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On Chaos Control in Uncertain Nonlinear Systems

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Abstract: Chaotic behavior of uncertain nonlinear systems offers a rich variety of orbits, which can be controlled by bounding the signals involved in closed-loop systems. In this paper, systems with nonlinear uncertainties with no prior knowledge of their bounds, unmodeled dynamic law and rapidly varying disturbances are analyzed in order to propose a stabilization controller of the chaotic behavior via the fuzzy logic systems.

Keywords: Uncertainties, unmodeled dynamics, disturbances, chaos, double pendulum.

1 Introduction

The utility of structured adaptive control formulations is important for a large class of nonlinear oscillations, aircraft control, spacecraft control, and cooperative robotic system control [Junkins, Subbarao and Verma (2000); Lin (2009)]. Chaotic behavior has been extensively analyzed and the concept of chaos control becomes desirable in various applications [Pereira-Pinto, Ferreira and Savi (2004)]. Chaos control methods are related to feedback linearization and backstepping techniques applied to design a wide variety of nonlinear controllers for systems with unknown parameters [Kristic, Kanellakopoulos and Kokotovic (1995); Kristic, Kanellakopoulos and Kokotovic (1992)]. The feedback linearization transforms a nonlinear system with matching conditions, into a linear system for which the linear control technique is used to acquire the desired performances [Isidori (1995)]. The backstepping technique requires the cancellation of the nonlinearities and it is used to design the controllers for the systems with known nonlinearities without satisfying the matching conditions [Kanellakopopoulos, Kokotovic and Morse (1991)]. For systems for which the precise knowledge of nonlinearities may not be available or nonlinearities may change in time, the models have been modified in order to

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introduce some prior knowledge of uncertainties, such as known bounds of static nonlinearities [Qu (1995)].

The case of uncertain nonlinear systems without satisfying the matching conditions has been extensively studied in the literature. However, the application of fuzzy logic systems (FLS) to model nonlinearities based on the feedback linearization have been addressed by Boulkroune, Tadjine, Saad and Farza(2008); Golea, Golea and Benmahammed (2003), whilst Chen, Liu and Tong (2007); Chen, Liu and Shi (2009); Tong, Li and Shi (2009) have applied the backstepping based adaptive FLS.

Lin (2010) proposed a direct method for the adaptive fuzzy-neural tracking control equipped with sliding mode and Lyapunov synthesis approach to analyze the training data corrupted by noise or rule uncertainties. Extension to multiple-input multiple-output (MIMO) nonlinear systems are discussed by Isidori (1995).

The Kalman synthesis dynamic compensation and Neural Network (NN) were utilized by Ursu, Toader and Tecuceanu (2009, 2010) to compensate the lack of system knowledge in conjunction with a unitary approach of adaptive output feedback control. A backstepping based robust adaptive NN control is also discussed by Li, Hong and Shi (2008) for strict-feedback nonlinear systems via a small gain theorem.

Polynomial chaos methods coupled with fictitious domain approach have been studied by Parussini and Pediroda (2008) for the investigation of multi geometric uncertainties.

We believe that controlled systems which only contain nonlinear uncertainties, without either the unmodeled dynamic law or dynamic disturbances, represent a limitation for obtaining the control design for real nonlinear systems with chaotic behavior. A relevant paper in this direction is due to Tong, He, Li and Zhang (2010). The authors have proposed an adaptive fuzzy robust control method, for single-input single-output nonlinear systems with nonlinear uncertainties, the unmodeled dynamic law and rapidly varying disturbances, by combining the backstepping technique with the nonlinear small-gain approach.

The chaos control method presented in this paper further extends the range of applications of the adaptive control method to systems with nonlinear uncertainties, unmodeled dynamic law and rapidly varying disturbances. The control objective is to apply fuzzy logic systems to determine a stabilization controller of the chaotic behavior. Numerical experiments are carried out for a driven double pendulum.

2 Preliminaries

In this section, the concepts of *Input-to State Stable* (ISS) and *ISS-Lyapunov function* proposed by Sontag (1989, 1990), Sontag and Wang (1995), Khalil (1996) and

Li, Hong and Shi (2008) are revisited from the chaos point of view. Chaos control usually involves two steps. In the first step, the unstable period trajectories that are embedded in the chaotic set are identified. After that, a control technique is employed in order to stabilize a desirable orbit [Pereira-Pinto, Ferreira and Savi (2004)].

A function $\gamma : R^+ \to R^+$ of class K is a continuous, strictly increasing function with $\gamma(0) = 0$. A function $\gamma : R^+ \to R^+$ of class K_∞ is a continuous, strictly increasing function with $\gamma(0) = 0$ and $\gamma(s) \to \infty$ as $s \to \infty$. The function $\beta : R^+ \times R^+ \to R^+$ is of class KL if $\beta(\cdot,t)$ is of class K for every $t \geq 0$ and $\beta(s,t) \to 0$ as $t \to \infty$ KL.

Definition 1 [Li, Hong and Shi (2008)]. *The system $\dot{x} = f(x,u)$ is Input-to-State practically Stable (ISpS) if there exists a function γ of class K, called the nonlinear L_∞ gain, and a function β of class KL, such that, for ansy initial condition $x(0)$, each measurable bounded control $u(t)$ defined for all $t \geq 0$ and a nonnegative constant d, the solutions $x(t)$are defined on* $[0,\infty)$ *and satisfy*

$$||x(t)|| \leq \beta(||x(0)||,t) + \gamma(||u_t||_\infty) + d. \tag{2.1}$$

For $d = 0$, the ISpS property reduces to the ISS property.

