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A Wavelet Numerical Method for Solving Nonlinear Fractional Vibration, Diffusion and Wave Equations

Zhou YH[1,2], Wang XM[2]; Wang JZ[1,2] and Liu XJ[2]

Abstract: In this paper, we present an efficient wavelet-based algorithm for solving a class of fractional vibration, diffusion and wave equations with strong nonlinearities. For this purpose, we first suggest a wavelet approximation for a function defined on a bounded interval, in which expansion coefficients are just the function samplings at each nodal point. As the fractional differential equations containing strong nonlinear terms and singular integral kernels, we then use Laplace transform to convert them into the second type Voltera integral equations with non-singular kernels. Certain property of the integral kernel and the ability of explicit wavelet approximation to the nonlinear terms of the unknown function in the equations enable us to numerically decouple complex spatial and temporal dependencies during solution of these equations, and eventually get a stable, high accuracy and efficient numerical method without involving any matrix inversions for numerically solving the nonlinear fractional vibration, diffusion and wave differential equations. Efficiency and accuracy of the proposed method are justified by numerical examples.

Keywords: wavelet; nonlinear fractional differential equation; Laplace transform; numerical method

Introduction

In recent years, fractional derivatives have been found to be very effective for describing many physical phenomena such as rheology, damping laws, anomalous random walk, fluid flow and many others [Carpinteri and Mainardi (1997); Podlubny (1999); Momani (2007); Metzler, Barkai, and Klafter (1999); Hanyga (2002); Momani (2005)]. These attract extensive studies regarding solutions of various fractional differential equations including but not limited to the fractional

[1] Corresponding author. E-mail addresses: zhouyh@lzu.edu.cn (Y. Zhou); jzwang@lzu.edu.cn (J. Wang).

[2] Key Laboratory of Mechanics on Disaster and Environment in Western China, the Ministry of Education of China, and Department of Mechanics and Engineering Sciences, School of Civil Engineering and Mechanics, Lanzhou University, Lanzhou, Gansu 730000, P.R. China

Fokker-Planck equations [Metzler, Barkai, and Klafter (1999)], the space-time fractional diffusion-wave equations [Hanyga (2002)], and the fractional KdV equations [Momani (2005)].

Exact closed form solutions of fractional differential equations are usually rare. Although based on techniques of integral transforms, we [Liu, Wang, Wang and Zhou (2011)] have derived a series-form exact solution for the general linear time-fractional diffusion-wave equations on a bounded space domain, yet there still exist no methods to analytically solve general nonlinear fractional partial differential equations. Approximation techniques play an indispensable role in complementing exact solutions. For example, both the homotopy perturbation [He (2000); He and Momani (2007)] and homotopy analysis methods [Song and Zhang (2007); Elsaid (2011)] provide an effective procedure for numerical solutions of differential equations. These methods are in principle based on Taylor series expansion with respect to an embedding parameter. However, the rate of convergence of the series solution depends on the determination of an auxiliary parameter and an initial guess of the solution, which are not always easy. The Adomian Decomposition Method (ADM) and the Variational Iteration Method (VIM) have been used by Odibat and Momani [Odibat and Momani (2006); Odibat and Momani (2008); Odibat and Momani (2009)] to solve the nonlinear partial differential equations of fractional order. However, application of the ADM needs to approximately replace a linear operator with fractional derives by that with integer derivatives, though the solution procedure is simple, but the ADM is not as effective or convenient as the VIM [Odibat and Momani (2009)]. Application of the VIM does not involve the calculation of the so-called Adomian polynomials, but a good guess to the initial approximation of the solution is crucial. Moreover, both the ADM and VIM need a large number of iterative calculations. By using a finite difference scheme in time and Legendre spectral methods in space, Lin and Xu [Lin and Xu (2007)] proposed a numerical procedure to quantitatively solve the fractional diffusion equation. We note that the adopted finite-difference method only has the accuracy of first order.

Despite the progresses outlined above, the literatures on high accuracy and easy to implement numerical techniques that are suitable for the solutions of nonlinear fractional vibration, diffusion and wave equations remain rather scarce due to the existence of strong nonlinearity and singularity in these equations. Wavelets represent a newly developed powerful mathematical tool, which has been broadly applied to signal decompositions and reconstructions, Laplace inversions [Wang, Zhou and Gao (2003)], differential equation solutions [Liu, Wang, Wang and Zhou (2011); Li (2010)] and active vibration control of piezoelectric smart structures [Zhou, Wang, Zheng and Jiang (2000)]. However, it is somewhat surprising that only very few studies have focused on the solution of fractional differential equations by using

wavelet methods [Lepik (2009)]. In this paper, an efficient wavelet based algorithm is proposed to solve the fractional differential equations with strong nonlinearities. This algorithm depends on an explicit wavelet approximation scheme for the nonlinear terms of unknown functions in the equation, in which series coefficients are just the function samplings at corresponding nodal points, and also depends on the elimination of singular integral kernels through converting the equations into equivalent nonsingular integral equations by using Laplace transform. Special property of the convolution kernel and the explicit form of wavelet approximations to nonlinear terms of unknown functions allow us to numerically decouple complex spatial and temporal dependencies in these equations, and eventually develop a stable and efficient numerical method with high accuracy for the solutions of nonlinear fractional differential equations.

1 Essential knowledge on wavelet theory

1.1 Approximating a function with Coiflets wavelets

Among compactly supported wavelets, a family known as Coiflets [Daubechies (1988); Mallat (1998)] has a number of properties that make it particularly useful in numerical analysis, one of which is the vanishing moments for both the scaling and wavelet functions. The shifted vanishing moments for scaling function can be used to construct one point quadratures in wavelet representation of a function. In this study, we choose the scaling functions of Coiflets that satisfy the following conditions:

$$\phi(x) = \sum_{k=0}^{3\gamma-1} p_k \phi(2x-k), \tag{1}$$

$$\int_{-\infty}^{\infty} \phi(x)dx = 1, \tag{2}$$

$$\int_{-\infty}^{\infty} (x-M_1)^j \phi(x)dx = 0 \tag{3}$$

where even integer γ is the maximum order of vanishing moments which insures that the compact support interval of $\phi(x)$ is $[0, 3\gamma-1]$, $j = 1, 2, \ldots, \gamma-1$, integer $M_1 = \int_{-\infty}^{\infty} x\phi(x)dx$ the first order moment of the scaling function, and p_k the filter coefficients, which have been listed in Table 1 [Wang, Zhou and Gao (2003)] for $\gamma = 2, 4, 6$. As an illustrating example, Fig. 1 shows the scaling function for $\gamma = 6$ and $M_1 = 7$. The general establishing method for Coiflets with different γ and M_1 has been given by Wang in [Wang (2001)].

