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# Investigation on the Singularities of Some Singular Integrals

**Zai You Yan**[1] **and Qiang Zhang**[1]

**Abstract:** In a boundary element method, the treatment of all the possible singular integrals is very important for the correctness and accuracy of the solutions. Generally, the directional derivative of a weakly singular integral is computed by an integral in the sense of Cauchy principal value if the directional derivative of the weakly singular integral kernel is strongly singular or in the sense of Hadamard finite part integral if it is hypersingular. In this paper, we try to discover how the strongly singular and hypersingular integrals are generated and propose an idea to avoid the appearance of such kind of strongly singular and hypersingular integrals. This idea is termed as the 'exact derivation' of the directional derivative of a weakly singular integral. Using some simple examples, we proof that the directional derivative of a weakly singular integral found by this idea can still be a weakly singular integral. That is none strongly or hypersingular integrals are generated in such a process. Therefore, Cauchy principal value and Hadamard finite part integral are not indispensable.

**Keywords:** weakly singular integral, strongly singular integral, hypersingular integral, Cauchy principal value, Hadamard finite part, improper integral

## 1 Introduction

To date, the boundary element method (BEM)[Banerjee (1994)] is a very popular numerical approach. It can be applied in many scientific and engineering fields. It has the merit that meshes need only to be generated on the boundary and the integral form will have better accuracy than the differential form. However, there are two main difficulties to generate a high efficiency and high accuracy BEM program.

One difficulty is due to the drawback that the final coefficient matrices in a conventional BEM are dense. Therefore, a great deal of computer memory and computational time is required to run a conventional BEM program. Fortunately, in the

[1] Department of Aerodynamics, Nanjing University of Aeronautics and Astronautics, Yu Dao Street 29, City of Nan Jing, Jiang Su Province, P. R.China, 210016. Email: jutsjtu@yahoo.com.cn

past few years, several fast BEMs had been developed to overcome this problem. Among these fast BEMs, the fast multipole expansion method (FMM) [Greengard & Rokhlin (1987), Liu & Nishimura (2006)] and the pre-corrected fast Fourier transform method (pFFT) [Phillips & White (1997), Ding & Ye (2004), Yan, Zhang, Ye & Yu (2010), Yan, Zhang & Ye (2010), Yan (2010)] are well-known and widely applied. Generally, the constant element is employed in such kinds of fast BEMs. Recently, Yan & Liu (2010) presented a pFFT method with higher order boundary elements. Their extension to the pFFT approach should be very important for the broad application of this method.

The other difficulty is the computation of the weakly singular, nearly singular, strongly singular and hypersingular integrals [De Klerk (2005)] appearing in the various kinds of boundary integral equations. To date, a lot of research has been done on the treatment of such kinds of integrals, especially the hypersingular integrals [Tanaka, Sladek & Sladek (1994), Chen & Hong (1999)]. One of the well-known hypersingular integrals is that occurring in the composite Helmholtz integral equation proposed by Burton & Miller (1971) for the exterior acoustic problems. In 2003, Yan, Hung & Zheng investigated the fast computation of this hypersingular integral using a regularization relationship. For the boundary integral equations in elasticity, hypersingular integrals may appear in the differentiation of the displacement boundary integral equations. To obviate the appearance of such kinds of hypersingular integrals, Okada, Rajiyah, & Atluri (1988, 1989, 1990, 1994) proposed some novel non-hypersingular boundary integral equations for velocity or displacement gradients that were directly derived from a weak form of the momentum balance equations. After that, Han & Atluri (2003) presented some weakly singular traction boundary integral equations for solids undergoing small deformations and then they presented a systematic derivation of the weakly singular boundary integral equations in 2007. The idea of directly deriving the gradient of the boundary integral equations was extended to acoustic problems by Qian, Han & Atluri (2004). In recent years, there are still many works on evaluation of various kinds of singular integrals. For example, Gao, Yang & Wang (2008) presented an algorithm for the evaluation of some weakly, strongly and hypersingular integrals in 2D problems by a semi-analytical method.

There are several books on the subject of the singular integrals. The book compiled by Sladek V. & Sladek J. (1998) presents many approaches to treat the singular, strongly singular and hypersingular integrals in the boundary element methods developed before 1998. The book written by Lifanov, Vainikko, Poltavskii & Vainikko (2004) focuses on the hypersingular integrals and their applications.

