Applications of Parameter-Expanding Method to Nonlinear Oscillators in which the Restoring Force is Inversely Proportional to the Dependent Variable or in Form of Rational Function of Dependent Variable

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Abstract: He's parameter-expanding method with an adjustment of restoring forces in terms of Chebyshev's series is used to construct approximate frequency-amplitude relations for a conservative nonlinear singular oscillator in which the restoring force is inversely proportional to the dependent variable or in form of rational function of dependant variable. The procedure is used to solve the nonlinear differential equation approximately. The approximate frequency obtained using this procedure is more accurate than those obtained using other approximate methods and the discrepancy between the approximate frequency and the exact one negligible.

Keywords: Parameter-expanding method, Non-linear oscillator

1   Introduction

In nonlinear systems, perturbation methods are well-known traditional tools to study various aspects of nonlinear problems. Surveys of the early literature with numerous references, and useful bibliographies, have been given by Nayfeh (1973), Mickens (1996). However, the use of perturbation theory in many significant practical problems is invalid, or it simply breaks down for parameters beyond a certain specified range. Therefore, new analytical techniques should be developed to overcome these shortcomings.

Such a new technique should work over a large range of parameters and yield accurate analytical approximate solutions beyond the coverage and ability of the classical perturbation methods.

For nonlinear oscillators, some of these techniques include harmonic balance method,

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[Wu, Sun, and Lim(2006); Wu, Li (2001)]; variational iteration method [He(1999); Ozis and Yildirim(2007)], homotopy perturbation method [He(2000); Ozis and Yıldırım(2007)].Also there are some other method Surveys of the literature with numerous references and useful bibliography and a review of these methods can be found in detail in [He(2006)]. In this paper, we adapted He's bookkeeping parameter method [He(2001)] with an adjustment of restoring forces in terms of Chebyshev's series for nonlinear oscillators in which the restoring force is inversely proportional to the dependent variable or in form of rational function of dependant variable to obtain analytic approximate solutions.

2   Parameter-expanding methods

In case no small parameter exists in a nonlinear equation, traditional perturbation methods cannot be directly applied. For this type of problem, [He (2001)] developed a technique where a bookkeeping parameter is introduced to the original differential equation. Recently parameter-expanding methods [Xu(2007); Mohyud-Din, Noor and Noor(2009)] including bookkeeping parameter method [He (2001)] and modified Lindstedt-Poincaré methods [Ozis and Yıldırım(2007);Ramos(2007)] have been caught much attention. The parameter expansion can also be applied to homotopy perturbation method [Ozis and Akci(2011)].

However, there is a class of nonlinear oscillators represents a new class of nonlinear oscillating systems which are called non-smooth oscillators play an important role in non-linear dynamics. Conservative non-smooth oscillators such as the ones considered here are governed by in which the restoring force is inversely proportional to the dependent variable or restoring forces are in form of rational function of dependent variable and they don't include any small parameter. In this paper, we apply He's bookkeeping parameter method with an adjustment of restoring forces in terms of Chebyshev's series to the nonlinear oscillators to the category state above.

Example 1: Mickens (1996) has recently analyzed the nonlinear differential equation

$$\frac{d^2y}{dt^2} + \frac{1}{y} = 0, \tag{1}$$

with initial conditions

$$y(0) = A, \quad y'(0) = 0 \tag{2}$$

This equation occurs in the modelling of certain phenomena in plasma physics [Mickens(1996)]. Equation (1) clearly indicates that acceleration is unbounded

when $y = 0$. What's more, the Eq.(1) corresponds to a conservative single-degree-of-fredom oscillator whose total energy is conserved, i.e.,

$$\frac{1}{2}\dot{y}^2 + \ln y = \ln A$$

which shows that the velocity is also unbounded when $y = 0$.

Mickens showed that all the motions corresponding to equation (1) are periodic [Mickens(2007)];

The system will oscillate within symmetric bounds $[-A, A]$, and the angular frequency and corresponding periodic solution of the nonlinear oscillator are dependent on the amplitude A. [Mickens(2007)] also provided comparisons between the results of harmonic balance method and those obtained by means of the application of the homotopy perturbation method to Eq.(1), i.e.,

$$y'' + w^2 y = p(w^2 y - y^{-1}) \tag{3}$$

$$0y'' + 1y = py''y \tag{4}$$

and

$$y'' + 0y + p(y'')^2 y = 0 \tag{5}$$

which coincides with Eq(1) for $p = 1$ and where p is a homotopy parameter which is set to unity at the end of the calculations.