Definition 2 [Jiang (1999)]. *A function V of class $C^1$is ISpS - Lyapunov function for $\dot{x} = f(x,u)$ if there exist the functions $\alpha_i, i = 1,2,...,4$ of classK_∞, and a constant* $d > 0$*such that*

$$\alpha_1(||x||) \leq V(x,t) \leq \alpha_2(||x||), \quad \forall x \in \mathrm{R}^n, \tag{2.2}$$

$$V_{,x} f(x,u) \leq -\alpha_3(||x||) + \alpha_4(||u||) + d.$$

For $d = 0$, the Lyapunov function is an ISS - Lyapunov function. Herein a comma in the subscript denotes differentiation with respect to the specified variable. The nonlinear L_∞ gain γ is derived from (2.2) as

$$\gamma(s) = \alpha_1^{-1} \circ \alpha_2^{\circ} \alpha_3^{-1} \circ \alpha_4, \quad \forall s > 0. \tag{2.3}$$

Proposition 1 [Sontag and Wang (1995); Praly and Wang (1996)]. *The system $\dot{x} = f(x,u)$is ISpS if and only if there exists an ISpS – Lyapunov function for the system.*

Theorem 1 [Jiang, Teel and Praly (1994); Jiang and Mareels (1997)]. *Consider two related ISpS systems in the feedback form*

$$\dot{x} = f(x,\omega), \quad \tilde{z} = H(x),$$

$$\dot{y} = g(y, \tilde{z}), \quad \omega = K(y, \tilde{z}). \tag{2.4}$$

Let the functions β_ω and β_ξ of class KL, and the functions γ_z, γ_ω of class K be such that all solutions $X(x;\omega,t)$ and $Y(y;\tilde{z},t)$ are defined on $[0,\infty)$ and satisfy the following relations for $t \geq 0$

$$||H(X(x;\omega,t)|| \leq \beta_\omega(||x||,t) + \gamma_z(||\omega_t||_\infty) + d_1,$$
$$||K(Y(y;\tilde{z},t)|| \leq \beta_\xi(||y||,t) + \gamma_\omega(||\tilde{z}_t||_\infty) + d_2, \tag{2.5}$$

for every $\omega \in L_\infty$, $\tilde{z} \in L_\infty$, $x \in \mathrm{R}^n$ and $y \in \mathrm{R}^m$, and $d_1 > 0$, $d_2 > 0$ two constants.

Under these conditions, if

$$\gamma_z(\gamma_\omega(s)) < s \text{ or } \gamma_\omega(\gamma_z(s)) < s, \quad \forall s > 0, \tag{2.6}$$

the solution of (2.4) is ISpS.

Proposition 2 [Tong, He, Li and Zhang (2010)]. *Given the interconnected systems*

$$\dot{x}_1 = f_1(x_1, x_2, u_1), \quad \dot{x}_2 = f_2(x_1, x_2, u_2), \tag{2.7}$$

where $x_i \in \mathrm{R}^{n_i}, u_i \in \mathrm{R}^{m_i}, i = 1,2,$ *and* $f_i : R^{n_1} \times R^{n_2} \times R^{m_i} \to R^{n_i}$ *locally satisfy the Lipschitz conditions, and assuming that there exist the ISpS - Lyapunov functions,*$V_i, i = 1,2,$ *then the following statements hold*:

1. *There exist the functions* $\vartheta_{i1}, \vartheta_{i2}$*of class* K_∞*, so that*

$$\vartheta_{i1}(||x_i||) \leq V_i(x_i) \leq \vartheta_{i2}(||x_i||), \quad \forall x_i \in \mathrm{R}^{n_i}, \tag{2.8}$$

2. *There exist the functions* α_i' *of class*K_∞*,the functions* χ_i *and* γ_i*of class K, and the constants* $c_i \geq 0$*, so that* $V_1(x_1) \geq \max(\chi_1(V_2(x_2)), \gamma_1(||u_1||) + c_1)$*implies that*

$$\nabla V_1(x_1) f_1(x_1, x_2, u_1) \leq -\alpha_1'(V_1), \tag{2.9}$$

and $V_2(x_2) \geq \max(\chi_2(V_1(x_1)), \gamma_2(||u_2||) + c_2)$*implies that*

$$\nabla V_2(x_2) f_2(x_1, x_2, u_2) \leq -\alpha_2'(V_2). \tag{2.10}$$

Theorem 2 [Jiang, Marels and Wang (1996)]. *In the conditions of the Proposition2, let us assume that the*x_i*-subsystems,* $i = 1,2,$ *have the ISpS - Lyapunov functions* $V_i, i = 1,2,$ *satisfying* (2.8)-(2.10). *If there exists* $c_0 \geq 0$ *such that*

$$\chi_1 \circ \chi_2(s) < s, \quad \forall s > c_0, \tag{2.11}$$

then the interconnected systems (2.7) *are ISpS. Futhermore, if* $c_0 = c_1 = c_2 = 0$*, the system is ISS.*

The robustness of the controller is obtained by checking the conditions of the theorem 2 for closed loop systems.

3 Formulation of the problem

In this section we study the following uncertain unmodeled nonlinear system

$$\dot{x}_1 = x_2 + f_1(x_1,x_2) + \Delta_1(z,x_1,...,x_n),\ \dot{x}_2 = x_3 + f_2(x_1,x_2,x_3) + \Delta_2(z,x_1,...,x_n),$$

$$\dots$$

$$\dot{x}_{n-1} = x_n + f_{n-1}(x_1,...,x_n) + \Delta_{n-1}(z,x_1,...,x_n),$$

$$\dot{x}_n = u + f_n(x_1,x_2,...,x_n,u) + \Delta_n(z,x_1,...,x_n),$$

$$\dot{z} = q(z,x_1),$$

$$y = x_1, \tag{3.1}$$

where $(x_1(t),x_2(t),...,x_n(t)) \in \mathrm{R}^n$ is the state, f_i, $i = 1,2,...,n$, are nonlinear functions that define the uncertainties of the system, $\Delta_i(z,x_1,...,x_n)$, $i = 1,2,...,n$ are rapidly varying disturbances, $u \in \mathrm{R}^m$ (m positive integer) is the input of the system, $z(t) \in D \subset \mathrm{R}^n$ (n positive integer) represents the dynamic unmodeled law for the unmeasured portion of the state, $q(z,x_1)$ is an uncertain Lipschitz continuous function, and $y \in \mathrm{R}^l$ (l positive integer) is the output of the system.

