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Application of the Differential Transform Method for Solving Periodic Solutions of Strongly Non-linear Oscillators

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Abstract: This paper presents the application of the differential transform method to solve strongly nonlinear equations with cubic nonlinearities and self-excitation terms. First, the equations are transformed by the differential transform method into the algebra equations in terms of the transformed functions. Secondly, the higher-order transformed functions are calculated in terms of other lower-order transformed functions through the iterative procedure. Finally, the solutions are approximated by the n-th partial sum of the infinite series obtained by the inverse differential transform. Two strongly nonlinear equations with different coefficients and initial conditions are given as illustrative examples.

Keywords: Taylor series, differential transform, nonlinear equation

1 Introduction

Perturbation methods have been the main methods to solve the weakly nonlinear equations of the following form [Nayfeh (2000)]

$$\ddot{x}+c_1x=\varepsilon g(\mu,x,\dot{x}), \tag{1}$$

where c_1 is a positive constant, ε is a small positive parameter, g is a polynomial function, and μ is a control parameter. Classical perturbation methods such as the harmonic balance (HB) method, the Lindstedt-Poincare (LP) method, the Krylov-Bogolioubov-Mitropolski (KBM) method, the averaging method and the multiple-scale method (MSM) are widely used to obtain approximate periodic solutions of Eq. (1). However, the aforementioned methods are seldom used to solve strongly nonlinear equations with nonlinear terms in unperturbed systems. The main reason

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is that it may become too complicated and difficult when applying the perturbation methods to strongly nonlinear equations. The Jacobian elliptic functions are introduced in many classic perturbation methods to extend their applications to strongly nonlinear systems. For example, Lakrad and Belhaq (2000) presented the generalized elliptic multiple scales method to solve the strongly nonlinear equation as below

$$\ddot{x}+c_1x+c_2f(x)=\varepsilon g(\mu,x,\dot{x}), \tag{2}$$

where c_1, c_2 are constants, $f(x)$ includes quadratic or cubic terms and g is a polynomial function. The solutions are expressed in terms of Jacobian elliptic functions.

The integral transform methods such as Laplace transform and Fourier transform are also widely used for solving engineering problems by transforming governing differential equations into algebraic equations which are easier to deal with [Tsai and Chen (2009)]. However, they often require an elaborate process when applying these integral transform techniques to nonlinear systems. The differential transformation method provides a different approach to solve strongly nonlinear equations [Chen and Liu (1998); Chen and Chen (2009); Ho and Chen (2006); Jang, Wang and Shie (2004)]. The differential transform method is a numerical method based on Taylor's series expansion. It can construct analytical solutions in the form of polynomials. Not like traditional high order Taylor series method that requires symbolic computation, the differential transform method relies on an iterative procedure to obtain Taylor's series solutions. In this paper, the differential transform method is applied to solve the strongly nonlinear equations in the form of Eq. (2) with different coefficients and initial conditions. The characteristics of solutions for each case are illustrated and discussed.

2 Differential Transform Method

The differential transform method is applied mainly to solve initial value problems. The basic principles of the differential transformation method are briefly described as follows: Let $x(t)$ be an analytic function in the time domain D. The Taylor series expansion of $x(t)$ is of the form

$$x(t)=\sum_{k=0}^{\infty}\frac{(t-t_0)^k}{k!}\left(\frac{d^kx(t)}{dt^k}\right)_{t=t_0}, \quad \forall t\in D. \tag{3}$$

When $t_0=0$, Eq. (3) is called the Maclaurin series of $x(t)$ which takes the form of

$$x(t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}\left(\frac{d^kx(t)}{dt^k}\right)_{t=0}, \quad \forall t\in D. \tag{4}$$