Table 1: Coefficients p_k for γ=2, 4 and 6.

k	(γ=2)	(γ=4)	(γ=6)
0	5.456145913796356e-02	1.689380907695821e-03	−2.392638657280051e-03
1	-1.795614591379636e-01	-1.816639282073453e-02	−4.932601854180402e-03
2	-1.091229182759271e-01	3.507862062605389e-02	2.714039971139949e-02
3	83591229182759271e-01	7.074394036809258e-02	3.064755594619984e-02
4	1.054561459137964e+00	-2.197082915811749e-01	−1.393102370707997e-01
5	3.204385408620364e-01	-1.013118304071172e-01	−8.060653071779983e-02
6		8.067593419102440e-01	6.459945432939942e-01
7		1.061135780078056e+00	1.116266213257999e+00
8		3.968448038803485e-01	5.381890557079980e-01
9		-1.047986487449172e-02	−9.961543386239989e-02
10		-2.066385574316280e-02	−7.992313943479994e-02
11		-1.921632058008399e-03	5.149146293240031e-02
12			1.238869565706006e-02
13			−1.583178039255944e-02
14			−2.717178600539990e-03
15			2.886948664020020e-03
16			6.304993947079994e-04
17			−3.058339735960013e-04

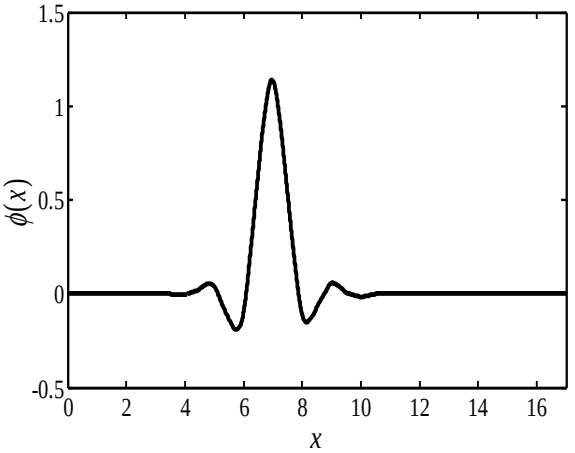


Figure 1: Scaling function of Coiflet wavelet with support interval $[0, 17]$.

Any function $f(x) \in L^2(\mathbf{R})$ can be approximated by its orthogonal projection to the subspace $\mathbf{V}_n$ formed by the orthonormal basis $\{\phi_{n,k}(x) = 2^{n/2}\phi(2^n x - k), k \in \mathbf{Z}\}$ as

$$f(x) \approx \mathbf{P}_n f(x) = \sum_{k=-\infty}^{\infty} c_{n,k}\phi_{n,k}(x) \tag{4}$$

in which

$$c_{n,k} = \int_{\mathbf{R}} f(x)\phi_{n,k}(x)dx. \tag{5}$$

Approximation accuracy of Eq. (4) can be estimated as [Sweldens and Piessens (1994)]

$$\|f(x) - \mathbf{P}_n f(x)\| = O(2^{-n\gamma}). \tag{6}$$

Following Eqs. (1-3), we have a good approximation of coefficients of $f(x)$ in the expansion, i.e., Eq. (5) can be approximated as [Zhou and Wang (1999)]

$$c_{n,k} = \int_{\mathbf{R}} f(x)\phi_{n,k}(x)dx \approx 2^{-n/2} f(\frac{M_1 + k}{2^n}). \tag{7}$$

Further, we have

$$f(x) \approx \sum_{k=-\infty}^{\infty} f(\frac{M_1 + k}{2^n})\phi(2^n x - k). \tag{8}$$

Here, we have assumed that the function $f(x)$ is smooth enough. It can be seen from Eq. (6) that the approximation error decays very fast as the resolution level n and integer γ increase. Moreover, such a single-point reconstruction formula of $f(x)$ has several very interesting characteristics [Zhou and Wang (1999)]: For any composite function of $f(x)$, $\Pi[f(x)] \in \mathbf{L}^2(\mathbf{R})$, by treating $\Pi[f(x)]$ as a new function and applying Eq. (8), we have

$$\Pi[f(x)] \approx \sum_{k=-\infty}^{\infty} \Pi[f(\frac{M_1 + k}{2^n})]\phi(2^n x - k). \tag{9}$$

Notice that approximations like (8) or (9) are not valid for Fourier or Fourier-like bases. Such form of series expansion is very useful when we use Galerkin type method to solve nonlinear differential/integral equations. For example, if $u(x)$ is an unknown function in an equation with nonlinear term $\exp[u(x)]$, when we use

Galerkin method to numerically solve such a nonlinear equation, we first approximate the unknown function $u(x)$ as

$$u(x) \approx \sum_k a_k h_k(x). \tag{10}$$

When we substitute Eq. (10) into the nonlinear term $\exp(u(x))$, we have

$$\exp(u(x)) \approx \exp(\sum_k a_k h_k(x)). \tag{11}$$

It can be seen from this equation that it will be very difficult to convert the governing equation into algebraic equations of the unknown coefficients a_k by performing weighted integration as the Galerkin method usually does. However, if we use Eq. (9) to approximate both the function $u(x)$ and its nonlinear composite $\exp[u(x)]$, we have

$$\exp(u(x)) \approx \sum_k \exp(u(x_k))\phi(2^n x - k) \tag{12}$$

where $x_k = (k + M_1)/2^n$. After performing weighted integration, algebraic equations containing $\exp(u(x_k))$can thus be obtained. Then unknown coefficients $u(x_k)$ can be determined by solving the nonlinear algebraic equations. This analysis shows that approximation in the form of Eqs. (8) and (9) is very useful when applying Galerkin method to solve differential/integral equations with nonlinear terms.