If $y(t)$ in Eq.(3) is expanded as

$$y(t) = y_0 + py_1 + ... \tag{6}$$

the homotopy perturbation yealds the same frequency as that of harmonic balance method of first order approximation[Mickens(2007)] but if $y(t)$ is expanded in Eq.(6) and the coefficients 1 and 0 of Eq.(4) are expanded as [Ozis and Akçi(2011)]

$$1 = w^2 + pa_1 + ... \tag{7}$$

$$0 = 1 + pb_1 + ... \tag{8}$$

respectively, then the homotopy perturbation method calculates the frequency as

$$w^2 = 4/3A^2 \tag{9}$$

Moreover, if $y(t)$ is expanded as in Eq.(6) and the coefficient 0 in Eq.(5) is expanded as in Eq.(8), the homotopy perturbation calculates the same frequency as that of Eq.(9).

Recently, [Shou(2009)] construct the homotopy of Eq.(1) as $y'' + w^2 y + p[y(y'')^2 - w^2 y]$ and calculates the frequency as in[Mickens(2007)].

It must be emphasized that the "expansion of constants" was first proposed by [He(2006)]. However, various methods introduce an artificial parameter such as modified Linstedt-Poincaré techniques , Homotopy perturbation method and book-keeping parameter method etc. In these methods and their variants one first introduces linear stiffness term and the artificial parameter then expands both the solution and the frequency of oscillation in terms of this artificial parameter which is set to unity at the end of the calculations. Here we adapted bookkeeping parameter method with an adjustment of restoring forces in terms of Chebyshev's series.

We re-write the Eq.(1) in the form

$$y''y + 0.y + 1.y^{-1} = 0 \tag{10}$$

Assume that the solution can be expressed as a power series in p:

$$y = y_0 + py_1 + p^2 y_2 + ... \tag{11}$$

where p is a bookkeeping parameter.

We also assume that the coefficients 0 and 1 in the left side of Eq. (10) can be, respectively, expanded into a series in p:

$$0 = w^2 + pw_1 + p^2 w_2 + ... \tag{12}$$

$$1 = a_1 p + a_2 p^2 + ... \tag{13}$$

Substituting Eqs.(12) and (13) into Eq(10) and equating the terms with the identical powers of p , we have

$$p^0 : y_0'' + w^2 y_0 = 0, \quad y_0(0) = A, \quad y_0'(0) = 0 \tag{14}$$

$$p^1 : y_1'' + w^2 y_1 + y_0 w_1 + a_1 \frac{1}{y_0} = 0, \tag{15}$$

The solution of (14) easily be obtained and is

$$y_0 = A\cos(wt) \tag{16}$$

Substituting the solution (16) into Eq.(15) gives

$$y_1'' + w^2 y_1 + (A\cos(wt))w_1 + a_1.A^{-1}.(\cos(wt))^{-1} = 0 \tag{17}$$

The last term in the equation (17) is proportional to $f(wt) = 1/\cos(wt)$ which is neither absolutely nor square integrable in $[0, 2\pi]$. Moreover, $f(wt)$ is unbounded at $wt = \pi/2$ and $wt = 3\pi/2$. Therefore, the Fourier series expansion of $f(wt)$ does not convergence to $f(wt)$ in the classical sense. This may be eased by using Chebyshev series expansion. Therefore, we have

$$f(x) = \sum_{n=0}^{\infty} c_{2n+1} T_{2n+1}(x),$$

where $x = \cos(wt)$, $T_{2n+1}(x) = \cos[(2n+1)wt]$ and the coefficients yield

$$c_{2n+1} = \frac{2}{\pi} \int_{-1}^{1} (1-x^2)^{-1/2} f(x) T_{2n+1}(x) dx.$$

For $f(x) = \frac{1}{x}$ where $x = \cos\theta$ yields the coefficients

$$c_{2n+1} = \frac{2}{\pi} \int_{0}^{\pi} (\cos\theta)^{-1} . \cos[(2n+1)\theta] d\theta$$

and for $n = 0$ gives

$$c_1 = \frac{2}{\pi} \int_{0}^{\pi} \frac{1}{\cos\theta} . \cos\theta . d\theta = \frac{2}{\pi} . \theta]_0^\pi = \frac{2}{\pi} . \pi = 2$$

Hence, it is easy to validate that $c_{2n+1} = 2$ for $n = 2, 4, 6, ...$ and $c_{2n+1} = -2$ for $n = 1, 3, 5, ...$