We assume that there exist the unknown positive constantsp_i, $i = 1,2,...,n$, and known nonnegative smooth functions$\psi_{i1}(||x_i||)$, $\psi_{i2}(||z||)$ such that for $t > t^*$, with a known t^*, to have [Jiang and Praly (1998); Bernardo and Stoten (2006)]

$$||\Delta_i|| \leq p_i[\psi_{i1} + \psi_{i2}], \quad \psi_{i2}(0) = 0, \quad i = 1,2,...,n. \tag{3.2}$$

Moreover, we assume that $q(z,x_1)$admits a Lyapunov function $V_0(z,x_1)$ of the form (2.2)

$$\alpha_1(||z||) \leq V_0(z,x_1) \leq \alpha_2(||z||), \quad \forall x \in \mathrm{R}^n,$$

$$V_{0,z}q(z,x_1) \leq -\alpha_0(||z||) + \gamma_0(||x_1||) + d_0, \tag{3.3}$$

where $\alpha_1(||z||)$, $\alpha_2(||z||)$, $\alpha_0(||z||)$ and $\gamma_0(||x_1||)$ are of class K_∞ and d_0 is a nonnegative constant. The unmodeled dynamic law $\dot{z} = q$ which verifies (3.3) characterizes an ISpS system [Jiang (1999)].

Under conditions (3.2) and (3.3), the adaptive fuzzy control scheme proposed by Tong, He, Li and Zhang (2010) can guarantee that closed-loop systems are semi-globally uniformly ultimately bounded.

4 Adaptive fuzzy state feedback control

A fuzzy logic system (FLS) fuzzifies the input vector $x = [x_1, x_2, ..., x_n]^T$ using prescribed membership functions, goes to a table-lookup rule base to determine associated output values, then defuzzifies the output values into a single output value y. The FLS is a collection of fuzzy *If-Then* rules

$$R^l: \text{ If } x_1 \text{ is } F_1^l \text{ and } \dots \text{ and } x_n \text{ is } F_n^l, \text{ then } y \text{ is } G^l,\ l = 1, 2, ..., N, \tag{4.1}$$

where F_i^l and G^l are fuzzy sets of the fuzzy functions $\mu_{F_i^l}(x_i)$ and $\mu_{G^l}(y)$, respectively, and N is the number of rules. The FLS can be expressed as

$$y(x) = \sum_{l=1}^{N} \bar{y}_l \prod_{i=1}^{n} \mu_{F_i^l}(x_i) / \sum_{l=1}^{N} \left(\prod_{i=1}^{n} \mu_{F_i^l}(x_i) \right), \tag{4.2}$$

where $\bar{y}_l = \max_{y \in \mathrm{R}} \mu_{G^l}(y)$, or in the form

$$y(x) = \varsigma^T \varphi(x). \tag{4.3}$$

In (4.3), $\varsigma^T = [\bar{y}_1, \bar{y}_2, ..., \bar{y}_N] = [\varsigma_1, \varsigma_2, ..., \varsigma_N]$ and $\varphi(x) = [\varphi_1(x), \varphi_2(x), ..., \varphi_N(x)]^T$ are the fuzzy basis functions defined as

$$\varphi_l = \prod_{i=1}^{n} \mu_{F_i^l}(x_i) / \sum_{l=1}^{N} \left(\prod_{i=1}^{n} \mu_{F_i^l}(x_i) \right). \tag{4.4}$$

Lemma 1 [Wang (1994)]. *Let $f(x)$ be a continuous function defined on a compact set Ω. Then for any constant $\varepsilon > 0$, there exists a FLS given by* (4.3) *such that*

$$\sup ||f(x) - \varsigma^T \varphi(x)|| \le \varepsilon, \tag{4.5}$$

Let us suppose that the unknown smooth functionsf_i, $i = 1, 2, ..., n$, defined on the compact sets U_i, can be approximated by the FLS (4.3)

$$\hat{f}(x_i|\varsigma_i) = \varsigma_i^T \varphi(x_i), \quad i = 1, 2, ..., n, \tag{4.6}$$

where the adaptation functions $\varsigma_i$are defined on the compact sets Ω_i. In this case, equation $(3.1)_3$can be written as

$$\dot{x}_{n-1} = x_n + \hat{f}_{n-1} + [\hat{f}_{(n-1)opt} - \hat{f}_{n-1}] + [f_{n-1} - \hat{f}_{(n-1)opt}] + \Delta_{n-1}, \tag{4.7}$$

where $\hat{f}_{iopt} = \hat{f}_i(x_i|\varsigma_{iopt})$. The optimal parameters ς_{iopt} are defined as

$$\varsigma_{iopt} = \arg \min_{\theta_i \in \Omega_1} \left[\sup{}_{x_i \in U_i} |\hat{f}_i - f_i| \right], \quad i = 1, 2, ..., n. \tag{4.8}$$

The fuzzy approximation errors $\varepsilon_i(x_i) = f_i - \hat{f}_{iopt}$, verify the following relation

$$||\varepsilon_i(x_i)|| \leq \varepsilon_{i0}, \quad i = 1,2,...,n,$$

with ε_{i0} positive constants. By inserting (4.6) into (4.7) it results

$$\dot{x}_{n-1} = x_n + \varsigma_{n-1}^T \varphi_{n-1} + (\varsigma_{(n-1)opt} - \varsigma_{n-1})^T \varphi_{n-1} + \varepsilon_{n-1} + \Delta_{n-1}. \tag{4.9}$$

If x_{i+1}, $i = 1,2,...,n$, are assumed to be the virtual controls, the stabilization functions $\pi_i(x_1,...,x_i,\varsigma_1,...,\varsigma_i,\hat{p}_1,...,\hat{p}_i)$ are defined by

$$\bar{x}_{i+1} = x_{i+1} - \pi_i, \quad i = 1,2,...,n, \tag{4.10}$$

where $\hat{p}_i$ are adaptation functions representing the estimate functions of p_i, $i = 1,2,...,n$ defined by (3.2). The time differentiation of (4.10) gives

$$\dot{\bar{x}}_{i+1} = \\ x_{i+2} + f_{i+1} - \sum_{j=1}^{i} \pi_{i,x_j} f_j - \sum_{j=1}^{i} \pi_{i,x_j} x_{j+1} - \sum_{j=1}^{i} \pi_{i,\theta_j} \dot{\theta}_j - \sum_{j=1}^{i} \pi_{i,\hat{p}_j} \dot{\hat{p}}_j + \Delta_i - \sum_{j=1}^{i} \pi_{i,j} f_j, \tag{4.11}$$

where the unknown smooth functions $f_{i+1} - \sum_{j=1}^{i} \pi_{i,x_j} f_j$ are defined on the compact sets Ω_i, $i = 1,2,...,n.$ Assume that these unknown functions can be approximated by FLS

$$f_i - \sum \pi_{i-1,x_j} f_j = \varsigma_i^T \varphi_i + (\varsigma_{iopt} - \varsigma_i)^T \varphi_i + \varepsilon_i. \tag{4.12}$$