Generally, the so-called strongly and hypersingular boundary integrals are created from the derivation of a directional derivative of a boundary integral with weakly singular integral kernels. In the past, nearly all the researchers derived the directional derivative of a boundary integral with a weakly integral kernel by finding the limit of the directional derivative from the domain to its boundary. That is to find the directional derivative in the domain firstly and then to find its limit to the boundary of the domain. They said that on the boundary the differential symbol could not be taken into the integral symbol directly due to the weakly singular integral kernel [Gray, Glaeser & Kaplan (2004)]. Therefore, few researchers had investigated how to take the differential symbol into the integral symbol just on the boundary directly.

While we doubt the above reason and insist that there must have some approaches to take the differential symbol into the integral symbol just on the boundary directly, because it is well known that the Dirac Delta function $\delta$ is derivable even though it is singular. Besides, since a weakly singular integral is actually integrable, if we find the boundary integral first then the result should be derivable provided that the derivative is a finite value (That is to exclude the cases such as $d\sqrt{x}/dx$ at $x = 0$). In this process, no limit to the boundary is applied yet. We know that if the numerator is a finite value then in the common sense the denominator cannot be a zero (except the generalized function Dirac Delta). Otherwise an unlimited value $\infty$ will appear. For a weakly singular boundary integral, no $\infty$ will appear in the integral kerkel because it is actually integrable. For example, for the weakly singular integral $\int_0^1 \ln x dx$, the integral kernel $\ln x \to \infty$ as $x \to 0$. However, with a variable substitution $x = t^2$, we have $\int_0^1 \ln x dx = \lim_{\varepsilon\to 0} \int_\varepsilon^1 \ln x dx = \lim_{\varepsilon\to 0} \int_\varepsilon^1 \ln t^2 dt^2 = 4\lim_{\varepsilon\to 0} \int_\varepsilon^1 t \ln t dt = -1$. Obviously, as $t \to 0$ the integral kernel $t\ln t$ already limits to 0 too. That is the integral is none singular after a variable transformation and a weakly singular integral is integrable in the sense of improper integral. In the improper integral, the limit is taken to the singular point along the integral boundary rather than the limit to the boundary technique usually applied in the treatments of hypersingular integrals. While for strongly singular integrals and hypersingular integerals, $\infty$ must appear in the integral kernel if none regularizations are implemented. Therefore we tend to investigate the problem of how to take the differential symbol into the integral symbol for the directional derivative of a weakly singular integral in the sense of the generalized functions, improper integral and generalized derivatives [Franssens (2009)].

In this paper, we will derive the directional derivatives of several boundary integrals with weakly singular kernels to see how to avoid the so-called strongly singular integral and hypersingular integrals. The idea is different from that proposed by Okada, Rajiyah, & Atluri (1988, 1989) for the derivation of the direct boundary integral equation for displacement gradients. They avoided finding the directional derivatives directly by deriving the boundary integral equation from a weak form of the momentum balance equations. However, we will just focus on the direct derivation of the directional derivatives of the boundary integral equations.

## 2 Cauchy principal integral and Hadamard finite part integral

Usually, a strongly singular integral should be integrated in the sense of a Cauchy principal value (*CPV*) integral. For instance, for a one dimensional strongly singular boundary integral $\int_a^b f(x)(x-y)^{-1}dx, \quad a<y<b$, its *CPV* [Martin & Rizzo (1996), Monegota (2009)] is defined as,

$$CPV\int_a^b \frac{f(x)}{x-y}dx = \lim_{\varepsilon\to 0^+}\left[\int_a^{y-\varepsilon}\frac{f(x)}{x-y}dx+\int_{y+\varepsilon}^b\frac{f(x)}{x-y}dx\right] \tag{1}$$

It can also be expressed by the form of the directional derivative of a weakly singular integral as [Carley (2007), Monegota (2009)],

$$CPV\int_a^b \frac{f(x)}{x-y}dx = -\frac{\partial}{\partial y}\int_a^b \ln|x-y|\, f(x)dx, \quad a<y<b \tag{2}$$