1.2 Approximating functions defined on a bounded interval

As we know, orthogonal scaling functions originally form a function basis on the whole real line. If one wants to use them in the solution of a boundary value problem defined on a bounded interval by simply taking restrictions of each basis function to the interval, some instability problems may arise, so one needs to introduce extra treatments to avoid this drawback. Typical treatments include zero-, symmetric- and periodic- extensions near the boundaries of the interval, which usually introduce an artificial "jump" to the function value or its derivatives at the edges. In this paper, we consider a natural extension treatment on the function defined on a finite interval by using Taylor series expansion at each boundary [Wang (2001)]. This method can make the extension to be smooth enough. In addition, boundary values and boundary derivatives of the function to be approximated can be explicitly embedded in the resulting scaling function expansions, when boundary conditions need to be imposed during the solution of a differential equation.

Consider a function $g(x) \in L^2[0,b]$ without loss of generality. Applying Taylor series expansions at each side of the interval $[0,b]$ leads to

$$g(x) = \begin{cases} \sum_{i=0}^{M} \frac{1}{i!} \frac{d^i g(0)}{dx^i} x^i & x \in (-\infty,\ 0) \\ g(x)\ x \in [0,\ b] & \\ \sum_{i=0}^{M} \frac{1}{i!} \frac{d^i g(b)}{dx^i} (x-b)^i & x \in (b,\ \infty) \end{cases} \tag{13}$$

In the case that some of the values $d^i g(0)/dx^i$ and $d^i g(b)/dx^i$ are known by the boundary conditions or/and initial conditions if applicable, we can simply insert them into Eq. (13). For those unknown derivatives at each border, applying numerical difference with equidistant knots at endpoints 0, b gives

$$\frac{d^i g(0)}{dx^i} = \sum_{k=0}^{\alpha} \zeta_{L,i,k} g(\frac{k}{2^n}), \quad \frac{d^i g(b)}{dx^i} = \sum_{k=0}^{\alpha} \zeta_{R,i,k} g(b - \frac{k}{2^n}) \tag{14}$$

where $\zeta_{L,i,k}, \zeta_{R,i,k}$ are the coefficients associated with the numerical difference. In the case that $\alpha = 3$, $M = 3$, according to the four-point-Malkoff numerical difference formulae [Sweldens (1995)], these coefficients can be given by

$$\begin{pmatrix} \zeta_{L,0} \\ \zeta_{L,1} \\ \zeta_{L,2} \\ \zeta_{L,3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{11}{6} & 3 & -\frac{3}{2} & \frac{1}{3} \\ 2 & -5 & 4 & -1 \\ -1 & 3 & -3 & 1 \end{pmatrix}, \quad \begin{pmatrix} \zeta_{R,0} \\ \zeta_{R,1} \\ \zeta_{R,2} \\ \zeta_{R,3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{11}{6} & -3 & \frac{3}{2} & -\frac{1}{3} \\ 2 & -5 & 4 & -1 \\ 1 & -3 & 3 & -1 \end{pmatrix} \tag{15}$$

where $\zeta_L = \{2^{-in}\zeta_{L,i,k}\}$, $\zeta_R = \{2^{-in}\zeta_{R,i,k}\}$, $i,k = 0,1,2,3$. Substituting the induced results or Eq. (15) into Eq. (14) then the resulting results into Eq. (13), one gets

$$g(x) = \begin{cases} \sum_{k=0}^{3} g(\frac{k}{2^n}) T_{L,k}(x, \boldsymbol{\beta}_L), & x \in (-\infty,\ 0) \\ g(x),\ x \in [0,\ b] & \\ \sum_{k=0}^{3} g(b - \frac{k}{2^n}) T_{R,k}(x, \boldsymbol{\beta}_R), & x \in (b,\ \infty) \end{cases} \tag{16}$$

in which

$$T_{L,k}(x) = \sum_{i=0}^{M} \beta_{L,i} \frac{\zeta_{L,i,k}}{i!} x^i, \quad T_{R,k} = \sum_{i=0}^{M} \beta_{R,i} \frac{\zeta_{R,i,k}}{i!} (x-b)^i. \tag{17}$$

$$\boldsymbol{\beta}_L = \{\beta_{L,i}\}, \quad \boldsymbol{\beta}_R = \{\beta_{R,i}\}, \quad i = 0,1,2,3.$$

We note that the introduction of $\boldsymbol{\beta}_L$ and $\boldsymbol{\beta}_R$ aims to assign boundary conditions to the function $g(x)$. For example, in the case that $x=0$: $d^i g(0)/dx^i = 0$, and other boundary derivatives are unknown, we can just need to set $\beta_{L,i} = 0$, and all other elements of $\boldsymbol{\beta}_L$ and $\boldsymbol{\beta}_R$ equal to 1.

Under the consideration of extension function introduced above, we use the extension function of Eq. (16) into the wavelet transform of Eq. (8). Then, one obtains

$$g(x) \approx \sum_{k=2-3N+M_1}^{b2^n+M_1-1} g\left(\frac{k}{2^n}\right) \phi(2^n x + M_1 - k). \tag{18}$$

Further rearrangement gives

$$g(x) \approx \sum_{k=0}^{b2^n} g_{n,k} \Phi_{b,n,k}(x) \tag{19}$$

where

$$g_{n,k} = g(\frac{k}{2^n}), \tag{20}$$

$$\Phi_{b,n,k}(x) = \begin{cases} \phi(2^n x + M_1 - k) + \sum\limits_{j=2-3N+M_1}^{-1} T_{L,k}(\frac{j}{2^n}, \boldsymbol{\beta}_L)\phi(2^n x + M_1 - j), & 0 \le k \le 3 \\ \phi(2^n x + M_1 - k), & 4 \le k \le b2^n - 4 \\ \phi(2^n x + M_1 - k) + \sum\limits_{j=1+b2^n}^{b2^n+M_1-1} T_{R,b2^n-k}(\frac{j}{2^n}, \boldsymbol{\beta}_R)\phi(2^n x + M_1 - j), & b2^n - 3 \le k \le b2^n \end{cases} \tag{21}$$

Eq. (19) provides a modified wavelet expansion of a function defined on a bounded interval with specified boundary values and derivatives. The corresponding coefficients appeared in Eq. (17) is just the function samplings at each nodal point, which is very convenient for dealing with nonlinear differential equations.