Now, consider the Lyapunov functions

$$V_i = V_{i-1} + \frac{1}{2}\bar{x}_i^2 + \frac{1}{2}(\varsigma_{iopt} - \varsigma_i)^T \Gamma_i^{-1} (\varsigma_{iopt} - \varsigma_i) + \frac{1}{2}\lambda_i^{-1}(\hat{p}_i - p_i)^2, \tag{4.13}$$

where $V_0 = \frac{1}{2}\eta(x_1^2)$ is a smooth function of the class K_∞, $\Gamma_i = \Gamma_i^T$ are known adaptation gain matrices, $\lambda_i > 0$ are design parameters. The function $\eta(x_1^2)$ is chosen such that its derivative with respect to x_1 is strictly positive. By using (4.11) and (4.12), the time differentiation of V_i becomes

$$\begin{aligned} \dot{V}_i &= \dot{V}_{i-1} + \bar{x}_i \dot{\bar{x}}_i - (\varsigma_{iopt} - \varsigma_i)^T \Gamma_i^{-1} \dot{\theta}_i + \lambda_i^{-1}(\hat{p}_i - p_i)\dot{\hat{p}}_i \\ &= \dot{V}_{i-1} + \bar{x}_i \left[x_{i+1} + H_{i-1} + (\varsigma_{iopt} - \varsigma_i)^T \varphi_i + \varepsilon_i + \Delta_i - \sum_{j=1}^{i} \pi_{i,j} f_j \right] \\ &- (\varsigma_{iopt} - \varsigma_i)^T \Gamma_i^{-1} \dot{\varsigma}_i + \lambda_i^{-1}(\hat{p}_i - p_i)\dot{\hat{p}}_i, \end{aligned} \tag{4.14}$$

where

$$H_{i-1} = \varsigma_i^T \varphi_i - \sum_{j=1}^{i-1} \pi_{i-1,x_j} x_{j+1} - \sum_{j=1}^{i-1} \pi_{i-1,\theta_j} \dot{\varsigma}_j - \sum_{j=1}^{i-1} \pi_{i-1,\hat{p}_j} \dot{\hat{p}}_{j+1}. \tag{4.15}$$

The adaptive fuzzy state feedback control design is based on the following theorem proven by Tong, He, Li and Zhang (2010).

Theorem 3 [Tong, He, Li and Zhang (2010)]. *Given the uncertain unmodeled nonlinear system* (3.1), *if* $\pi_i(x_1,...,x_i,\varsigma_1,...,\varsigma_i,\hat{p}_1,...,\hat{p}_i)$,$\hat{p}_i$ *and* θ_i,$i = 1,2,...,n$, *are defined as*

$$\pi_i = -\bar{x}_{i-1} - c_i\bar{x}_i - H_{i-1} - \varepsilon_{i0} \tanh\frac{\bar{x}_i\varepsilon_{i0}}{k} - \hat{p}_i W_i, \tag{4.16}$$

$$\dot{\hat{p}}_i = \lambda_i \bar{x}_i W_i - \lambda_i \sigma_p (\hat{p}_i - p_{i0}), \quad \hat{p}_i(0) = p_{i0}, \tag{4.17}$$

$$\dot{\varsigma}_i = \Gamma_i \bar{x}_i \varphi_i - \Gamma_i \sigma_\theta (\varsigma_i - \varsigma_{i0}), \quad \varsigma_i(0) = \varsigma_{i0}, \tag{4.18}$$

$$W_i = \bar{x}_i \phi_{i1} + \frac{\bar{x}_i}{4} + \frac{1}{4}\sum_{j=1}^{i-1} \left(\pi_{i-1,x_j}\right)^2 \bar{x}_i, \tag{4.19}$$

$$\phi_{i1} = \frac{1}{4} + \left(||x_i|| \int_0^1 \psi_{i1,s}(s||x_i||\mathrm{d}s \right)^2 + \left(\sum_{j=1}^{i-1} ||\pi_{i-1,x_j}|| \psi_{j1}(||x_j||) \right)^2, \tag{4.20}$$

where $c_i > 0$ *are design parameters,* $\sigma_p > 0$,$\sigma_\theta > 0$ *are known design parameters, kis a given arbitrary constant and* ψ_{i1}*are given by* (3.2).

then (4.13) *are ISpS - Lyapunov functions.*

The robustness of the adaptive fuzzy controller is obtained for a proper choice of $V_0 = \frac{1}{2}\eta\left(x_1^2\right)$ if the conditions of Theorem 1 are verified.

5 Double pendulum: A case study in chaos

In this section, we analyze the double pendulum with driven forces to reveal the chaos control performance of the proposed algorithm. The forced double pendulum is a well-known chaotic system [Chen and Yu (2003); Hsu (2000)]. Fig.5.1 shows a double pendulum subject to periodic non-conservative loads [Munteanu and Donescu (2004)]. For large motions it is a chaotic system, but for small motions it is a simple linear system. This pendulum consists of two straight rods O_1O_2 and O_3O_4 of massesM_1, M_2, lengths $2l_1$, $2l_2$, and mass centers C_1,C_2, respectively. The rods are articulated in O_3 and suspended in O_1, so that they can move in the vertical plane xO_1y without friction. Other notations from Fig.5.1 are $l = O_1O_3$,

$l_1 = O_1C_1$, $l_2 = O_3C_2$. We denote byθ_1, θ_2 the displacement angles with respect to the vertical axis O_1x, I_1 the moment of inertia of O_1O_2 with respect to C_1, I_2 the moment of inertia of O_3O_4 with respect to C_2, and g the gravitational constant. The forces acting upon the pendulum are, firstly, the weights of bars. The generalized forces are

$$G_1 = -M_1l_1g\sin\theta_1 - M_2gl\sin\theta_1, \quad G_2 = -M_2gl_2\sin\theta_2. \tag{5.1}$$

Secondly, we consider the case of a non-conservative force $P\cos\omega t$ along O_3O_4 (Fig.5.1).