This expression tells us that the direcitonal derivative $\frac{\partial}{\partial y}\int_a^b \ln|x-y|\, f(x)dx$ as $a<y<b$ should be computed by the *CPV* of a strongly singular integral, $-CPV\int_a^b \frac{f(x)}{x-y}dx$. However, a hypersingular boundary integral should be generally integrated in the sense of a Hadamard finite part (*FP*) integral [Martin & Rizzo (1996), Monegota (2009)]. For example, for a one dimensional hypersingular boundary integral $\int_a^b f(x)(x-y)^{-2}dx$, $a<y<b$, its *FP* is defined as [Martin & Rizzo (1996), Monegota (2009)],

$$FP\int_a^b \frac{f(x)}{(x-y)^2}dx \quad = \lim_{\varepsilon\to 0}\left[\int_a^{y-\varepsilon}\frac{f(x)}{(x-y)^2}dx+\int_{y+\varepsilon}^b\frac{f(x)}{(x-y)^2}dx-\frac{2f(y)}{\varepsilon}\right] \tag{3}$$

It can also be expressed by the form of a directional derivative of the *CPV* of a strongly singular integral as [Carley (2007), Monegota (2009)],

$$FP\int_a^b \frac{f(x)}{(x-y)^2}dx = -\frac{\partial}{\partial y}\left[CPV\int_a^b \frac{f(x)}{(x-y)}dx\right], \quad a<y<b \tag{4}$$

Substituting the expression (2) into the above formulation, then the above *FP* integral can be further expressed by the form of a second-order directional derivative of a weakly singular boundary integral as,

$$FP\int_a^b \frac{f(x)}{(x-y)^2}dx = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\int_a^b \ln|x-y| f(x)dx\right]$$
$$= \frac{\partial^2}{\partial y^2}\int_a^b \ln|x-y| f(x)dx, \quad a<y<b \quad (5)$$

Monegota (2009) also presented the finite part expressions for boundary integrals with supersingular integral kernels. However, in this paper only the boundary integrals with at most hypersingular integral kernels will be concerned, because supersingular boundary integrals seldom appear in the practical boundary element applications.

From the above description, we can see that a weakly singular integral is integrable in the normal sense; A strongly singular integral should be integrated in the sense of the *CPV*; And a hypersingular integral should be integrated in the sense of Hadamard *FP* integral.

Obviously, both of the integrals $\int_a^b f(x)(x-y)^{-1}dx$ in the formulation (2) and $\int_a^b f(x)(x-y)^{-2}dx$ in the formulation (5) will have at least strongly singular integral kernels. That is when the numerators of the integral kernels are non-zero, the corresponding denominators will sometimes be zero and the final integral does not exist in the normal sense. This is just the reason why the *CPV* and the *FP* were proposed.

However, we know a fact that the denominator cannot be a zero when the numerator is a finite value (Dirac Delta function is an exception). Then we wonder whether the integral forms, such as the strongly singular integral $\int_a^b f(x)(x-y)^{-1}dx$ and the hypersingular integral $\int_a^b f(x)(x-y)^{-2}dx$ exist indeed. If they do not exist, then how are they created and can we avoid the appearance of such kind of singular integrals? In the following, we will try to answer this question through some simple derivations. It should be pointed out that we do not want to deny the importance of the CPV and the Hadamard FP and we just try to find a direct means to avoid the appearance of such kinds of strongly singular and hypersingular integrals.

## 3 The weakly singular forms of the expressions on the right hand side of the formulations (2) and (5)

Now let us consider how to take the differential symbols on the right hand side of the formulation (2)

$$\frac{\partial}{\partial y}\int_a^b \ln|x-y| f(x)dx, \quad a<y<b$$

and (5)

$$\frac{\partial^2}{\partial y^2}\int_a^b \ln|x-y|\, f(x)dx, \quad a<y<b$$

into the integral symbols directly. That is how to exchange the differential symbol and the integral symbol as the integral kernel is weakly singular.

In 1992, Guiggiani, Krishnasamy, Rudolphi & Rizzo proposed a method to compute the hypersingular integrals directly without using the Hadamard finite part integral. The main difference between the present method and their approach is that in the present method, all the processes of differential and integral are carried out on the boundary only and no limit to the boundary is needed.

