 CMES, vol.75, no.1, pp.33-42, 2011

Generalized Method Based on Nodal and Mesh Analysis for Computation of Time Constants of Linear Circuits

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Abstract: The generalized method for determination of time constants of linear circuits is introduced. Nodal and mesh analysis, conventional methods whose applications are simpler than the state-space formulation, are used in obtaining the system equations. The approach is based on the use of the relationship between transfer functions and system equations of linear circuits, obtained by the conventional methods. The examples of active and passive circuits are given to illustrate the method.

Keywords: Time-constant, nodal analysis, mesh analysis, linear electrical circuit

1 Introduction

The time-constant is one of the important parameters in circuit analysis applications. It refers to the transient response of circuits and determines the duration of the transient-state. The time-constants are obtained from the eigenvalues, referred to as poles of transfer functions, of circuits. Determination of time-constants provides valuable information for circuit designers. In engineering applications [Shu and Johnson (1988); Paul and Huper (1994); Lee and Liu (2001); Sodagar (2007); Lindberg et al. (2009); Shiyu, Zhinong and Xinghuo (2009)], the time constants, indirectly eigenvalues, are particularly useful in the design of feedback systems in which relative stability, dynamics and other complete response characteristics are important functions.

The difficulty of determining the time-constants arises from obtaining the system equations. In general, the time constants of circuits are obtained from the state-space formulation, a form of differential equations, in circuit analysis [Thomas and Rosa (2006), Nilsson and Riedel (2000), Vlach and Singhal (1983), Yildiz (2006)]. But, this method has some restrictions in obtaining the system equations, as will be explained in Section II. Moreover, it is not very suitable for computer applications.

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In this paper, it is shown how the time constants can be computed according to nodal and mesh analysis, in which the system equations are easily obtained. Nodal and mesh analysis provide a systematic framework, a form of algebraic equations, to apply Kirchhoff's current and voltage laws to the circuit problem.

The time-constants relate to the eigenvalues in time-domain and the poles of transfer functions in s-domain. Therefore, another method used to determine the time-constants is based on obtaining the poles of transfer functions relating to any circuit. Cochran and Grabel (1973) calculated the time-constants associated with each reactive element under different combinations of shorting and opening of other reactive elements in the circuit. A method for determination of transfer functions of lumped RC circuits using a combination of the time-constants and low frequency transfer functions under different combinations of shorting and opening of the capacitors is developed in [Davis and Moustakas (1980)]. The transfer function of a first-order system is determined using the extra element theorem presented in late 1980s by Middlebrook (1989). The approach was generalized to N extra elements in the 1990s in [Middlebrook, Vorperian and Lindal (1998)].

Haley (1988) introduced a modification-decomposition (MD) method to compute linear system transfer function poles and zeros. A method for estimation poles and zeroes of linear active circuit transfer functions is described in [Haley and Hurst (1989)]. Hauksdottir and Hjaltadottir (2003) gave closed-form expressions, for real and/or complex eigenvalues, of transfer function responses. Hagiwara (1992) used the eigenvalue approach to calculate the zeros of the system. In [Hajimiri (2010)], the transfer functions of circuits are expressed in terms of time and transfer constants calculated under different combinations of shorted and opened energy-storing elements.

The main contribution of the paper is that it gives a systematic and generalized method to compute the time-constants of linear circuits according to nodal and mesh analysis. The rest of the paper is organized as follows: In Section 2, the structure of the proposed method and system equations are explained. In Section 3, illustrative examples of the approach are given. Section 4 is the conclusion.

2 Description of the Method

In this paper, a generalized method for computation of time-constants is proposed according to the relationship between transfer functions and system equations obtained by nodal and mesh analysis.

2.1 Transfer Functions

For a single-input single-output linear time-invariant (LTI) circuit, the transfer function is defined as the ratio of the voltages and/or currents of any two arbitrary ports of the circuit, including the ratio of the voltage and current of the same port, in s-domain. We designate the input and output variables as U(s) and Y(s). For example, when U(s) is an input voltage due to a voltage source and Y(s) is the voltage of another node in the circuit, the transfer function, $H(s)$, would correspond to a voltage gain. It is called the voltage transfer function. On the other hand, if the input variable is the voltage and the output variable is the current through any element in the circuit, the transfer function is called the transfer admittance. Another type of the transfer function is the driving point impedance or input impedance. It is related to the voltage and the current at the same port.