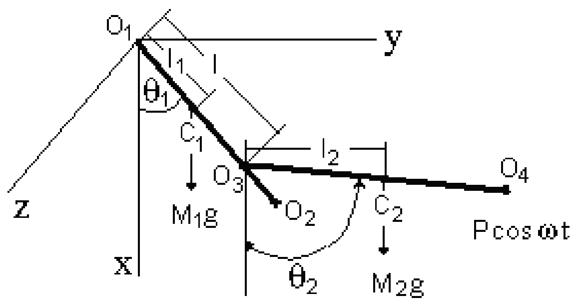

Figure 1: Geometry of the system.

Therefore, the generalized force is

$$Q = Pl\sin(\theta_2 - \theta_1)\cos\omega t. \tag{5.2}$$

The equations of motion obtained from the Lagrange equations, namely,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot\theta_1}\right) - \frac{\partial T}{\partial\theta_1} = G_1 + Q, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot\theta_2}\right) - \frac{\partial T}{\partial\theta_2} = G_2, \tag{5.3}$$

are non-dimensionalized in the form

$$\ddot\theta_1 + \alpha[\ddot\theta_2\cos(\theta_2-\theta_1) - \dot\theta_2^2\sin(\theta_2-\theta_1)] + \beta\sin\theta_1 = R,$$

$$\ddot\theta_2 + \gamma[\ddot\theta_1\cos(\theta_2-\theta_1) + \dot\theta_1^2\sin(\theta_2-\theta_1)] + \sin\theta_2 = 0. \tag{5.4}$$

Here, the dimensionless variables and coefficients are given by

$$t \to t\sqrt{\eta}, \quad \omega \to \frac{\omega}{\sqrt{\eta}}, \quad \eta = \frac{gM_2l_2}{I_2 + l_2^2M_2}, \quad \alpha = \frac{M_2ll_2}{I_1 + l_1^2M_1 + l^2M_2}, \ \gamma = \frac{M_2ll_2}{I_2 + l_2^2M_2},$$

$$\beta = \frac{M_1 l_1 + M_2 l}{I_1 + l_1^2 M_1 + l^2 M_2} \cdot \frac{I_2 + l_2^2 M_2}{M_2 l_2}, \quad \tilde{\mu} = \frac{P}{gM_2}, \ \ \nu = \frac{4}{3}\alpha\tilde{\mu},$$

$$R = \nu \sin(\theta_2 - \theta_1) \cos \omega t. \tag{5.5}$$

A dot denotes differentiation with respect to the dimensionless variable$t$. By setting

$$m = \frac{M_1}{M_2}, \ r = \frac{l}{l_1}, \quad s = \frac{l_2}{l_1}, \tag{5.6}$$

we assume that the motion of the double pendulum with driven forces depends on the unknown parameters r, s, m, $\tilde{\mu}$ and ω. The motion of the double pendulum is studied without any knowledge of the above parameters. It is easy to show that α, β, γ become

$$\alpha = \frac{3rs}{3r^2 + 4m}, \ \beta = \frac{4s(r+m)}{3r^2 + 4m}, \ \gamma = \frac{3r}{4s}. \tag{5.7}$$

By introducing the new variables

$$\theta_1 = x_1, \ \dot{\theta}_1 = x_2, \ \theta_2 = x_3, \ \dot{\theta}_2 = x_4, \quad u = \nu \sin(\theta_2 - \theta_1) \cos \omega t, \tag{5.8}$$

equations (5.4) are rewritten as

$$\dot{x}_1 = x_2 + \Delta_1,$$

$$\dot{x}_2 = f_2(x_1, x_3, u) + \Delta_2,$$

$$\dot{x}_3 = x_4 + \Delta_3,$$

$$\dot{x}_4 = f_4(x_1, x_3, x_4, u) + \Delta_4,$$

$$\dot{z} = -3z + 0.25x_1^2,$$

$$y = x_1,$$

where

$$f_1 = 0,$$

$$\begin{aligned} f_2(x_1, x_3, u) = & \frac{1}{1 - \alpha\gamma\cos^2(x_3 - x_1)} [-\beta \sin x_1 + \alpha \sin x_3 \cos(x_3 - x_1) + \\ & + \alpha x_4^2 \sin(x_3 - x_1) + \alpha\gamma x_2^2 \sin(x_3 - x_1)\cos(x_3 - x_1) - u] \end{aligned}$$

$$f_3 = 0,$$

$$f_4(x_1,x_3,x_4,u) =$$
$$\frac{1}{1-\alpha\gamma\cos^2(x_3-x_1)}[-\sin x_3+\beta\gamma\sin x_1\cos(x_3-x_1)-$$
$$-\gamma x_2^2\sin(x_3-x_1)-\alpha\gamma x_4^2\sin(x_3-z_1)\cos(x_3-x_1)+\gamma u cos(x_3-x_1)],$$
$$\Delta_1=(\sin x_1+z)^2,\quad \Delta_2=x_2 z\sin x_1,\quad \Delta_3=x_3 z\sin x_1,\quad \Delta_4=(\sin x_4+z)^2,$$
$$u=-\frac{4}{3}\alpha\tilde{\mu}\cos\omega t\sin(x_3-x_1),\quad 1-\alpha\gamma\cos^2(x_3-x_1)\neq 0. \tag{5.9}$$