Now, let us derive the following three directional derivatives of three weakly singular integrals respectively to show how to take the differential symbols into the integral symbols directly. Cases (b) and (c) are just the objectives in this session.

(a)

$$\frac{\partial}{\partial x_0}\int_{x_1}^{x_2} \frac{1}{\sqrt{|x-x_0|}}dx, \quad x_1<x_0<x_2 \tag{6}$$

(b)

$$\frac{\partial}{\partial y}\int_a^b \ln|x-y|\, f(x)dx, \quad a<y<b \tag{7}$$

(c)

$$\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\int_a^b \ln|x-y|\, f(x)dx\right], \quad a<y<b \tag{8}$$

### *3.1 Derivation of the formulation (6)*

Generally, this form of integral should be computed by a strongly singular integral in the sense of *CPV*. In the following, we will derive this formulation regardless of the *CPV*.

For convenience, we set,

$$f(x_0) = \int_{x_1}^{x_2} \frac{1}{\sqrt{|x-x_0|}}dx, \quad x_1<x_0<x_2.$$

Using the definition of the improper integral, we find that

$$f(x_0) = 2\left(\sqrt{x_2-x_0}+\sqrt{x_0-x_1}\right).$$

Therefore, the directional derivative of $f(x_0)$ with respect to $x_0$ is,

$$\frac{\partial f(x_0)}{\partial x_0} = \frac{1}{\sqrt{x_0 - x_1}} - \frac{1}{\sqrt{x_2 - x_0}} \tag{9}$$

This result is obtained through the process of finding the weakly singular integral first and then finding the final directional derivative. For simplicity, we term this process as ID. In practical boundary element application, the weakly singular integrals are very hard or impossible to be integrated analytically. Therefore, the above process may not be finished easily in most of the practical applications. Then the problem is can we find the result through the process of taking the differential symbol into the integral symbol first and then finding the final integral? Yes, we can. For simplicity, we term this process as DI. Now we will finish this process using two approaches.

In the first approach of DI, we will find the directional derivative of the integral $f(x_0)$ through the process of finding the derivative to both the integral kernel and the upper and lower limits. To do so, we must emphasize that the variable $x_0$ is in the range $(x_1, x_2)$. Therefore, the integral can be divided into two parts as,

$$f(x_0) = \int_{x_1}^{x_2} \frac{1}{\sqrt{|x - x_0|}} dx = \int_{x_1}^{x_0} \frac{1}{\sqrt{x_0 - x}} dx + \int_{x_0}^{x_2} \frac{1}{\sqrt{x - x_0}} dx, \quad x_1 < x_0 < x_2$$

In this formulation, each part of the integrals is a weakly singular integral and its value exists in the normal sense or in the sense of the improper integral. In the following derivation, we will neglect the limit process in the calculation of the improper integrals for simplicity. Then, the directional derivative of $f(x_0)$ with respect to $x_0$ can be found as,

$$\frac{\partial}{\partial x_0} \int_{x_1}^{x_2} \frac{1}{\sqrt{|x - x_0|}} dx = \frac{\partial}{\partial x_0} \int_{x_1}^{x_0} \frac{1}{\sqrt{x_0 - x}} dx + \frac{\partial}{\partial x_0} \int_{x_0}^{x_2} \frac{1}{\sqrt{x - x_0}} dx$$

$$= \left.\frac{1}{\sqrt{x_0 - x}}\right|_{x=x_0} + \int_{x_1}^{x_0} \frac{\partial}{\partial x_0} \left[\frac{1}{\sqrt{x_0 - x}}\right] dx - \left.\frac{1}{\sqrt{x - x_0}}\right|_{x=x_0} + \int_{x_0}^{x_2} \frac{\partial}{\partial x_0} \left[\frac{1}{\sqrt{x - x_0}}\right] dx$$