The transfer function of a LTI circuit can be written as

$$H(s) = \frac{Y(s)}{U(s)} \tag{1}$$

The numerator and denominator of the transfer function in (1) have polynomial form. The roots of the denominator give the poles of the transfer function in s-domain or the eigenvalues of the circuit in t-domain. The roots of the numerator give the zeroes of the transfer function. The order of the denominator, n, determines the number of natural frequencies, poles, of a LTI circuit and is equal to the number of independent energy storage elements. Natural frequencies, poles, or eigenvalues are independent of the choice of the input and output variables and are intrinsic characteristics of the circuit. Therefore, all transfer functions relating to a LTI circuit have the same poles, eigenvalues. On the contrary, the zeroes of the transfer function depend on the choice of the input and output.

The eigenvalues, indirectly poles, determine the stability of circuits. The stability concept is necessary for the dynamics of circuits. For stability, all eigenvalues of circuits must be located on the left hand of the complex s plane. Knowing the eigenvalues of the transfer function of a LTI circuit, we can predict its dynamics and transient response. Especially, the eigenvalues are very important to determine the time-constants, the main purpose of the paper. The time-constants are obtained from the inverse of the eigenvalues. The duration of the transient-state depends on the max. time constant (τ_{max}) : $5\tau_{max}$.

2.2 System Equations

In general, the system equations used for determining the time-constants are expressed according to the state-space formulation. Although the state-space method, based on the graph theoretical approach, has min. variables, it involves intensive mathematical process and has major limitations in the formulation of circuit equations. Some of these limitations arise because the state variables are capacitor voltages and inductor currents. Every circuit element cannot be easily included into the state equations. It has a structure of differential equations. Especially, there are some restrictions in the analysis of active circuits. It is not suitable to use always this method for obtaining both time-constants and transfer functions. In this study, it is shown that the time-constants can be easily computed according to nodal and mesh equations in s-domain, more efficient and systematic methods in circuit analysis applications. It has a structure of algebraic equations. They have no restrictions in the formulation of circuit equations. In [Chatzarakis, Tortoreli and Tziolas (2003), Chatzarakis and Tortoreli (2004)], nodal and mesh analysis methods with virtual sources for some special cases in circuit analysis are used.

The system and the output equations in s-domain, obtained by using nodal or mesh analysis method, relating to a LTI circuit are

The system equations:

$$AX(s) = BU(s) \tag{2}$$

The output equations:

$$Y(s) = CX(s) + DU(s) \tag{3}$$

Where, A, B, C, D are coefficient matrices, U(s) is the input vector, the voltage or current source, X(s) is the unknown vector, Y(s) is the output vector. The frequency-dependent elements (inductor, capacitor) can be entered in the form having s (sL, sC) or 1/s (1/sL, 1/sC) into the system equations. Matrix A is also called the characteristic matrix. Solutions of the system equations and the output equations are given in (4) and (5), respectively.

$$X(s) = A^{-1}BU(s) = \left(\frac{1}{\det(A)}Adj(A)\right)BU(s) \tag{4}$$

$$Y(s) = CA^{-1}BU(s) + DU(s) = \left[CA^{-1}B + D\right]U(s)$$

$$Y(s) = \left[C\left(\frac{1}{\det(A)}Adj(A)\right)B + D\right] = BU(s) \tag{5}$$

It is obvious that solutions of (4) and (5) are fractional. The determinant of the characteristic matrix (A) has also fractional and polynomial form as in (6).

$$\det(A) = \frac{Q(s)}{R(s)} \tag{6}$$

The determinant expression in (6) is substituted into (4) and (5).

$$X(s) = \left(\frac{1}{\frac{Q(s)}{R(s)}} Adj(A) \right) BU(s)$$

$$X(s) = \frac{Adj(A) * B * R(s)}{Q(s)} U(s) = \frac{M(s)}{Q(s)} U(s) \tag{7}$$

$$Y(s) = \left[C * \frac{1}{\frac{Q(s)}{R(s)}} Adj(A) * B + D \right] U(s)$$

$$Y(s) = \frac{N(s)}{Q(s)} U(s) \tag{8}$$

Where,

$$N(s) = C * Adj(A) * B * R(s) + D * Q(s) = C * M(s) + D * Q(s)$$

The transfer functions of the system are obtained from (9). All transfer functions relating to a LTI circuit have the same denominator, Q(s).

$$H(s) = \frac{Y(s)}{U(s)} = \frac{N(s)}{Q(s)} = \frac{C * M(s) + D * Q(s)}{Q(s)} \tag{9}$$

The numerator of determinant of the coefficient matrix (A) in (6) is the same as the denominator of the transfer functions in (9), indirectly the denominator of the output vector in (8). In the paper, this relationship is used to compute the time-constants. Q(s) polynomial is also called the characteristic equation in circuit analysis applications.