The differential transform of $x(t)$, denoted by $X(k)$ or $T[x(t)]$, is defined as

$$X(k) = T[x(t)] = \frac{H^k}{k!}\left(\frac{d^k x(t)}{dt^k}\right)_{t=0}, \quad k = 1,2,3,\cdots \quad \forall k \in K, \tag{5}$$

where K is the set of non-negative integers. $X(k)$ is the transformed function, also called the spectrum function, of $x(t)$ in the transformation domain, or the K domain. $H$is the time interval. Furthermore, the original function $x(t)$, also called the inverse transform of $X(k)$, is expressed, by substituting Eq.(5) into Eq. (4). as

$$x(t) = \sum_{k=0}^{\infty}\left(\frac{t}{H}\right)^k X(k) \tag{6}$$

or

$$x(t) = \sum_{k=0}^{n}\left(\frac{t}{H}\right)^k X(k). \tag{7}$$

Nevertheless, in practice, $x(t)$ is usually approximated by the n-th partial sum of power series in Eq. (6). The concept of the differential transform method is derived from the Taylor series expansion. Therefore, let $X(k) = T[x(t)]$ and $Y(k) = T[y(t)]$, the differential transforms of basic function operations below follow immediately from the definition of the Taylor series expansion

Addition:

$$T[x(t) + y(t)] = X(k) + Y(k) \tag{8}$$

Derivative:

$$T\left[\frac{dx^m(t)}{dt^m}\right] = \frac{(k+1)(k+2)\cdots(k+m)}{H^m} X(k+m) \tag{9}$$

Convolution:

$$T[x(t)y(t)] = X(k) * Y(k) = \sum_{l=0}^{k} X(l)Y(k-l) \tag{10}$$

Basically, the procedure of applying the differential transform method to solving differential equations includes three major steps. Step 1: The differential equations to be solved are transformed by the differential transform method from the time domain into the transformation domain. The resultant equations are algebra equations in terms of the transformed function $X(k)$. Step 2: The values of the transformed function $X(k)$ for each k are calculated through an iterative procedure, starting with the first few values determined directly from initial conditions. Step 3: The original function $x(t)$ is approximated through the inverse differential transform.

3 Numerical Procedure

Two second-order nonlinear differential equations with cubic nonlinearity and different initial conditions are solved in the present study as numerical examples. The solving procedures are described for both examples.

Example 1:

$$\ddot{x}+x^3=\varepsilon_1\dot{x}+\varepsilon_2x^2\dot{x}, \tag{11}$$

where ε_1, ε_2 are constants. The initial conditions are $x(0)=\alpha$, $\dot{x}(0)=\beta$, where α,β are constants. Apply the differential transform to Eq. (11) with respect to time to obtain the transformed equation in the following iterative form

$$\begin{aligned}&\frac{(k+1)(k+2)}{H^2}X(k+2)+\sum_{l=0}^{k}X(k-l)\sum_{m=0}^{l}X(l-m)X(m)\\&=\varepsilon_1\frac{k+1}{H}X(k+1)+\varepsilon_2\sum_{l=0}^{k}X(k-l)\sum_{m=0}^{l}X(l-m)\frac{m+1}{H}X(m+1)\end{aligned} \tag{12}$$

The above equation gives the algebraic relationship between the transformed functions $X(k)$ for $k=1,2,\cdots,k+2$. The highest order transformed function, $X(k+2)$, is expressed in terms of other lower order transformed functions as

$$\begin{aligned}X(k+2)=&\frac{H^2}{(k+1)(k+2)}\Big(-\sum_{l=0}^{k}X(k-l)\sum_{m=0}^{l}X(l-m)X(m)+\varepsilon_1\frac{k+1}{H}X(k+1)\\&+\varepsilon_2\sum_{l=0}^{k}X(k-l)\sum_{m=0}^{l}X(l-m)\frac{m+1}{H}X(m+1)\Big)\end{aligned} \tag{13}$$