2 Solution method

By using the modified wavelet approximation scheme for functions defined on bounded intervals developed in the previous section, here, we consider the numerical solutions of both fractional initial-value and initial-boundary-value problems.

2.1 Initial-value problem

For simplicity and without loss of generality, we consider a single degree of freedom vibration system with fractional derivative type damping and a nonlinear restoring force. Governing equation and initial conditions for such a system can be given by [Enelund and Olsson (1999)]

$$c_1\ddot{y}(t)+c_2D_t^{\alpha}y(t)+g(y(t))=f(t),\ t>0, \tag{22}$$

$$y(0)=y_0,\ \dot{y}(0)=y_1 \tag{23}$$

where $c_1\ddot{y}(t)$ represents the inertia force, $c_2D_t^{\alpha}y(t)$ is the damping force, $f(t)$ is the external excitation, $g(y(t))$ is the nonlinear restoring force, $0<\alpha<2$, and $D_t^{\alpha}y(t)$ is the Caputo fractional derivative of $y(t)$, defined as [Caputo and Mainardi (1971)]

$$D_t^{\alpha}y(t)=\begin{cases} y^{(n)}(t), & \alpha=n\in\mathbf{N} \\ \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{y^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau, & n-1<\alpha<n \end{cases} \tag{24}$$

in which $y^{(n)}(t)=d^ny(t)/dt^n$, and $\Gamma(\cdot)$ is the Gamma function. By denoting the Laplace transform of $y(t)$ by $Y(s)$, i.e., $L[y(t)]=Y(s)$, the Laplace transform of $D_t^{\alpha}y(t)$ can be expressed as [Caputo and Mainardi (1971)]

$$L[D_t^{\alpha}y(t)]=s^{\alpha}L[y(t)]-\sum_{m=0}^{n-1}s^{\alpha-1-m}y^{(m)}(0^+),\ n-1<\alpha\le n,\ n\in\mathrm{N}. \tag{25}$$

Applying Laplace transform to the governing Eq. (22) and taking the initial conditions of Eq. (23) into account, we have

$$(c_1s^2+c_2s^{\alpha})Y(s)+L[g(y(t))-f(t)]=\sum_{n=0}^{1}c_1s^ny_n+\sum_{n=0}^{[\alpha]}c_2s^{\alpha-1-n}y_n, \tag{26}$$

or

$$Y(s)+R(s)\{L[g(y(t))-f(t)]\}=Q(s) \tag{27}$$

in which $R(s)=1/(c_1s^2+c_2s^{\alpha})$, $Q(s)=R(s)(\sum_{n=0}^{1}c_1s^ny_n+\sum_{n=0}^{[\alpha]}c_2s^{\alpha-1-n}y_n)$, $[\alpha]$ represents the maximal integer less than α, and Eq. (25) has been used.

Applying inverse Laplace transform to Eq. (27), we get

$$y(t)+\int_0^t r(t-\tau)[g(y(\tau))-f(\tau)]d\tau=q(t). \tag{28}$$

From the Ref. [Hong, Kim and Wang (2006)], we know

$$r(t) = L^{-1}[R(s)] = \frac{1}{c_1} t E_{2-\alpha,2}(-\frac{c_2}{c_1} t^{2-\alpha}), \quad q(t) = L^{-1}[Q(s)] \tag{29}$$

where $E_{2-\alpha,2}(-c_2 t^{2-\alpha}/c_1)$ is the generalized Mittag–Leffler type functions defined in the power series

$$E_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+b)}. \tag{30}$$

It can be seen from Eq. (29) that the integral kernel $r(t)$ is a non-singular smooth function with property of $r(0) = 0$ when $0 < \alpha < 2$.

Applying Eq. (19) to approximate the term $r(t-\tau)[g(y(\tau)) - f(\tau)]$ with $\tau \in [0,t]$ in Eq. (28), we have

$$r(t-\tau)[g(y(\tau)) - f(\tau)] \approx \sum_{k=0}^{t2^n} r(t-t_k)[g(y_k) - f_k]\Phi_{t,n,k}(\tau) \tag{31}$$

where $t_k = k/2^n$, $g(y_k) = g(y(t_k))$, and $f_k = f(t_k)$. Integrating both sides of Eq. (31), we obtain

$$\int_0^t r(t-\tau)[g(y(\tau)) - f(\tau)]d\tau \approx \sum_{k=0}^{t2^n} r(t-t_k)[g(y_k) - f_k]\Phi^{\int}_{t,n,k}(t) \tag{32}$$

where $\Phi^{\int}_{t,n,k}(t_i) \equiv \int_0^{t_i} \Phi_{t,n,k}(\tau)d\tau$ can be exactly obtained according to Zhou and Wang [Liu, Wang and Zhou (2011)]. Inserting Eq. (32) into Eq. (28), and then set $t = t_i$ yields

$$y_i + \sum_{k=0}^{i} r_{i-k}[g(y_k) - f_k]\Phi^{\int}_{t_i,n,k}(t_i) \approx q_i \tag{33}$$

Where $y_i = y(t_i)$, $t_i = i/2^n$ $(i = 1,2,3,...)$, $r_{i-k} = r(t_i - t_k)$, and $q_i = q(t_i)$. According to Eq. (29), there should be $r(0) = 0$ for $0 < \alpha < 2$, which implies that $r(t_i - t_k)[g(y_k) - f_k]\Phi^{\int}_{t_i,n,k}(t_i) = 0$ when $k = i$, thus, Eq. (33) can be further simplified into the form

$$y_i \approx \sum_{k=0}^{i-1} r_{i-k}[f_k - g(y_k)]\Phi^{\int}_{t_i,n,k}(t_i) + q_i. \tag{34}$$

It can be seen from Eq. (34), the solution y_i can be directly obtained step-by-step as the index i increases. In this process, no matrix inversion is needed.

2.2 Initial-boundary-value problem

To verify if the above method is still valid for the solution of a fractional differential equation with both space and time variables, here, we consider a fractional diffusion-wave equation with a nonlinear term and an inhomogeneous source terms as follows

$$c_1\frac{\partial^2}{\partial t^2}u(x,t)+c_2D_t^{\alpha}u(x,t)-\frac{\partial^2 u}{\partial x^2}+\chi(x,t,u)=\psi(x,t),\ \ 0<x<b,\ t>0, \tag{35}$$

with the initial and boundary conditions,

$$u(x,0)=g_0(x),\ u_t(x,0)\ =g_1(x),\ 0\le x\le b, \tag{36}$$

$$u(0,t)=0,\ u(b,t)=0,\ t\ge 0 \tag{37}$$

where $0<\alpha<2$, c_1 and c_2 are constants, $\chi(x,t,u)$ stands for a nonlinear term of unknown function $u(x,t)$, and $\psi(x,t)$ is an inhomogeneous source term.