The initial conditions are

$$x_1(0)=x_{10},\ x_2(0)=x_{20},\ x_3(0)=x_{30},\ x_4(0)=x_{40}, \tag{5.10}$$

with the known constants x_{i0}, $i=1,2,...,4$. By choosing

$$\psi_{i1}(s)=\psi_{i2}(s)=\psi_{i3}(s)=s^2,\quad \psi_{i4}(s)=0,\quad i=1,2,\quad V_0=z^2,$$
$$\alpha_0(s)=1.85s^2,\quad \alpha_1(s)=0.2s^2,\quad \alpha_2(s)=2s^2,\quad \gamma_0(s)=s^4,\quad d_0=0, \tag{5.11}$$

conditions (3.2) and (3.3) are satisfied. Therefore, the adaptive fuzzy control scheme proposed by Tong, He, Li and Zhang (2010) can guarantee that closed-loop systems are semi-globally uniformly ultimately bounded. The FLS are

$$\theta_m^T\varphi_m=\sum_{j=1}^{9}\theta_{mj}^T\varphi_{mj},\quad m=2,3,4, \tag{5.12}$$

with the fuzzy functions chosen as

$$\begin{aligned}\varphi_{mj}(x_1,x_2,...x_m)&=\left(\Pi_{i=1}^m\exp\left(-0.25(x_i-0.5j+2)^2\right)\right)\\ &\quad/\left(\Pi_{l=1}^9\Pi_{i=1}^m\exp\left(-0.25(x_i-0.5l+2)^2\right)\right).\end{aligned} \tag{5.13}$$

Regarding the small-gain conditions, we chose

$$\varepsilon_i=0.001,\quad i=1,2,...,4,\quad \beta(s)=s^2,\quad \gamma(s)=\sqrt{s},$$
$$2V_0=\eta(x_1^2)=10^5x_1^8+4x_1^2,\quad \sigma_p=0.2,\quad \sigma_\theta=0.2,\quad \Gamma_2=\Gamma_3=\Gamma_4=I_9.$$

The Lyapunov functions V_i, $i=1,2,...,4$, are ISpS - Lyapunov functions if the conditions of Theorem 3 are satisfied. The adaptive functions and the fuzzy controller are

$$\pi_1=-x_1v_1(x_1^2)-\hat{p}_1\phi_{11}(x_1)x_1\eta',$$

$$\pi_j = -\bar{x}_{j-1} - c_j\bar{x}_j - H_{j-1} - \varepsilon_{j0}\tanh(\bar{x}_j\eta'\varepsilon_{j0}/k) - \hat{p}_j\bar{x}_j\left(\phi_{j1} + 0.25 + 0.25\pi^2_{j-1,x_j}\right), \quad j = 2,3,4,$$

$$\dot{\hat{p}}_1 = \lambda_1 x_1^2 \eta^2_{,x_1}\phi_{11} - \lambda_1\sigma_p(\hat{p}_1 - p_{10}),$$

$$\dot{\hat{p}}_j = \lambda_j\bar{x}_j\left(\bar{x}_j\phi_{j1} + 0.25 + 0.25\pi^2_{j-1,x_j}\right) - \lambda_j\sigma_p(\hat{p}_j - p_{j0}), \quad \hat{p}_j(0) = p_{j0}, \quad j = 2,3,4$$

$$\dot{\theta}_2 = \Gamma_2\bar{x}_2\varphi_2 - \Gamma_2\sigma_\theta(\theta_2 - \theta_{20})\ \dot{\theta}_i = \Gamma_i\bar{x}_i\varphi_i - \Gamma_i\sigma_\theta(\theta_i - \theta_{i0}), \quad \theta_i(0) = \theta_{i0}, \quad i = 3,4,$$

$$u = -\bar{x}_2 - c_4\bar{x}_4 - (H_2 + H_4) - \varepsilon_{20}\tanh(\bar{x}_2\varepsilon_{20}/k) - \varepsilon_{40}\tanh(\bar{x}_4\varepsilon_{40}/k).$$

where H_i are given by (4.15) and $\varepsilon_{j0} = 0.1$, $\lambda_j = 1$, $j = 2,3,4$, $c_i = 2$, $k = 0.01$.

We assess the efficiency of our analysis by computing the representations of the solutions for a forced double-pendulum. Initial conditions are chosen in the interval [−1.5, 1.5], i.e. $\{\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2\} \in [-1.5, 1.5]$. Numerical experiments on the solutions have shown that for $\omega = \tilde{\mu} = 0$ the motion of the pendulum is bounded and stable. For certain values of the parameters $\omega \neq 0$, $\tilde{\mu} \neq 0$ we also obtain bounded motions, but we depict other regions of these parameters for which the corresponding solutions may sudden change to irregular, chaotic type motions. The initial conditions are given by

$$\hat{p}_j(0) = p_{j0} = 1, \quad j = 1,2,3,4, \quad c_i = 2, \quad \theta_{i0} = [0.1,...,0.1]^T, \quad i = 2,3,4.$$

To analyze the effect of the rapidly varying disturbances, let us consider first that

$$\Delta_i(z, x_1, ..., x_n) = 0, \quad i = 1,2,...,n.$$

The first step is to construct the chaotic attractor which has a dense set of unstable periodic trajectories. The chaotic response has a sensitive dependence on the initial conditions, which implies that small perturbations in the initial conditions may dramatically alter the evolution of the motion [Miller (2005)]. The second step is to identify a set of unstable trajectories. Once the desired unstable trajectories to be stabilized are chosen, the control will be initialized to require the pendulum to move towards the equilibrium position.

The resulting phase portraits of $\theta_i, i = 1,2$ and their corresponding time derivatives divided by ω, for $0 \geq t \leq 50\,\text{sec}$. are displayed in Figs. 5.2 and 5.3, respectively.

Fig. 5.4 displays the trajectories of θ_i, $i = 1,2$ for $0 \geq t \leq 50\,\text{sec}$. The trajectory of the control is presented in Fig. 5.5. All the signals in closed-loop system are

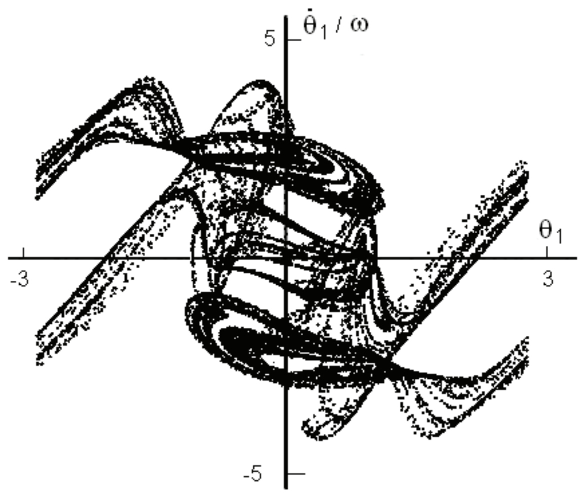

Figure 2: Phase portrait of $\dot{\theta}_1/\omega$ and θ_1.