The iterative procedure based on the above equation can be used to calculate $X(k)$ for $k>2$ once the first two transformed functions, $X(0)$ and $X(1)$, are determined directly from the following initial conditions

$$X(0)=\alpha \quad \text{and} \quad X(1)=H\beta. \tag{14}$$

After the first n+1 transformed functions are solved, the solution of $x(t)$ is approximated by the n-th partial sum of Eq. (6) as

$$x(t)=\sum_{k=0}^{n}\left(\frac{t}{H}\right)^kX(k) \tag{15}$$

Usually, it is necessary to increase n to achieve a good approximation when the time interval H is large. In practical applications, in order to avoid a large n value and speed up the convergence rate, the entire time domain is often split into several sub-intervals such that the solutions are solved sequentially in each sub-interval. The values of the transformed function at the end of previous sub-interval are then used as the initial conditions of the next sub-interval. By repeating the similar procedure, the differential equation over the entire time domain can be solved.

Example 2:

$$\ddot{x} - x + x^3 = \varepsilon_1 \dot{x} + \varepsilon_2 x^2 \dot{x}, \tag{16}$$

where ε_1 and ε_2 are also arbitrary constants. The initial conditions are $x(0) = \alpha$, $\dot{x}(0) = \beta$, where α, β are constants. Apply the differential transform to Eq. (16) with respect to time to obtain the transformed equation in the following iterative form

$$\begin{aligned} &\frac{(k+1)(k+2)}{H^2} X(k+2) - X(k) + \sum_{l=0}^{k} X(k-l) \sum_{m=0}^{l} X(l-m) X(m) \\ &= \varepsilon_1 \frac{k+1}{H} X(k+1) + \varepsilon_2 \sum_{l=0}^{k} X(k-l) \sum_{m=0}^{l} X(l-m) \frac{m+1}{H} X(m+1) \end{aligned} . \tag{17}$$

Again, the above equation can be rearranged in the way such that the highest order transformed function, $X(k+2)$, is expressed in terms of other lower order transformed functions as

$$\begin{aligned} X(k+2) = &\frac{H^2}{(k+1)(k+2)} (X(k) - \sum_{l=0}^{k} X(k-l) \sum_{m=0}^{l} X(l-m) X(m) \\ &+ \varepsilon_1 \frac{k+1}{H} X(k+1) + \varepsilon_2 \sum_{l=0}^{k} X(k-l) \sum_{m=0}^{l} X(l-m) \frac{k+1}{H} X(m+1)) \end{aligned} \tag{18}$$

The rest of procedures are similar to those of example 1.

4 Results and Discussions

Two self-excitation terms are included in the nonlinear equations, Eq. (11) and Eq. (16), of both examples. Since no external forces exist in both examples, the responses of these two equations solely depends on their initial conditions. The time response and the phase plot of the solutions of example 1 with $\varepsilon_1 = 0$, $\varepsilon_2 = 0$ and initial conditions $\alpha = 1$, $\beta = 0$ are shown in Fig. 1 and Fig. 2, respectively. Initial point (α, β) is defined by the initial conditions $x(0) = \alpha$, $\dot{x}(0) = \beta$. Fig. 3

is the phase plot of the solution of example 1 with the same coefficients, $\varepsilon_1 = 0$, $\varepsilon_2 = 0$, but with different initial conditions, $\alpha = 0.5$, $\beta = 0$. Fig. 2 and Fig. 3 show the phase plot within the same time interval, $0 \le t \le 10$. A smaller initial value α reduces the size and the oscillating frequency of the solution curves. As shown in Fig. 2 and Fig. 3, within the same period of time, the solution curve for $\alpha = 1$ forms a larger loop and goes about one and a half cycles around the loop, on the other hand, the solution curve for $\alpha = 0.5$ can not circle around a smaller loop completely. Fig. 4 shows the phase plot of example 1 with large initial values, $\alpha = 5$, $\beta = 5$. As expected, the solution curve forms a large loop and encircles it many times when $0 \le t \le 10$. The folds of the path are due to the discrete sampling and regenerating of the data curve.