Clearly, we have taken the differential symbol into the integral symbol successfully. One may say that there are still two indefinite terms $\left.\frac{1}{\sqrt{x_0 - x}}\right|_{x=x_0}$ and $\left.\frac{1}{\sqrt{x - x_0}}\right|_{x=x_0}$ in this formulation. Never mind, these terms can be canceled by the integral by part of the integrals following them. Then we can find the result by an integral as,

$$\frac{\partial}{\partial x_0} \int_{x_1}^{x_2} \frac{1}{\sqrt{|x - x_0|}} dx = \left.\frac{1}{\sqrt{x_0 - x}}\right|_{x=x_0} - \int_{x_1}^{x_0} \frac{\partial}{\partial x} \left[\frac{1}{\sqrt{x_0 - x}}\right] dx$$

$$- \left.\frac{1}{\sqrt{x-x_0}}\right|_{x=x_0} - \int_{x_0}^{x_2} \frac{\partial}{\partial x}\left[\frac{1}{\sqrt{x-x_0}}\right] dx$$

$$= \left.\frac{1}{\sqrt{x_0-x}}\right|_{x=x_0} - \left[\frac{1}{\sqrt{x_0-x}}\right]_{x=x_1}^{x=x_0} - \left.\frac{1}{\sqrt{x-x_0}}\right|_{x=x_0} - \left[\frac{1}{\sqrt{x-x_0}}\right]_{x=x_1}^{x=x_0}$$

$$= \frac{1}{\sqrt{x_0-x_1}} - \frac{1}{\sqrt{x_2-x_0}}$$

Obviously, we find the same result using the different processes. In the above process, we have emphasized that the critical step is separating the range of the integral into two parts about the variable $x_0$.

However, in the second approach of DI, we will firstly do coordinate transformation about the integral $f(x_0)$, then carry out the directional derivative and finally find the integral.

Similarly, since $x_1 < x_0 < x_2$, we have

$$\frac{\partial}{\partial x_0}\int_{x_1}^{x_2} \frac{1}{\sqrt{|x-x_0|}} dx = \frac{\partial}{\partial x_0}\int_{x_1}^{x_0} \frac{1}{\sqrt{x_0-x}} dx + \frac{\partial}{\partial x_0}\int_{x_0}^{x_2} \frac{1}{\sqrt{x-x_0}} dx \tag{10}$$

Now begin to investigate the part $\frac{\partial}{\partial x_0}\int_{x_1}^{x_0}\frac{1}{\sqrt{x_0-x}}dx$ firstly. Let us do a variable substitution $x = (1-\eta)x_1 + \eta x_0$ and set $J = x_0 - x_1$.

Then we have $dx = Jd\eta$. And therefore,

$$\frac{\partial}{\partial x_0}\left[\int_{x_1}^{x_0} \frac{1}{\sqrt{x_0-x}} dx\right] = \int_0^1 \frac{\partial}{\partial x_0}\left[\frac{J}{\sqrt{x_0-x}}\right] d\eta$$

$$= \int_0^1 \left[J\frac{\partial}{\partial x_0}\left(\frac{1}{\sqrt{x_0-x}}\right) + \frac{1}{\sqrt{x_0-x}}\frac{\partial J}{\partial x_0}\right] d\eta$$

Clearly, we have taken the differential symbol into the integral symbol successfully. Then, the result can be found by an integral as,

$$\frac{\partial}{\partial x_0}\left[\int_{x_1}^{x_0} \frac{1}{\sqrt{x_0-x}} dx\right] = \int_0^1 \left[J\frac{-1}{2\sqrt{(x_0-x)^3}}\frac{\partial}{\partial x_0}(x_0-x) + \frac{1}{\sqrt{x_0-x}}\right] d\eta$$

$$= \int_0^1 \left[J\frac{-1}{2\sqrt{(x_0-x)^3}}\left(1-\frac{\partial x}{\partial x_0}\right) + \frac{1}{\sqrt{x_0-x}}\right] d\eta$$

$$= \int_0^1 \left[J\frac{-1}{2\sqrt{(x_0-x)^3}}(1-\eta) + \frac{1}{\sqrt{x_0-x}}\right] d\eta$$

$$= \frac{1}{\sqrt{x_0 - x_1}} \int_0^1 \frac{1}{2\sqrt{1-\eta}} d\eta = \frac{1}{\sqrt{x_0 - x_1}} \tag{11}$$

Similarly, we can find the other part,

$$\frac{\partial}{\partial x_0} \int_{x_0}^{x_2} \frac{1}{\sqrt{x - x_0}} dx = -\frac{1}{\sqrt{x_2 - x_0}} \tag{12}$$

Finally, we obtained,

$$\frac{\partial}{\partial x_0} \int_{x_1}^{x_2} \frac{1}{\sqrt{|x - x_0|}} dx = \frac{1}{\sqrt{x_0 - x_1}} - \frac{1}{\sqrt{x_2 - x_0}}$$

Again, we find the same result through different processes.