Similar to our previous treatment for the initial value problem, we denote the Laplace transform of $u(x,t)$ by $U(x,s)$, i.e., $L[u(x,t)]=U(x,s)$. Applying Laplace transform to the fractional diffusion-wave Eq. (35) and taking into account the initial conditions in Eq. (36), we have

$$(c_1s^2+c_2s^{\alpha})U(x,s)-L[H(x,t)]=\sum_{n=0}^{1}c_1s^ng_n(x)+\sum_{n=0}^{[\alpha]}c_2s^{\alpha-1-n}g_n(x) \tag{38}$$

Where we denote

$$H(x,t)=\frac{\partial^2}{\partial x^2}u(x,t)-\chi(x,t,u)+\psi(x,t). \tag{39}$$

Further, Eq. (38) can be rewritten as

$$U(x,s)-R(s)\{L[H(x,t)]\}=Q(x,s). \tag{40}$$

By denoting $Q(x,s)=R(s)(\sum_{n=0}^{1}c_1s^ng_n(x)+\sum_{n=0}^{[\alpha]}c_2s^{\alpha-1-n}g_n(x))$, and $R(s)=1/(c_1s^2+c_2s^{\alpha})$. Apply inverse Laplace transform to Eq. (40), we have

$$u(x,t)-\int_0^t r(t-\tau)H(x,\tau)d\tau=q(x,t) \tag{41}$$

where $r(t)$ is given by Eq. (29), which is a non-singular smooth function with property of $r(0)=0$ due to $0<\alpha<2$ and $q(x,t)=L^{-1}[Q(x,s)]$. Obviously, function $q(x,t)$ can be easily obtained once the initial conditions in Eq. (36) are specified.

Using Eq. (29) to approximate the function $r(t-\tau)H(x,\tau)$ with $\tau \in [0,t]$, we have

$$r(t-\tau)H(x,\tau) \approx \sum_{k=0}^{t2^n} r(t-t_k)H(x,t_k)\Phi_{t,n,k}(\tau). \tag{42}$$

Inserting Eq. (42) into Eq. (41), and set $t = t_i$, we obtain

$$u(x,t_i) \approx \sum_{k=0}^{i-1} r_{i-k}H(x,t_k)\Phi^{\int}_{t_i,n,k}(t_i) + q(x,t_i). \tag{43}$$

On the other hand, we have

$$u(x,t_k) \approx \sum_{l=1}^{b2^m-1} u(x_l,t_k)\Phi_{b,m,l}(x) \tag{44}$$

where $x_l = l/2^m$, and $\beta_{L,0} = 0$, $\beta_{R,0} = 0$ and all other elements of $\boldsymbol{\beta}_L$ and $\boldsymbol{\beta}_R$ being equal to 1 have been assigned into Eqs. (19) and (21) to specify $\Phi_{b,m,l}(x)$. Denote

$$H_{j,k} \equiv H(x_j,t_k) \approx \sum_{l=1}^{b2^m-1} u_{l,k}\Phi^{''}_{b,m,l}(x_j) + \psi(x_j,t_k) - \chi(x_j,t_k,u_{j,k}) \tag{45}$$

where $u_{l,k} = u(x_l,t_k)$, $f^{''}(x) \equiv d^2f(x)/dx^2$. We further have

$$u_{j,i} \approx \sum_{k=0}^{i-1} r_{i-k}H_{j,k}\Phi^{\int}_{t_i,n,k}(t_i) + q_{j,i}. \tag{46}$$

To clearly show the method applied in the time-space problem, let us summarize the procedure of the method as follows

When $t = 0$ or $i = 0$, we have $u_{j,0} = g_0(x_j)$.

When $t = 1/2^n$ or $i = 1$, we have

$$H_{j,0} \approx \sum_{l=1}^{b2^m-1} u_{l,0}\Phi''_{b,m,l}(x_j) + \psi(x_j,t_0) - \chi(x_j,t_0,u_{j,0}),$$

$$u_{j,1} \approx r_1H_{j,0}\Phi^{\int}_{1/2^n,n,0}(\frac{1}{2^n}) + q_{j,1}. \tag{47}$$

When $t = i_0/2^n$ or $i = i_0$, we have already known $u_{j,i<i_0}, H_{j,i<i_0-1}$, and

$$H_{j,i_0-1} \approx \sum_{l=1}^{b2^m-1} u_{l,i_0-1}\Phi''_{b,m,l}(x_j) + \psi(x_j,t_{i_0-1}) - \chi(x_j,t_{i_0-1},u_{j,i_0-1}),$$

thus, we can obtain

$$u_{j,i_0} \approx \sum_{k=0}^{i_0-1} r_{i_0-k} H_{j,k} \Phi^{f}_{t_{i_0},n,k}(t_{i_0}) + q_{j,i_0}. \tag{48}$$

The process a), b), c) can be proceeded to any time t.

It can be seen that such a solution procedure decouples the complicated spatial-temporal dependences, making the solution of the nonlinear fractional differential Eqs. (35-37) very simple. And no inversion of matrix and special solution technique are needed to deal with the nonlinear spatial and temporal operators.

3 Numerical examples

In this section we shall give a few numerical examples to demonstrate the efficiency of the proposed method.