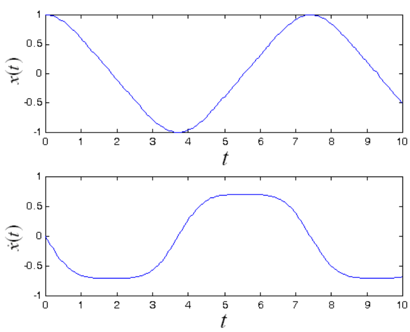

Figure 1: Example 1, time response with $\varepsilon_1 = 0$, $\varepsilon_2 = 0$ and $\alpha = 1$, $\beta = 1$.

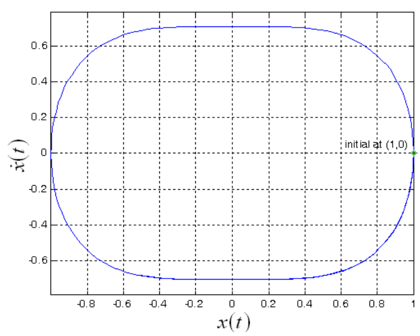

Figure 2: Example 1, phase plot with $\varepsilon_1 = 0$, $\varepsilon_2 = 0$ and $\alpha = 1$, $\beta = 0$ for $0 \le t \le 10$.

Fig. 5-8 show the solution curves of example 1 with different ε_1, ε_2. Although they all start from the same initial point (1,0), the solution curves exhibit quite different characteristics. For $\varepsilon_1 = -1$ and $\varepsilon_2 = 0$, the solution converges to the origin gradually from the initial point (1,0) as shown in Fig. 5. Fig.6 shows the solutions goes along a twisted path from the initial point outward to infinite when $\varepsilon_1 = 1$, $\varepsilon_2 = 0$. If $\varepsilon_1 = 0$, $\varepsilon_2 = -1$, the solution converges to the origin again as shown in Fig. 7. But, unlike the solution in Fig. 5, in which the solution move directly toward to the origin in the third quadrant, the solution encircles the origin and moves along a spiral curve toward the origin. The computation time span is $0 \le t \le 5$ in Fig. 6 and $0 \le t \le 100$ in Fig.7. It is noted that, hereafter, in order to clearly demonstrate the characteristics of the solutions, different computation time span may be used in each individual figure. Fig.8 is the phase plot of the solution of example 1 with $\varepsilon_1 = 1$, $\varepsilon_2 = -1$ and initial conditions, $\alpha = 1$, $\beta = 0$. The solution

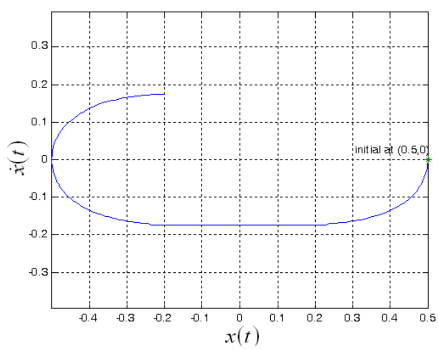

Figure 3: Example 1, phase plot with $\varepsilon_1 = 0$, $\varepsilon_2 = 0$ and $\alpha = 0.5$, $\beta = 0$ for $0 \leq t \leq 10$.

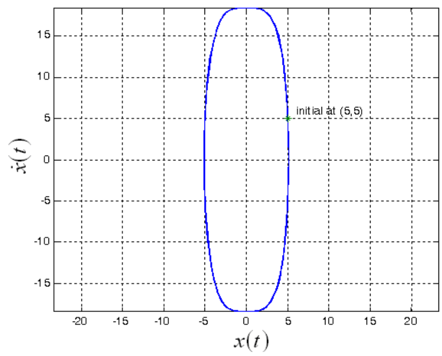

Figure 4: Example 1, phase plot with $\varepsilon_1 = 0$, $\varepsilon_2 = 0$ and $\alpha = 5$, $\beta = 5$ for $0 \leq t \leq 10$.

curve converges to a twisted limit orbit from inside. In Fig. 9, if the initial point is moved outside of the limit orbit, say, $\alpha = 5$, and $\beta = 5$, then the solution curves, starting from the initial point (5,5), converges from outside toward the same limit orbit as in Fig.8.