The second approach is very important because of its broadly potential application in the numerical implementation of the boundary element method. This process also tells us how to avoid the appearance of the strongly singular integral. That is after a coordinate transformation both the integral kernel and the Jacobian determinant should be differentiated when the differential symbol is taken into the integral symbol [Yan (2009)].

### *3.2 Derivation of the formulation (7)*

Generally, this form of integral should be computed by a strongly singular integral in the sense of Cauchy principal value. In the following, we will derive this formulation regardless of the CPV.

For convenience, we set,

$$g(y) = \int_a^b \ln|x - y| f(x) dx, \quad y \in (a, b) \tag{13}$$

Then, because of $y \in (a, b)$, we have

$$\frac{\partial g(y)}{\partial y} = \frac{\partial}{\partial y} \int_a^y \ln|x - y| f(x) dx + \frac{\partial}{\partial y} \int_y^b \ln|x - y| f(x) dx \tag{14}$$

Using the processes similar to those applied in the previous case, for the first part on the right hand side of the above formulation we have,

$$\frac{\partial}{\partial y} \int_a^y \ln|x - y| f(x) dx = [\ln|x - y| f(x)]_{x=y} + \int_a^y \frac{\partial \ln|x - y|}{\partial y} f(x) dx$$

$$= [\ln|x - y| f(x)]_{x=y} + \int_a^y \left[-\frac{\partial \ln|x - y|}{\partial x}\right] f(x) dx$$

$$= [\ln|x-y|\, f(x)]_{x=y} - [\ln|x-y|\, f(x)]_{x=a}^{x=y} + \int_a^y \ln|x-y|\, f'(x)dx$$

$$= \int_a^y \ln|x-y|\, f'(x)dx + \ln|a-y|\, f(a) \tag{15}$$

Similarly, for the second part we have,

$$\frac{\partial}{\partial y}\int_y^b \ln|x-y|\, f(x)dx = \int_y^b \ln|x-y|\, f'(x)dx - \ln|b-y|\, f(b) \tag{16}$$

Therefore, the final result for formulation (7) is,

$$\frac{\partial}{\partial y}\int_a^b \ln|x-y|\, f(x)dx = \int_a^b \ln|x-y|\, f'(x)dx - [\ln|x-y|\, f(x)]_{x=a}^{x=b} \tag{17}$$

This expression is very interesting and important, because it is a weakly singular integral and can be integrated normally regardless of the sense of the Cauchy principal value. While in the past, most of the researchers regarded that the formulation (7) must be integrated in the sense of the Cauchy principal value.

The following derivation shows that the formulation (17) is the same as the expression (2) presented in the reference Monegota (2009) with a minus sign (notice that the expression just below the equation (2) in that reference missed a minus sign).

$$\begin{aligned}
&\int_a^b \ln|x-y|\, f'(x)dx - [\ln|x-y|\, f(x)]_{x=a}^{x=b} \\
&= [\ln|x-y|\, f(x)]_{x=a}^{x=b} - \int_a^b \frac{f(x)}{x-y}dx - [\ln|x-y|\, f(x)]_{x=a}^{x=b} \\
&= -\int_a^b \frac{f(x)-f(y)}{x-y}dx - f(y)\int_a^b \frac{1}{x-y}dx \\
&= -\int_a^b \frac{f(x)-f(y)}{x-y}dx - f(y)\,[\ln|x-y|]_{x=a}^{x=b} \\
&= -\int_a^b \frac{f(x)-f(y)}{x-y}dx - f(y)\ln\frac{b-y}{y-a}
\end{aligned} \tag{18}$$

The difference of the minus sign is due to that,

$$\frac{\partial}{\partial y}\int_y^b \ln|x-y|\, f(x)dx = -CPV\int_a^b \frac{f(x)}{x-y}dx$$

### 3.3 Derivation of the formulation (8)

Generally, this form of integral should be computed by a hypersingular integral in the sense of Hadamard finite part integral. In the following, we will derive this formulation to an integral form regardless of the Hadamard finite part integral and the CPV.