In this study, the system equations of a LTI circuit are obtained algebraically first by nodal or mesh analysis in s-domain. The transfer function, H(s), is expressed for any desired output variable. The characteristic equation is determined in terms of the numerator of determinant of the coefficient matrix (A) relating to system equations. The eigenvalues, indirectly poles, of circuit are obtained from the roots of the characteristic polynomial, seen from both the transfer function and the determinant of the coefficient matrix. Later, the time constants are computed by taking inverse of the eigenvalues.

3 Illustrative Examples

Two examples are given in this section to present the method. The system equations and the time constants are obtained by mesh analysis in the first example and nodal analysis in the second example.

Example 1: Consider the magnetic coupling circuit shown in Fig.1. Element values are R_1=4Ω, R_2=5Ω, L_1=4H, L_2=2H, M=0.5H, C=0.1F. The output variable is I_o current.

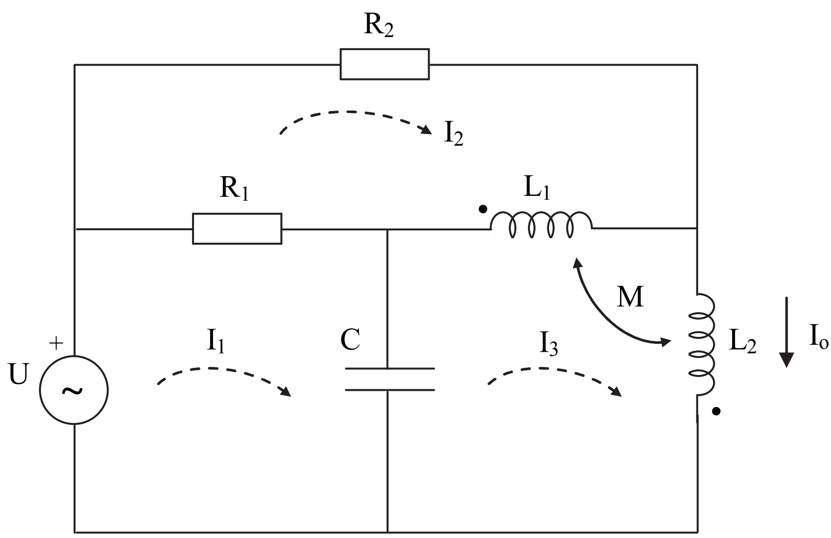

Figure 1: Circuit for Example 1

Mesh equations: $AX(s) = BU(s)$

Output equations: $Y(s) = I_o(s) = CX(s) + DU(s)$

Where,

$$A = \begin{bmatrix} R_1 + \frac{1}{sC} & -R_1 & -\frac{1}{sC} \\ -R_1 & R_1 + R_2 + sL_1 & -sL_1 + sM \\ -\frac{1}{sC} & -sL_1 + sM & sL_1 + sL_2 + \frac{1}{sC} - 2sM \end{bmatrix}$$

$$X(s) = \begin{bmatrix} I_1(s) \\ I_2(s) \\ I_3(s) \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad D = [0]$$

After substituting the element values into the system equations, the determinant of the coefficient matrix (A) is obtained as follows.

$$\det(A) = \frac{Q(s)}{R(s)} = \frac{14s^3 + 115s^2 + 360s + 400}{2s} \quad (10)$$

The characteristic equation: $Q(s) = 14s^3 + 115s^2 + 360s + 400$

After solving the system equations, the transfer function for the desired output is obtained as follows.

$$H(s) = \frac{I_o(s)}{U(s)} = \frac{I_3(s)}{U(s)}$$

$$H(s) = \frac{4s^2 + 20s + 180}{14s^3 + 115s^2 + 360s + 400} \quad (11)$$

It can be easily seen that the numerator of determinant of the coefficient matrix, called the characteristic equation (Q(s)), is the same as the denominator of the transfer function. The roots of characteristic equation give the eigenvalues (poles) of the third order circuit: $\alpha_1 \cong -2.85 + 1.8i$, $\alpha_2 \cong -2.85 - 1.8i$, $\alpha_3 = -2.5$. The circuit is stable because all eigenvalues are located on the left hand of the complex s plane. Time constants are computed by taking inverse of real part of the eigenvalues.