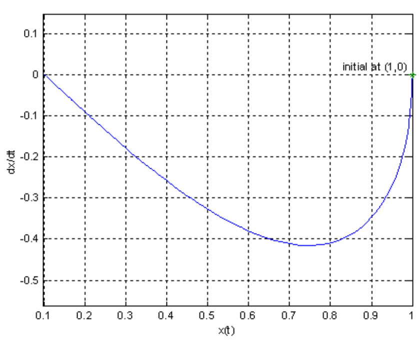

Figure 5: Example 1, phase plot with $\varepsilon_1 = -1$, $\varepsilon_2 = 0$ and $\alpha = 1$, $\beta = 0$, $0 \leq t \leq 10$

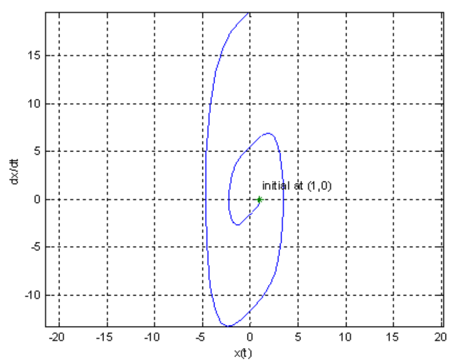

Figure 6: Example 1, phase plot with $\varepsilon_1 = 1$, $\varepsilon_2 = 0$ and $\alpha = 1$, $\beta = 0$, $0 \leq t \leq 5$

The phase plots of example 2 are shown in Fig. 10-18 with different constants ε_1, $\varepsilon_{2,}$.and initial conditions α, β. Example 2 has trivial solutions if the initial conditions are chosen to be $\alpha = 1$, $\beta = 0$ or $\alpha = -1$, $\beta = 0$ because Eq. (15) reduces to $\ddot{x} = \dot{x} = 0$. No system response will occur under such initial conditions. Fig. 10 and Fig. 11 show two phase plots with $\varepsilon_1 = 0$, $\varepsilon_2 = 0$. Initial conditions are

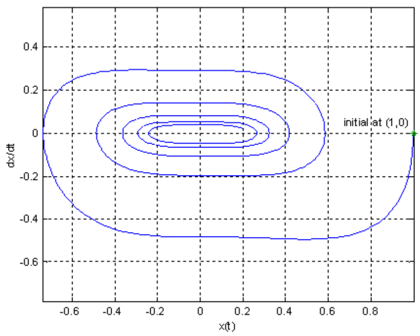

Figure 7: Example 1, phase plot with $\varepsilon_1 = 0$, $\varepsilon_2 = -1$ and $\alpha = 1$, $\beta = 0$, $0 \leq t \leq 100$

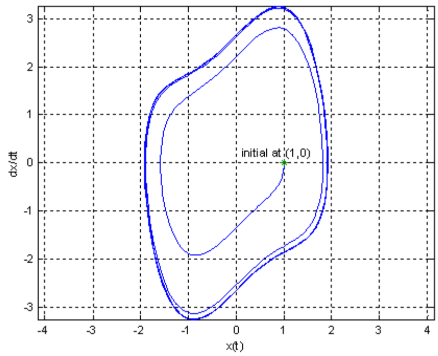

Figure 8: Example 1, phase plot with $\varepsilon_1 = 1$, $\varepsilon_2 = -1$ and $\alpha = 1$, $\beta = 0$, $0 \leq t \leq 20$