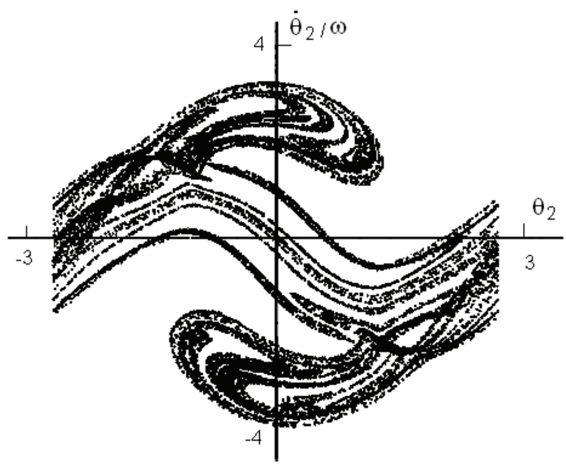

Figure 3: Phase portrait of $\dot{\theta}_2/\omega$ and θ_2.

bounded, and the system output converges to a small neighborhood of the origin eventhough the exact information on the nonlinear functions in the controlled systems is not available.

Next, to make the system chaotic during rapid oscillations, we select

$\Delta_i(z, x_1, ..., x_n) \neq 0, \quad i = 1, 2, ..., n,$

and u as a sinusoidal force of appropriate amplitude and frequency.
The resulting chaotic trajectories of θ_i, $i = 1, 2$, for closed-loop systems is presented in Fig.5.6. The discontinuities of the angular evolutions are due to the fact that the angular values are projected into a short interval of time. From Fig. 5.6 it is clearly visible that the control requires the pendulum to move towards the equilibrium position at about $t = 5$ ($\theta_i \dot{\theta}_i \leq 0$, $i = 1, 2$).

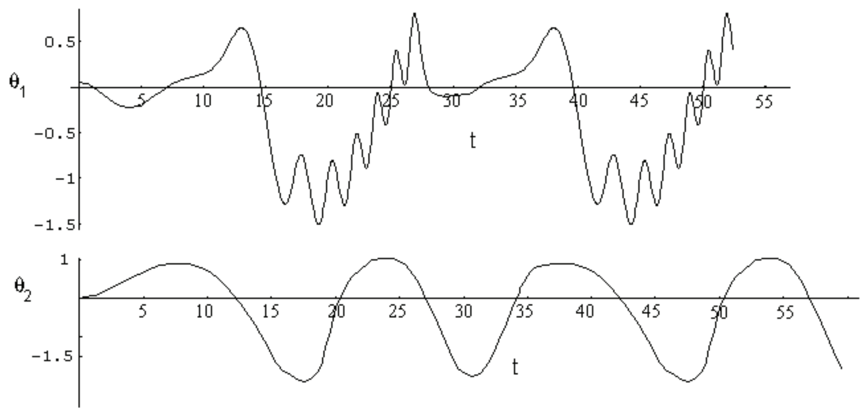

Figure 4: Trajectories of θ_1 and θ_2.

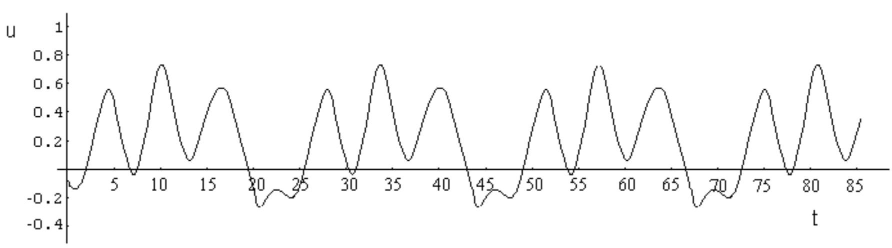

Figure 5: Trajectory of the control u.

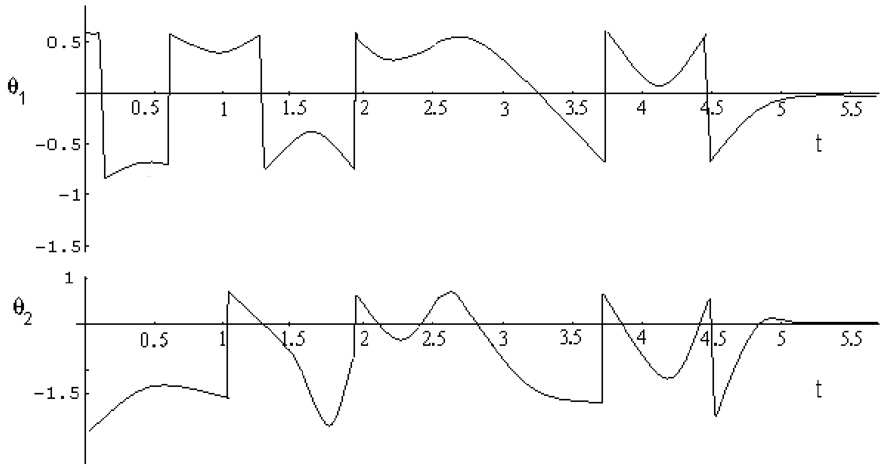

Figure 6: Trajectories of θ_1 and $\theta_2$under the action of the rapidly varying disturbances.

6 Conclusions

The chaos control presented in this paper further extends the range of applications of the adaptive robust control method (Tong, He, Li and Zhang (2010)) to systems with various nonlinear uncertainties, unmodeled dynamic law and rapidly varying disturbances. The model of a stable adaptive controller is obtained by combining the backstepping and small-gain approaches. The method was used to control the chaotic motion of the double pendulum without knowledge of the parameters. Once the desired unstable trajectories to be stabilized are chosen, the control will be initialized to require the pendulum to move towards the equilibrium position.

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References

Bernardo, M.; Stoten, D.P. (2006): Minimal control synthesis adaptive control of nonlinear systems: utilizing the properties of chaos, *Phil. Trans. R. Soc.* A, 364: 2397–2415.