Example 1: We consider the following fractional equation given by Li [Li (2010)],

$$D^2 y(t) + D^{1.234} y(t) + D^{0.333} y(t) + y^3(t) = f(t),\ t > 0, \tag{49}$$

$$y(0) = 0, \dot{y}(0) = 0, \tag{50}$$

where

$$f(t) = \frac{2}{\Gamma(2)} t + \frac{2}{\Gamma(2.766)} t^{1.766} + \frac{2}{\Gamma(3.667)} t^{2.667} + (\frac{1}{3} t^3)^3,$$

and the exact solution $y(t) = \frac{1}{3} t^3$. Eq. (49) can be converted into the form of Eq. (27) and the corresponding convolution kernel $r(t)$ can be calculated as

$$r(t) = L^{-1}[R(s)], \quad R(s) = \frac{1}{s^2 + s^{1.234} + s^{0.333}}. \tag{51}$$

Here, we adopt an efficient and robust method of Laplace inversion via wavelet expansion of functions in the transform domain developed by Wang et al. [Wang, Zhou and Gao (2003)]. This wavelet-based approach of Laplace inversion has been justified by successful applications in vibration problems associated with randomness [Koziol and Hryniewicz (2006); Koziol, Cristinel and Ibrahim (2008); Hryniewicz (2011)] and fractional damping [Liu, Wang, Wang and Zhou (2011); Wang, Zhou and Gao (2003)]. For Eq. (51), we have

$$r(t) = \lim_{q \to \infty} \left\{ \frac{e^t}{2^{q+1} \pi} \hat{\phi}(-\frac{t}{2^q}) \sum_{k=-\infty}^{\infty} R(1 + i\frac{M_1 + k}{2^q}) e^{itk/2^q} \right\} \tag{52}$$

where $\hat{\phi}(\xi)$ is the Fourier transform of the scaling function $\phi(x)$, which can be obtained by a product formula as shown in [Wang, Zhou and Gao (2003)].

By considering Eqs. (34) and (52), we obtain

$$y_i \approx \sum_{k=0}^{i-1} r_{i-k}(f_k - y_k^3)\Phi^{\int}_{t_i,n,k}(t_i) \tag{53}$$

where $r(t)$ is given by Eqs. (51) and (52), and $\Phi_{t,n,k}(t)$ is given by Eq. (21).

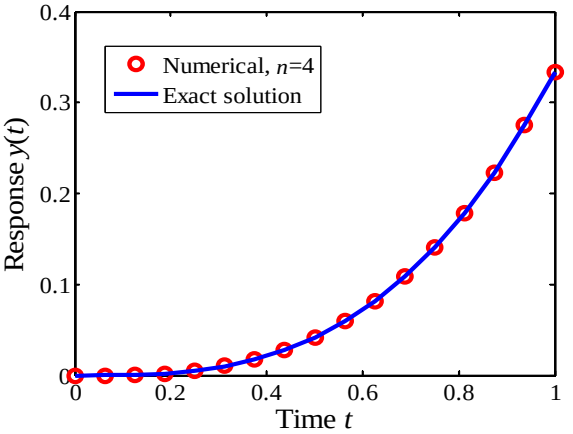


Figure 2: Comparison between the numerical and exact solutions of Example 1.

Fig. 2 shows the comparison between the exact result of this problem and its numerical approximation according to Eq. (53) when $n = 4$ corresponding to a time step of 1/16. It can be seen from Fig. 2 that the approximate result almost coincides with the exact one. Li [Li (2010)] solved the same equation by using a method based on the so-called Chebyshev wavelets. Such a method needs numerical solution of a set of nonlinear algebraic equations. In [Li (2010)], the absolute error of the approximate solution at $t = 0.5$ obtained by numerically solving 48 nonlinear algebraic equations is 1.269211e-04, which is larger than the absolute error by using Eq. (53) when $n \geq 4$ as shown in Table 2. We should note that the approximate solution Eq. (53) does not involve any inversion of matrixes and solutions of algebraic equations, saving a lot of computing time.

Example 2: Following Wang et al. [Wang, Hong and Huang (2002)], we consider a single degree of freedom nonlinear vibration system with fractional damping

$$\mathbf{D}^2 y(t) + \frac{8}{10}\mathbf{D}^{1/2} y(t) + y^3(t) = f(t),\ t > 0, \tag{54}$$

Table 2: Absolute error of numerical solutions of Example 1 under different resolution level n.

t	$n=4$	$n=5$	$n=6$
0.25	6.1702945e-05	1.0564871e-06	1.9098283e-07
0.50	5.2252928e-06	2.1596250e-06	9.0243531e-08
0.75	8. 2326125e-06	4.2263352e-06	8.2589488e-08
1.00	1.0274991e-05	7.6263066e-06	1.1919017e-07

$$y(0)=0, \dot{y}(0)=0 \tag{55}$$

where

$$f(t)=2(t-\tfrac{9}{10})(t-\tfrac{7}{10})+4t(t-\tfrac{7}{10})+4t(t-\tfrac{9}{10})+2t^2+ \frac{8}{10\Gamma(1/2)}(\tfrac{128}{35}\sqrt{t^7}-\tfrac{128}{25}\sqrt{t^5}+\tfrac{42}{25}\sqrt{t^3})+[t^2(t-\tfrac{9}{10})(t-\tfrac{7}{10})]^3 \ .$$

For this equation, one easily verifys that the exact solution is of the form $y(t)=t^2(t-9/10)(t-7/10)$ [Wang, Hong and Huang (2002)]. Following the solution procedure suggested above, we convert Eq. (54) into the form of Eq. (34). The corresponding convolution kernel $r(t)$ in Eq. (34) can be given by Eq. (28) with parameters $c_1=1, c_2=8/10, \alpha=1/2$, which is

$$r(t) \ = tE_{1.5,2}(-\frac{8}{10}t^{1.5}). \tag{56}$$

Then, the approximate solution of Eqs. (54) and (55) can be expressed as

$$y_i \approx \sum_{k=0}^{i-1} r_{i-k}(f_k-y_k^3)\Phi^{\int}_{t_i,n,k}(t_i). \tag{57}$$

Fig. 3 shows the comparison between the exact result and the approximate result obtained by Eq. (57) for $n=7$. A good agreement between the exact and approximate results can be observed from Fig. 3. Saha Ray et al. [Saha Ray, Chaudhuri and Bera (2006)] also solved Eqs. (54) and (55) by using the modified Adomain Decomposition Method (ADM), and Atanackovic et al. [Atanackovic and Stankovic (2008)] solved the same equation by using a different numerical scheme with an expansion formula for fractional derivative. Table 3 gives a quantitative comparison between the exact results and some existing approximate results. From Table 3, we can see that the approximate results given in this paper have accuracy of at

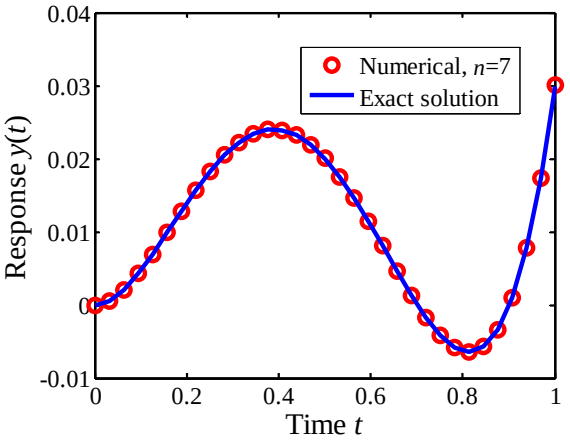


Figure 3: Comparison between the numerical and exact solutions of Example 2.