No secular terms in y_1 requires

$$w_1 = -w^2, \quad a_1 = 1$$

and

$$Aw_1 + a_1 A^{-1} c_1 = 0 \text{ or } A(-w^2) + A^{-1} c_1 = 0.$$

Hence, we obtain first –order approximation as

$$w^2 = \frac{1}{A^2} . 2 \text{ or } w = \frac{\sqrt{2}}{A} = \frac{1.414214}{A} \tag{18}$$

which agrees with Mickens' first-order harmonic balance solution [Mickens(2007)] where exact value is

$$w_{ex} = \frac{\sqrt{2\pi}}{2A} = \frac{1.2533141}{A} \tag{19}$$

By utilizing modified generalized rational harmonic balance method [Beléndez , Méndez , Beléndez, Hernández and Alvarez(2008)] and [Beléndez, Gimeno, Fernández,I Méndez and Alvarez (2008)] determined the second- order approximate frequency as $w_2^{(1)} \approx 1.2193273A^{-1}$ and for the second order approximation, $w_2^{(2)} \approx 1.2482546.A^{-1}$

Example 2: We, now, consider the Duffing-harmonic oscillator:

$$u'' + \frac{u^3}{1+u^2} = 0, \tag{20}$$

with initial conditions

$$u(0) = A, \quad u'(0) = 0 \tag{21}$$

Note that for small u values, Eq. (20) reduces to the equation of motion of the Duffing type nonlinear oscillator, while for large u values it reduces to the equation of motion of a linear harmonic oscillator. Therefore, Eq. (20) is called a Duffing-harmonic oscillator equation of motion. For example [Shou (2009)], obtained the frequency via the application of homotopy perturbation method by constructing the homotopy in form of

$$u" + \frac{1.u}{1+(p^{1/2}u)^2} = 0$$

and expanding the coefficient 1 as $1 = w^2 + pw_1 + p^2w_2 + ...$ and calculate the frequency as $w = \frac{1}{\sqrt{1+\frac{3A^2}{4}}}$.

We re-write the Eq. (20) in the form

$$1.u'' + 0.u + 1.(u''.u^2 + u^3) = 0 \tag{22}$$

Assume that the solution can be expressed as a power series in p:

$$u = u_0 + pu_1 + p^2u_2 + ... \tag{23}$$

where p is a bookkeeping parameter.

We also assume that the coefficients 0 and 1 in the left side of Eq. (22) can be, respectively, expanded into a series in p:

$$0 = w^2 + pw_1 + p^2w_2 + \ldots \tag{24}$$

$$1 = a_1p + a_2p^2 + \ldots \tag{25}$$

Substituting Eqs.(24) and (25) into Eq(22) and equating the terms with the identical powers of p, we have

$$p^0 : u_0'' + w^2u_0 = 0, \quad u_0(0) = A, \quad u_0'(0) = 0 \tag{26}$$

$$p^1 : u_1'' + w^2u_1 + u_0w_1 + a_1u_0''.u_0^2 + a_1.u_0^3 = 0, \tag{27}$$

The solution of (26) easily be obtained and is

$$u_0 = A\cos(wt) \tag{28}$$

Substituting the solution (28) into Eq.(27) gives

$$u_1'' + w^2u_1 + w_1.A\cos(wt) + (-a_1.w^2.A^3 + a_1.A^3)\cos^3(wt) = 0 \tag{29}$$

Using Chebyshev series expansion,
$f(x) = \sum\limits_{n=0}^{\infty} c_{2n+1}T_{2n+1}(x)$, where $x = \cos(wt)$, $T_{2n+1}(x) = \cos[(2n+1)wt]$ and the

coefficients yield $c_{2n+1} = \frac{2}{\pi} \int\limits_{-1}^{1} (1-x^2)^{-1/2}f(x)T_{2n+1}(x)dx$.

For $f(x) = x^3$ where $x = \cos\theta$ yields the coefficients

$$c_{2n+1} = \frac{2}{\pi} \int\limits_{0}^{\pi} \cos^3\theta.\cos[(2n+1)\theta]d\theta$$

and for $n = 0$ gives

$$c_1 = \frac{2}{\pi} \int\limits_{0}^{\pi} \cos^4\theta.d\theta = \frac{2}{\pi}.\frac{3}{8}.\pi = \frac{3}{4}$$

No secular terms in u_1 requires
$w_1A + (-a_1w^2A^3 + a_1A^3)c_1 = 0$ and $w_1 = -w^2, a_1 = 1$

Hence, we obtain first –order approximation as

$$-w^2 A - c_1 w^2 A^3 + c_1 A^3 = 0, \quad w^2 = \frac{c_1 A^2}{1 + c_1 A^2}$$

or

$$w = \sqrt{\frac{\frac{3}{4} A^2}{1 + \frac{3}{4} A^2}} \tag{30}$$

which coincides with the one by [Lim and Wu(2003)], and [Belendez, Hernandez,. Belendez, Fernandez, Alvarez, and Neipp (2007)].