Based on the formulation (17), we have,

$$\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\int_a^b \ln|x-y|\, f(x)dx\right] = \frac{\partial}{\partial y}\left[\int_a^b \ln|x-y|\, f'(x)dx - [\ln|x-y|\, f(x)]_{x=a}^{x=b}\right]$$
$$= \int_a^b \ln|x-y|\, f''(x)dx - \left[\ln|x-y|\, f'(x)\right]_{x=a}^{x=b} - \left[\frac{f(x)}{y-x}\right]_{x=a}^{x=b}, \quad a<y<b \tag{19}$$

This expression is more important due to that a so-called hypersingular integral is now proved to be at most a weakly singular integral and no CPV and Hadamard FP are required in the derivation. However, in the past, most of the researchers concluded that such kind of integrals must integrate in the sense of Hadamard FP [Carley (2007), Monegota (2009)].

The following derivation shows that the formulation (19) is the same as the expressions (27) and (28) presented in the reference Monegota (2009) with a minus sign.

$$\begin{aligned}
&\int_a^b \ln|x-y|\, f''(x)dx - \left[\ln|x-y|\, f'(x)\right]_{x=a}^{x=b} - \left[\frac{f(x)}{y-x}\right]_{x=a}^{x=b}\\
&= -\int_a^b \frac{f'(x)}{x-y}dx - \left[\frac{f(x)}{y-x}\right]_{x=a}^{x=b}\\
&= -\int_a^b \frac{f(x)}{(x-y)^2}dx\\
&= -\int_a^b \frac{f(x)-f(y)}{(x-y)^2}dx - f(y)\int_a^b \frac{1}{(x-y)^2}dx
\end{aligned} \tag{20}$$

The minus sign is due to that,

$$\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\int_a^b \ln|x-y|\, f(x)dx\right] = -\int_a^b \frac{f(x)}{(x-y)^2}dx, \quad a<y<b$$

As an example, $f(x)=1$ , $a=-1$, $b=1$ and $|y|\neq 1$, we can easily found the value of expression (8) from the formulation (19). That is,

$$\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\int_{-1}^{1}\ln|x-y|dx\right]=-\left[\frac{1}{y-x}\right]_{x=-1}^{x=1}$$
$$=-\frac{2}{y^2-1},\quad -1<y<1 \tag{21}$$

And compared with the formulation (20), we have,

$$\int_{-1}^{1}\frac{1}{(x-y)^2}dx=-\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\int_{-1}^{1}\ln|x-y|dx\right]$$
$$=\frac{2}{y^2-1},\quad -1<y<1$$

Therefore,

$$\int_{-1}^{1}\frac{1}{(x-y)^2}dx=\frac{2}{y^2-1},\quad -1<y<1 \tag{22}$$

Formulation (22) is just identical to the equation (2.2b) presented by Carley (2007) or the equation (25) presented by Monegota (2009) as $-1<y<1$.

## 4 Validation of the expression (19) by several numerical computations

To validate the expression (19), two special cases about the integral $\frac{1}{2\pi}\int_{-a}^{a}\frac{f(x)}{x^2}dx,\quad a=\frac{1}{30}$ will be computed by numerical methods.
case 1: $f(x)=1$;
case 2: $f(x)=\sqrt{x+2a}$.
According to the expression (19), for the case one we have

$$\frac{1}{2\pi}\int_{-a}^{a}\frac{1}{x^2}dx=\frac{1}{a\pi}=\frac{30}{\pi}$$

While in the numerical simulation, we use the fundamental solution of 2D potential problems $G_0=-\ln r/(2\pi)$. Therefore, on an element located on the $x$ axis, the hypersingular integral operator is reduced to,

$$N_0 f=\int_{-a}^{a}\frac{\partial^2 G_0}{\partial n_p\partial n_q}f(q)\,dx_q=\frac{1}{2\pi}\int_{-a}^{a}\frac{f(x)}{x^2}dx \tag{23}$$

To calculate the hypersingular integral (23) numerically, the regularization relationship technique presented by Yan, Hung and Zheng (2003) is employed with the constructed boundary as follows,