Table 3: Comparison of numerical solutions of Example 2 with the exact results.

t	Exact	This paper	Saha Ray et al. (2006)	Atanackovic et al. (2008)
0.25	0.01828130	0.01828127	0.0182813	0.0182531908
0.50	0.02000000	0.02000003	0.020026	0.0198515236
0.75	-0.00421875	-0.00421871	-0.0419593	-0.004492587
1.00	0.03000000	0.0300000	0.0300995	0.0293800105

least 5 effective numbers, and the other results obtained by, e.g., Saha Ray et al. [Saha Ray, Chaudhuri and Bera (2006)] and Atanackovic et al. [Atanackovic and Stankovic (2008)] have only two effective numbers.

Example 3: As has been discussed in Rashidinia et al. [Rashidinia, Ghasemi and Jalilian (2010)], we consider a nonlinear fractional diffusion-wave equation.

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \mu \frac{\partial^2 u}{\partial x^2} + \chi(x,t,u) = \psi(x,t),\ 0 \le \mathrm{x} \le 1, 1 < \alpha \le 2, \mathrm{t} > 0 \tag{58}$$

which subjects to the initial and boundary conditions

$$u(x,0) = 0, u_t(x,0) = 0,\ 0 \le x \le 1, \tag{59}$$

$$u(0,t) = 0, u(1,t) = t^3,\ \mathrm{t} \ge 0, \tag{60}$$

where $\chi(x,t,u) = u^2$, $\psi(x,t) = 6tx^3 + (tx)^6 - 6t^3x$, $\mu = 1$, when $\alpha = 2$, the exact solution of Eqs. (58)-(60) in a closed form is $u(x,t) = x^3t^3$. In order to

change the inhomogeneous boundary condition in this example into homogeneous, we introduce an auxiliary function $v(x,t)$ that satisfies $u(x,t) = v(x,t) + x^5t^3$. Inserting $u(x,t) = v(x,t) + x^5t^3$ into Eqs. (58)-(60) yields an equation of unknown function $v(x,t)$ with homogeneous boundary conditions and a new source term $\tilde{\psi}(x,t) = 6x^3t + (xt)^6 - 6xt^3 - 6x^5t + 20x^3t^3$. For such a reduced equation of $v(x,t)$, we have

$$r(t) = L^{-1}[\frac{1}{s^\alpha}] = \frac{t^{-1+\alpha}}{\Gamma(\alpha)}, \tag{61}$$

and

$$v_{j,i} \approx \sum_{k=0}^{i-1} r_{i-k} H_{j,k} \Phi^{\int}_{t_i,n,k}(t_i) \tag{62}$$

in which $H_{j,k} \approx \sum_{l=1}^{2^m-1} v_{l,k} \Phi''_{1,m,l}(x_j) + \tilde{\psi}_{j,k} - (v_{j,k} + x_j^5 t_k^3)^2$, $\tilde{\psi}_{j,k} = \tilde{\psi}(x_j, t_k)$.

Fig. 4 (a) and (b) show, respectively, the calculated time responses at space point, $x = 0.25$, and the displacement configurations at the instant $t = 0.25$ under different fractional order α. Fig. 4 also shows the comparison of the exact results for the case of $\alpha = 2$ to corresponding numerical results when wavelet resolution levels for time- and space- variables chosen as $n = 8$ and $m = 4$, respectively. From Fig. 4, we find that the time-response (displacement configuration) increases as the factor α decreases from $\alpha = 2$ through $\alpha = 1.9$ to $\alpha = 1.75$, and the approximate solution for $\alpha = 2$ is in good agreement with the exact solution of $u(x,t) = x^3t^3$.

Example 4: We consider a nonlinear fractional Sine-Gordon equation with a non-polynomial nonlinear term as follows

$$D_t^\alpha u - u_{xx} + \sin u = f(x,t),\ 0 \le x \le 1, t > 0. \tag{63}$$

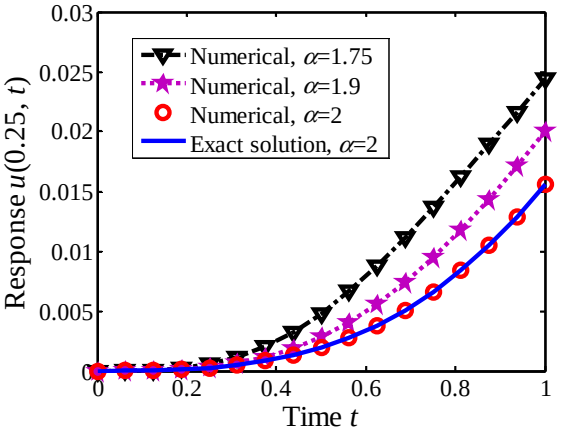
In the case that $\alpha = 1.8$, $f(x,t) = \frac{10}{\Gamma(0.2)} t^{0.2} \sin(\pi x) + \pi^2 t^2 \sin(\pi x) + \sin(t^2 \sin(\pi x))$, and the initial and the boundary conditions