Example 3: We, now, consider the following nonlinear oscillator;

$$u'' + \frac{u}{\sqrt{1 + u^2}} = 0, \tag{31}$$

with initial conditions

$$u(0) = A, \quad u'(0) = 0 \tag{32}$$

We re-write Eq.(31) in the form

$$u'' + 0.u + 1.\frac{u}{\sqrt{1 + u^2}} = 0 \tag{33}$$

Assume that the solution can be expressed as a power series in p:

$$u = u_0 + p u_1 + p^2 u_2 + \dots \tag{34}$$

where p is a bookkeeping parameter.

We also assume that the coefficients 0 and 1 in the left side of Eq.(33) can be respectively expanded to a series in p:

$$0 = w^2 + p w_1 + p^2 w_2 + \cdots \tag{35}$$

$$1 = a_1 p + a_2 p^2 + \cdots \tag{36}$$

Substituting Eqs.(35) and (36) into Eq.(33) and equating the terms with the identical powers of p, we have

$$p^0 : u_0'' + w^2 u_0 = 0, \quad u_0(0) = A, \quad u_0'(0) = 0 \tag{37}$$

$$p^1 : u_1'' + w^2 u_1 + w_1 u_0 + a_1 . \frac{u_0}{\sqrt{1 + u_0^2}} = 0 \tag{38}$$

The solution of Eq.(37) can be easily obtained as

$$u_0 = A \cos wt. \tag{39}$$

Substituting the result into Eq.(38) yields

$$u_1'' + w^2 u_1 + w_1 A \cos wt + a_1 . \frac{A \cos wt}{\sqrt{1 + (A \cos wt)^2}} = 0 \tag{40}$$

Using Chebyshev series expansion, we have

$$f(x) = \sum_{n=0}^{\infty} c_{2n+1} T_{2n+1}(x), \text{ where } x = \cos(wt), T_{2n+1}(x) = \cos[(2n+1)wt]$$

and the coefficients yield $c_{2n+1} = \frac{2}{\pi} \int_{-1}^{1} (1 - x^2)^{-1/2} f(x) T_{2n+1}(x) dx$.

For $f(x) = \frac{x}{\sqrt{1 + A^2 x^2}}$ where $x = \cos \theta$ and $n = 0$ yields the coefficient c_1

$c_1 = \frac{2}{\pi} \int_{-1}^{1} (1 - x^2)^{-1/2} \frac{x}{\sqrt{1 + A^2 x^2}} x dx$ and for $x = \cos \theta$ reads

$$c_1 = \frac{2}{\pi} \int_{0}^{\pi} \frac{\cos^2 \theta}{\sqrt{1 + A^2 \cos^2 \theta}} d\theta, \theta = wt \tag{41}$$

No secular terms in u_1 requires

$$w_1 A + a_1 c_1 A = 0 \tag{42}$$

If the first order approximation is adequate then set $p = 1$ and from (35) and (36) we have

$$0 = w^2 + w_1, \tag{43}$$

$$1 = a_1 \tag{44}$$

From Eqs.(42)-(44), we obtain

$$w = \sqrt{c_1} \tag{45}$$

where c_1 is defined by Eq.(41) which agrees well with ref. [Shou and He(2007)] as it is seen in Table 1. and it is noticeable that the obtained frequency is valid for $0 < A < \infty$.

Table 1: Comparisons of our results with [Shou and He (2007)] for various amplitudes.

A	w_{app} (Present method)	w_{app} [Shou and He(2007)]
0.1	0.9981	0.998
0.01	1.0000	0.99998
1.0	0.8733	0.873
10	0.3510	0.354
100	0.1115	0.1128
1000	0.0353	0.03568
10000	0.0112	0.01128

3   Conclusion

The parameter-expanding method with an adjustment of restoring forces in terms of Chebyshev's series for nonlinear oscillators in which the restoring force is inversely proportional to the dependent variable or in form of rational function of dependant variable proved to be a powerful mathematical tool to nonlinear oscillators. The technology can be easily extended to higher order approximate solutions but it is, for the time being, out of our consideration in this letter. The present approach can be used as prototype for many other applications of nonlinear oscillators with discontinuity in searching for period or frequency.

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