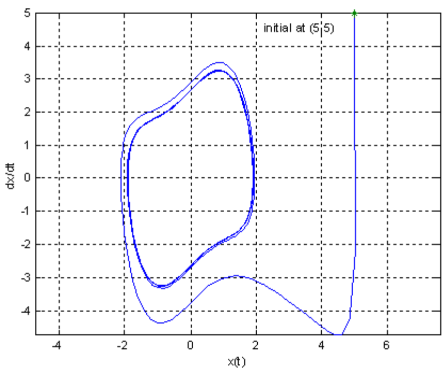

Figure 9: Example 1, phase plot with $\varepsilon_1 = 1$, $\varepsilon_2 = -1$ and $\alpha = 5$, $\beta = 5$, $0 \leq t \leq 20$.

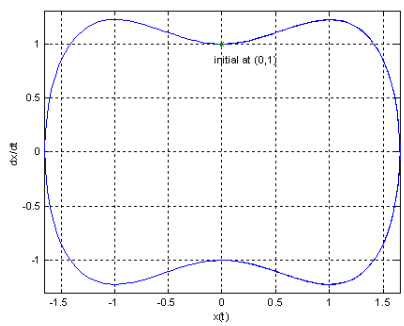

Figure 10: Example 2, phase plot with $\varepsilon_1 = 0$, $\varepsilon_2 = 0$ and $\alpha = 0$, $\beta = 1$, $0 \leq t \leq 10$.

$\alpha = 0$, $\beta = 1$ and $\alpha = 0.2$, $\beta = 0.2$ in Fig. 10 and Fig.11, respectively. The solution curves make a periodic motion along the close loops that are symmetric about the horizontal axis ($dx/dt = 0$). This results are obvious since the acceleration,$\ddot{x}$, is the function of x only. In Fig. 11, the center region shrinks to a small neck. Under certain conditions, the solution curves will break into two parts at $x = 0$. Fig.12 shows such a result for $\alpha = 0.5$, $\beta = 0$ that the solution curve remains in the region $x > 0$. It is expected that if the initial condition is changed to $\alpha = -0.5$, $\beta = 0$, the solution will follow a reflection path of Fig.12 with respect to the vertical axis ($x = 0$).

Fig. 13 shows the phase plot for coefficients $\varepsilon_1 = -1$, $\varepsilon_2 = 0$, and Fig.14 shows that for $\varepsilon_1 = 0$, $\varepsilon_2 = -1$, respectively. The solution curves converge to different points with different coefficients initial conditions. In Fig. 13, the solution curve, starting from the initial point (0,1), converges to the point (1,0), following a spiral path. In Fig. 14, if the initial point moves to (-2,1), the solution curve converges to a different point (0,-1) in the similar way. Fig. 15 and Fig. 16 show that the solution curves starting from the same initial point (0,1) converge to different limit orbits. When $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$, the solution curve moves outward to approach the twisted orbit in Fig. 15. On the other hand, when $\varepsilon_1 = -1$ and $\varepsilon_2 = 1$, the solution curve approaches a slightly deformed elliptic orbit from outside in Fig. 16.

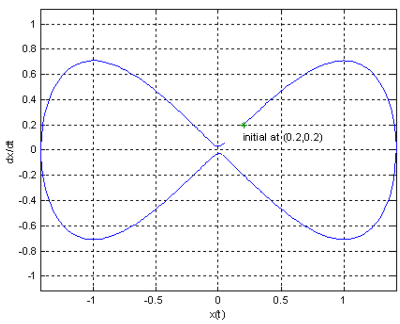

Figure 11: Example 2, phase plot with $\varepsilon_1 = 0$, $\varepsilon_2 = 0$ and $\alpha = 0.2$, $\beta = 0.2$, $0 \le t \le 20$.