Boulkroune, A.; Tadjine, M.; Saad, M.; Farza, M. (2008): How to design a fuzzy adaptive controller based on observers for uncertain affine nonlinear systems, *Fuzzy Sets and Systems* 159: 926–948.

Chen, B.; Liu, X.P.; Tong, S.C. (2007): Fuzzy approximate disturbance decoupling of MIMO nonlinear systems by backstepping approach, *Fuzzy Sets and Systems* 158: 1097–1125.

Chen, B., Liu, X.P.; Shi, P. (2009): Comment on adaptive fuzzy $H\infty$ stabilization for strict-feedback canonical nonlinear systems via back stepping and small-gain approach, *IEEE Transactions on Fuzzy Systems* 17: 732–733.

Chen, G.; Yu, X. (eds) (2003): *Chaos control: theory and applications*, Berlin, Germany: Springer.

Hsu, T.H.(2000): *Control of a chaotic double pendulum model for a ship mounted*, PhD thesis, Virginia Polytechnic Institute and State University.

Isidori, A. (1995): *Nonlinear Control Systems*, 3^{rd} edition Springer, Berlin, Germany,

Jiang, Z.P.; Teel, A.; Praly, L. (1994): Small-gain theorem for ISS systems and applications, *Math. Contr., Sig. Syst.* 7: 95–120.

Jiang, Z.P.; Marels, I.M.Y.; Wang, Y. (1996): A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems, *Automatica* 32: 1211–1215.

Jiang, Z.P.; Mareels, I.M.Y. (1997): A small gain control method for nonlinear cascaded systemns with dynamic uncertainties', *Automat. Contr.*, AC-42: 292–308.

Jiang, Z.P.; Praly, L. (1998): Design of robust adaptive controllers for nonlinear systems with dynamic uncertainties, *Automatica* 34: 825–840.

Jiang, Z.P. (1999): A combined backstepping and small-gain approach to adaptive output feedback control, *Automatica* 35:1131–1139.

Junkins, J.L.; Subbarao, K.; Verma, A. (2000): Structured adaptive control for poorly modeled nonlinear dynamical systems, CMES: Computer Modeling in Engineering & Sciences, 1(4): 99–118.

Khalil, H.K. (1996): *Nonlinear Systems*, Macmillan, New York.

Kanellakopopoulos, I.; Kokotovic, P.V.; Morse, A.S. (1991): Systematic design of adaptive controllers for feedback linearizable systems, *IEEE Transactions on Automatic Control* 36: 1141–1253.

Kristic, M.; Kanellakopoulos, I.; Kokotovic, P.V. (1995): *Nonlinear and Adaptive Control Design*, Wiley, NewYork.

Kristic, M; Kanellakopoulos, I.; Kokotovic, P.V. (1992): Adaptive nonlinear control without over parametrization, *System and Control Letters* 19: 177–185.

Li, T.S.; Hong, B.G.; Shi, G.Y. (2008): DSC-backstepping based robust adaptive NN control for strict-feedback nonlinear systems via small gain theorem, *Int. J. Systems, Control and Communications,* **1**(1): 124–145.

Lin, T.C. (2010): Based on interval type-2 fuzzy-neural network direct adaptive sliding mode control for SISO nonlinear systems, *Commun Nonlinear Sci Numer Simulat.*

Lin, S.M. (2009): Vibration suppression of a moving beam subjected to an active-control electrostatic force, *CMES: Computer Modeling in Engineering & Sciences*, 43(1): 73–90.

Golea, N.; Golea, A.; Benmahammed, K. (2003): Stable indirect fuzzy adaptive control, *Fuzzy Sets and Systems* 127: 353–366.

Miller, C. (2005): Analysis of the double pendulum, *Math 5360* Texas State University – San marcos, TX, 1–11.

Munteanu, L.; Donescu, St. (2004): *Introduction to Soliton Theory: Applications to Mechanics",* Book Series "Fundamental Theories of Physics", 143, Kluwer Academic Publishers.

Parussini, L.; Pediroda, V. (2008): Investigation of multi geometric uncertainties by different polynomial chaos methodologies using a fistitious domain solver, *CMES: Computer Modeling in Engineering & Sciences* 23(1): 29–52.

Pereira-Pinto, F.H.I.; Ferreira, A.M.; Savi, M.A. (2004): Chaos control in a nonlineat pendulum using a semi-continuous method, *Chaos, Solitons and Fractals,* 22: 653–668.

Praly, L.; Wang, Y. (1996): Stabilization in spite of matched unmodeled dynamics and an equivalent definition of input-to-state stability, *Math. Contr. Sig. Syst.*, 9: 1–33.

Qu, Z. (1995): Robust control of nonlinear uncertain systems without generalized matching conditions, *IEEE Transactionson Automatic Control* 40: 1453–1460.

Sontag, E.D. (1989): Smooth stabilization implies coprime factorization, *Automat. Contr.*, 34: 1411–1428.

Sontag, E.D. (1990): Further results about input-state stabilization, *IEEE Trans. Automat.Contr.*, 35: 473–476.

Sontag, E.D.; Wang, Y. (1995): On characterizations of input-to-state stability property with respect to compact sets, *Proc. Prep. IFAC NOLCOS*, Tahoe City, 226–231.

Tong, S.C.; Li, Y.; Shi, P. (2009): Fuzzy adaptive backstepping robust control for SISO nonlinear systems with dynamic uncertainties, *Information Sciences* 179: 1319–1332.

Tong, S.C.; He, X.; Li, Y.; Zhang, H. (2010): Adaptive fuzzy backstepping robust control for uncertain nonlinear systems based on small-gain approach, *Fuzzy Sets and Systems* 161: 771–796.

Ursu, I.; Toader, A.; Tecuceanu, G. (2009): Control of uncertain systems by feedback linearization with neural networks augmentation. Part I. Controller design, INCAS Bulletin, 1: 84–89.

Ursu, I.; Toader, A.; Tecuceanu, G. (2010): Adaptive control of uncertain systems: A unitary approach, *Proceedings of the Romanian Academy*, *Series A*, 11(3): 236–244.

Wang, L.X. (1994): *Adaptive Fuzzy Systems and Control:Design and Stability Analysis*, Prentice-Hall, Englewood Cliffs, NJ.