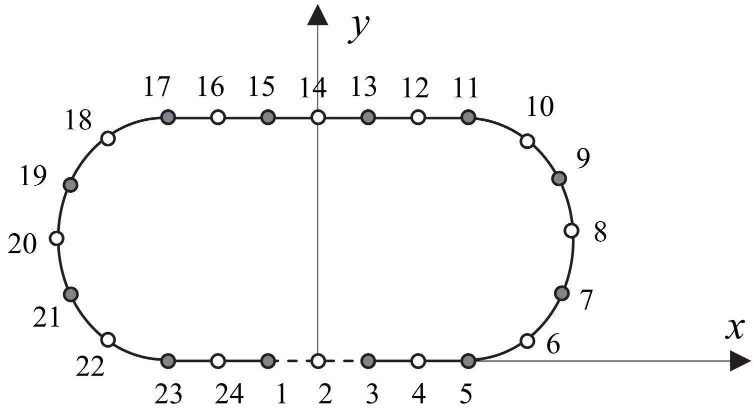


Figure 1: Illustration of a 2D boundary discretization with 24 nodes.

This boundary is composed of two segments of length $6a$ and two semi-circles with radius $3a$. It is discretized using three-nodded curvilinear quadratic elements. There are 12 elements and 24 nodes in total. The nodes marked by hollow circle are located at the centres of the corresponding elements. The element represented by dashed line is just the integral domain of the expression (23).

Then we use the regularization relationship technique presented by Yan, Hung and Zheng (2003) to find the matrix $D_0$ for the integral operator $N_0$ . On the same time, the direct integral about the integral operator $N_0$ as the point $p$ is located at the node 2 is computed on the boundary excepting the element marked by dashed line using Gaussian quadrature and stored in a vector $B$. As a result, the value of expression (23) is obtained by,

$$\frac{1}{2\pi}\int_{-a}^{a}\frac{f(x)}{x^2}dx=\sum_{j=1}^{24}\left(D_0[2,j]-B[j]\right)*f_j \tag{24}$$

Tab.1 presents the numerical results for two integrals $\int_{-a}^{a}1/x^2dx$ and $\int_{-a}^{a}\sqrt{x+2a}/x^2dx$ as $a=1/30$. Two kinds of numerical methods corresponding to the expressions (24) and (19) are employed to compute these integrals. To validate the convergence of the numerical methods, two discretizations of the model shown in Fig. 1 with 12 elements and 36 elements are applied in the simulation. For the integral $\int_{-a}^{a}1/x^2dx, \quad a=1/30$, it has an analytical solution similar to that given by the expression (22). Clearly, both of the numerical results with 36 elements are very close to the analytical solution. Moreover, in this case the numerical result found by the expression (19) is just the analytical solution due to that the second derivative of the function $f(x) = 1$ is 0. This example validates the numerical methods and its convergence. Then the integral $\int_{-a}^{a} \sqrt{x+2a}/x^2 dx$ further validate the convergence of the numerical methods.

## 5 Summary & Conclusion

We try to find a general way to avoid the occurring of the strongly singular and hypersingular integrals. On the same time, we try to discover the reason why such kind of singular integrals are created. Using the derivation of several simple examples, we find a way to avoid the occurring of the strongly singular and hypersingular integrals. That is, when the differential symbol is taken into the integral symbol of an integral with a weakly singular integral kernel, the range of this integral must be regarded as a function of the differential variable. The result will help us to form a new approach to overcome the strongly and hypersingular integrals occurring in the boundary element method. The description of the derivations presented in this paper may not be very strict mathematically. To form a general theory, a lot of research should be done in the future.

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Table 1: Numerical results of two integrals of the form $\int_{-a}^{a} f(x)/x^2 dx, \quad a = 1/30$

| | Numerical method | $\frac{1}{2\pi}\int_{-a}^{a}\frac{1}{x^2}dx = -\frac{30}{\pi} \approx -9.5493$ | Error | $\frac{1}{2\pi}\int_{-a}^{a}\frac{\sqrt{x+2a}}{x^2}dx$ |
|---|---|---|---|---|
| 12 elements | Regularization relationship | -9.5240 | 0.26% | -2.4504 |
| | Expression (19) | -9.5493 | 0 | -2.5653 |
| 36 elements | Regularization relationship | -9.5462 | 0.03% | -2.5402 |
| | Expression (19) | -9.5493 | 0 | -2.5453 |

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