Time constants:

$$\tau_1 = \frac{1}{|-2.85|} \cong 0.35s, \quad \tau_2 = \frac{1}{|-2.5|} = 0.4s \quad (= \tau_{\max})$$

The duration of the transient-state depends on the max. time constant: $5\tau_{\max} = 2s$

Example 2: Consider the Op−Amp circuit shown in Fig.2. R_1=1Ω, R_2=1Ω, R_a=1Ω, R_b=1Ω, C_1=2F, C_2=2F. The voltage and current constraints of ideal Op−Amp are I_p=0, I_n=0, $U_c - U_d$=0. The input terminals of Op-Amp are simultaneously short circuit and open circuit. It is an interesting property of Op-Amp. The output variable is U_o voltage.

Nodal equations: $AX(s) = BU(s)$

Output equations: $Y(s) = U_o(s) = CX(s) + DU(s)$

Where,

$$A = \begin{bmatrix} G_1 & -G_1 & 0 & 0 & 0 & 1 \\ -G_1 & G_1 + G_2 + sC_1 & -G_2 & 0 & -sC_1 & 0 \\ 0 & -G_2 & G_2 + sC_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & G_a + G_b & -G_a & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

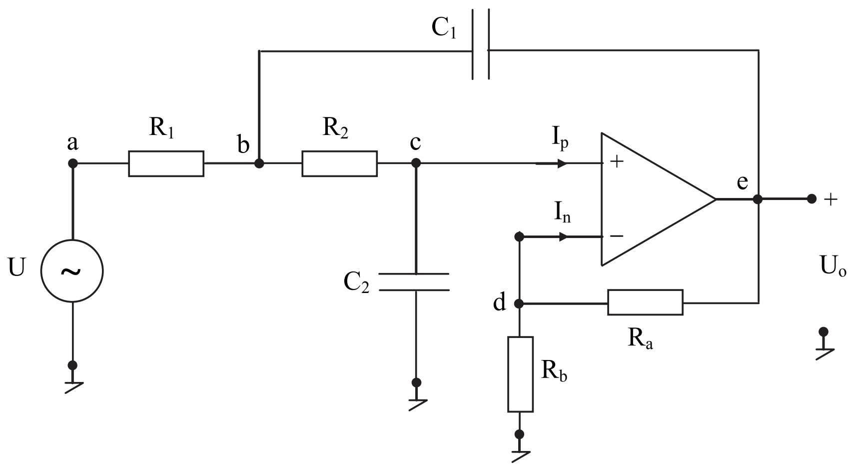

Figure 2: Circuit for Example 2

$$X(s) = \begin{bmatrix} U_a \\ U_b \\ U_c \\ U_d \\ U_e \\ I_u \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad D = [0]$$

After substituting the element values into the system equations, the determinant of the coefficient matrix (A) is obtained as follows.

$$\det(A) = \frac{Q(s)}{R(s)} = \frac{4s^2+2s+1}{1} \tag{12}$$

The characteristic equation: $Q(s) = 4s^2+2s+1$

After solving the system equations, the transfer function for the desired output is obtained as follows.

$$H(s) = \frac{U_o(s)}{U(s)} = \frac{U_e(s)}{U(s)} = \frac{2}{4s^2+2s+1} \tag{13}$$

The numerator of determinant of the coefficient matrix, (Q(s)), is equal to the denominator of the transfer function. The roots of characteristic equation give the eigenvalues (poles) of the second order circuit: $\alpha_1 = -0.25+0.433i$, $\alpha_2 = -0.25-0.433i$. The circuit is stable because all eigenvalues are located on the left hand of the complex s plane.

Time constant:

$$\tau = \frac{1}{|-0.25|} = 4s$$

The duration of the transient-state is $5\tau = 20s$.

4 Conclusions

A general and systematic method for computation of time-constants of linear circuits has been described. In general, it is not suitable to determine the time-constants from the state-space formulation because it has some restrictions in obtaining the system equations. In this paper, it is shown how to compute the time-constants by nodal and mesh analysis. The conventional methods have no limitations in the formulation of the system equations. The approach has been developed according to the relationship between the transfer function and the characteristic equation created by nodal and mesh analysis. The examples present the efficiency of the proposed method.

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