$$u(x,0) = 0,\ u_t(x,0) = 0,\ 0 \le x \le 1, \tag{64}$$

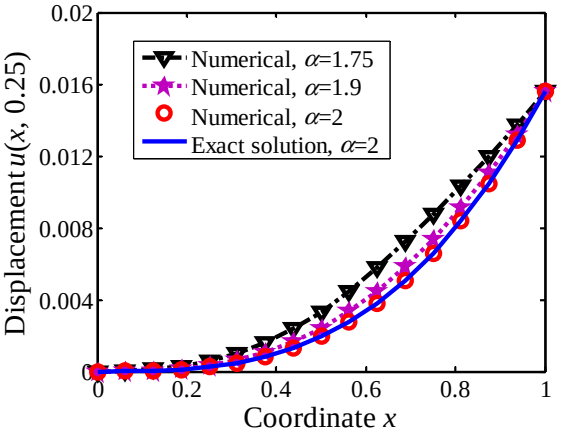
$$u(0,t) = 0,\ u(1,t) = 0,\ t \ge 0, \tag{65}$$

we can obtain an analytical solution $u(x,t) = t^2 \sin(\pi x)$ that satisfies Eqs. (63-65). Following a similar procedure as that has been applied to Eqs. (58-60), we can obtain the numerical solution of Eqs. (63-65) as

$$u_{j,i} \approx \sum_{k=0}^{i-1} r_{i-k} H_{j,k} \Phi^{\int}_{t_i,n,k}(t_i) \tag{66}$$

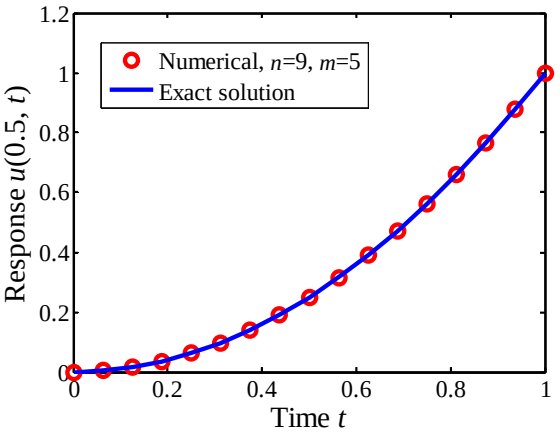


(a) Time-responses at space point $x = 0.25$

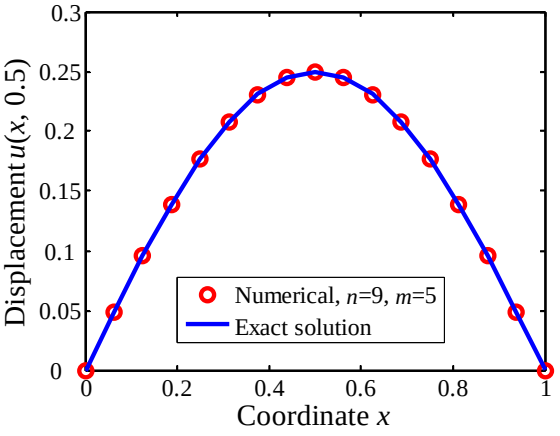


(b) Displacement configurations at instant $t = 0.25$

Figure 4: Numerical solutions of Example 3 for $\alpha = 1.75$, $\alpha = 1.9$, $\alpha = 2$ under scaling factors $n = 8, m = 4$.

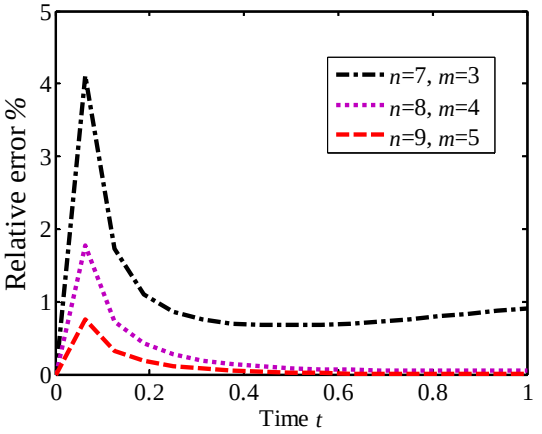


(a) Time-response at point $x = 0.5$

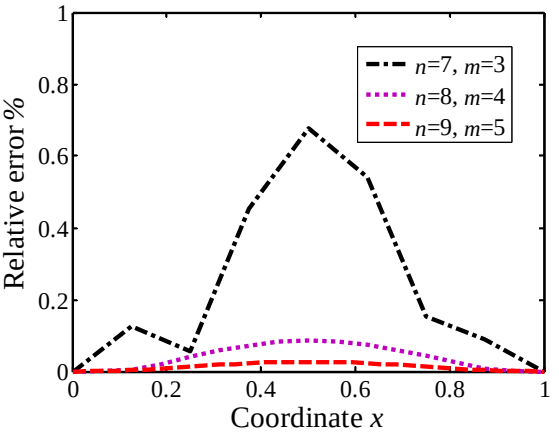


(b) Displacement configuration at instant $t = 0.5$

Figure 5: Comparison between the numerical and exact solutions of Example 4.



(a) For the time-response at point $x = 0.5$



(b) For the displacement configuration at instant $t = 0.5$

Figure 6: Relative error of the wavelet-based numerical solutions comparing with the exact solution to Example 4.

where $r(t) = t^{0.8}/\Gamma(1.8)$, $H_{j,k} \approx \sum_{l=1}^{2^m-1} u_{l,k}\Phi''_{1,m,l}(x_j) + f_{j,k} - \sin(u_{j,k})$ and $f_{j,k} = f(x_j,t_k)$.

Fig. 5 gives the comparison of numerical results obtained in terms of Eq. (66) under parameter choice of $n = 9$, $m = 5$ to the corresponding exact solution at space point $x = 0.5$ and instant $t = 0.5$. The relative error is below 1% as demonstrated by Fig. 6.

4 Conclusions

In this study, we have proposed a numerical method for the solution of a class of nonlinear fractional differential equations originated from engineering problems related to phenomena of vibration, diffusion and wave motion etc. This method has been realized by using a Coiflets-based function approximation scheme which enables nonlinear terms in an equation being explicated expressed, and by applying Laplace transform which converts the fractional differential equations with singular integral kernels into equivalent integral equations with non-singular convolution kernels, $r(t)$. The property of the non-singular integral kernel, $r(0) = 0$, together with the Coiflets-based function expansion scheme, allow us to establish a solution procedure that decouples the complex spatial and temporal dependencies in the equation. Such a solution procedure does not involve any matrix inversions and can be implemented like the linear multi-step method for initial value problems. We note that the method proposed in the present study can also be applied to more general nonlinear fractional differential equations.

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