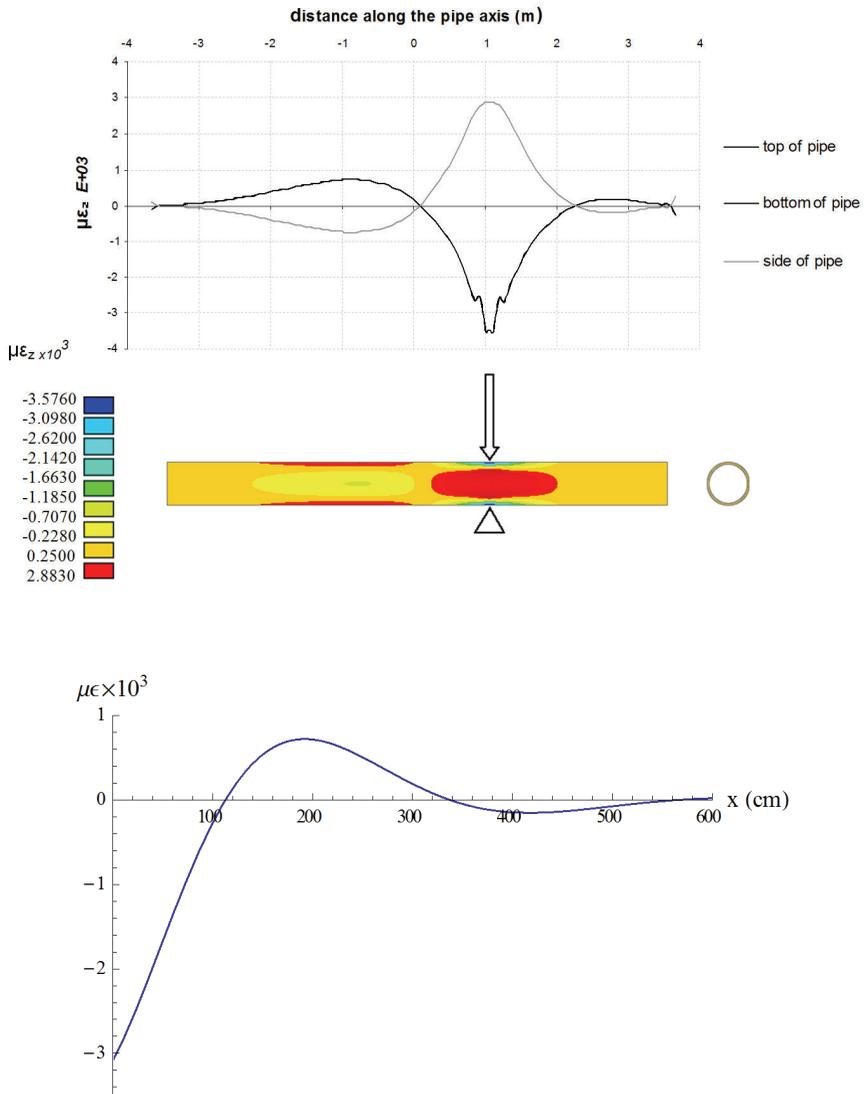


Figure 11: FEM vs. proposed formula results for a pipe with $D/t=64$ (diameter 609.6mm, wall thickness 9.45mm) and applied load of 1MN: mid-surface top and bottom strain values along the pipe axis

Finally, it is worth reaffirming that the proposed formula can also be applied to estimate the effects of preventing the natural ovalisation of the cross-section under bending, which takes place on account of the well-known von Kármán effect (von Kármán (1911)). In fact, if the ovalisation is restricted by any form of constraints, a marked asymmetry between the compressive and tensile strains occurs with a turn round with respect to what has been shown in the previous figures. This is the case of the results from a 152mm diameter pipe bend test test (Ellinas, Walker, Langfield and Vines (1985)), shown in Fig.3, where the averaged compressive axial strains are seen to be about 1.36 times the corresponding values of the tensile strains. The explanation of this phenomenon is quite simple: preventing the natural ovalisation under bending can be seen as applying a loading similar to those in Figs. 9-11, but reversed in sign. The resulting strains will be reversed in sign, too, and, in the case of Fig. 3, add to the bending strains causing an increase in the value of the compressive strains and a reduction in the value of the tensile strains.

6 Conclusions

In the present work a simple analytical formulation to evaluate the effects of test arrangement and boundary conditions on the level of apparent strain has been provided. Essentially, the formula is an extension of a classical work on circular cylindrical shells in the case of inextensional deformation (Timoshenko and Woinowsky-Kreiger (1959)). By means of the obtained result, the seemingly anomalous values of measured axial strain in tests can be explained and evaluated very straightforwardly. The proposed formulation also offers a physical insight into the mechanics of the problem in the fashion of many classical results still widely used in the engineering practice and can constitute a basis for accounting for these effects in design codes.

Acknowledgement: The author is grateful to the anonymous reviewers for the suggestions received to improve the paper and convey the results by refraining from excessively lengthy mathematical developments.

References

- ANSYS Inc.** (2007): ANSYS 11.0 user's documentation. Canonburg, PA 15317. USA.
- Calladine, C.R.** (1983): *Theory of Shell Structures*. Cambridge University Press.
- Courant, R.; Hilbert, D.** (1953): *Methods of mathematical physics. Vol. I*. Interscience.
- Ellinas, C.P.; Walker, A.C.; Langfield, G.N.; Vines, M.J.** (1985): A Develop-

ment in the Reeling Method for Laying Subsea Pipeline. *Proc. 1st Petroleum Technology Australian Conference*. Perth, Australia.

Guarracino, F.; Walker, A.C. (1999): *Energy Methods in Structural Mechanics*. Telford.

Guarracino, F. (2003) : On the analysis of cylindrical tubes under flexure: theoretical formulations, experimental data and finite elements analysis. *Thin-Walled Structures*, vol. 41, pp. 127–147.

Guarracino, F.; Fraldi, M.; Giordano, A.(2008): Analysis of testing methods of pipelines for limit state design. *Applied Ocean Research*, vol.30, pp.297-304.

Guarracino, F.; Walker, A.C.; Giordano, A.(2009): Effects of Boundary Conditions on Testing of Pipes and Finite Element Modelling. *Int. J. Press. Ves. Piping*, vol.86, pp.196-206.

Ritz, W. (1909): Über eine Neue Methode zur Lösung gewisser Variationsprobleme der Mathematischen Physik. *J. Reine Angew. Math.*, vol.135, pp. 1–61.

Timoshenko, S.P.; Woinowsky- Kreiger, S. (1959): *Theory of Plates and Shells*. McGraw-Hill.

Vlasov, V.Z. (1964): *General Theory of Shells and its Applications in Engineering*. NASA Technical Translation TTF-99.

von Kármán, Th. (1911): Ueber die Formänderung dünnwandiger Rohre, insbesondere federnder Ausgleichrohre. *Zeitschrift des Vereines deutscher Ingenieur*, vol.45, pp.1889-1895.

Walker, A.C.; Holt, A.; Wilmot, D.(2003): Test Procedure for Pipe and Pipeline Material. *Offshore Pipeline Technology Conference*. Amsterdam, Holland.

Wolfram, S. (1999): *The Mathematica Book*. Cambridge University Press.