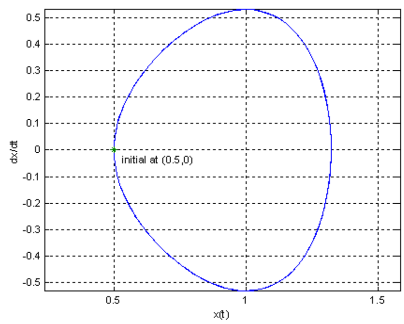

Figure 12: Example 2, phase plot with $\varepsilon_1 = 0$, $\varepsilon_2 = 0$ and $\alpha = 0.5$, $\beta = 0.2$, $0 \le t \le 10$.

Fig.17-18 show, starting with different initial conditions, the phase plots of Eq. (15) with smaller coefficients,$\varepsilon_1 = -1$, $\varepsilon_2 = 0$. If the initial conditions are $\alpha = 0$ and $\beta = 1$, the solution at first moves along a shrinking orbit, which has a small necking area at the center near x=0 and encircles the origin. Then, later on, the solution curve shifts to the left, approaching a limit egg-shaped orbit that encircles the point (-1,0), shown in Fig. 17. The phase plot with the initial conditions $\alpha = 0$, $\beta = 2$ is shown in Fig. 18. However, it is interesting that, instead of converging inward to a limit orbit, the solution moves outward if the initial point moves to (0,2) from (0,1). There seems to have a saddle line between these two sets of solutions shown in Fig.17 and Fig.18.

The solutions of the above two examples demonstrate the ability of the differential transform method to solve strongly nonlinear equations with cubic nonlinearities and self-excitation terms. The differential transformation method uses the itera-

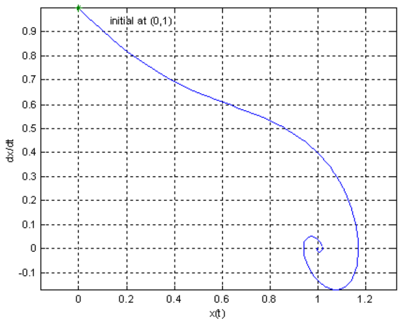

Figure 13: Example 2, phase plot with $\varepsilon_1 = -1$, $\varepsilon_2 = 0$ and $\alpha = 0$, $\beta = 1$, $0 \leq t \leq 20$.

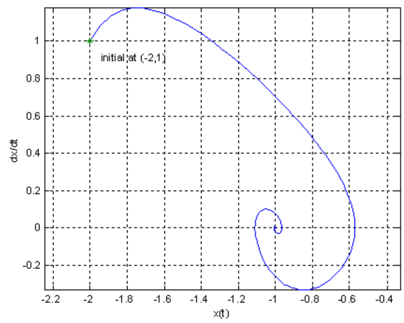

Figure 14: Example 2, phase plot with $\varepsilon_1 = 0$, $\varepsilon_2 = -1$ and $\alpha = -2$, $\beta = 1$, $0 \leq t \leq 20$.

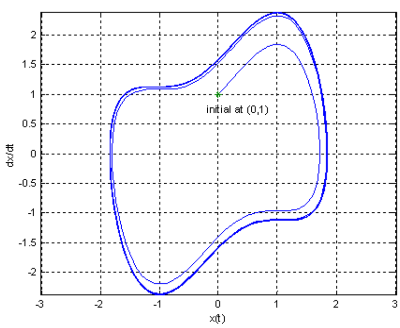

Figure 15: Example 2, phase plot with $\varepsilon_1 = 1$, $\varepsilon_2 = -1$ and $\alpha = 0$, $\beta = 1$, $0 \leq t \leq 100$.

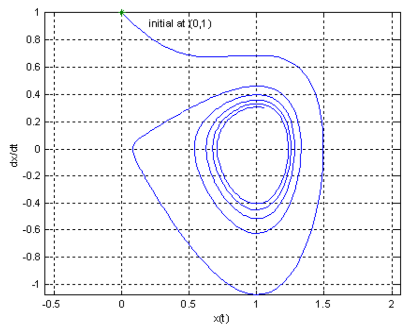

Figure 16: Example 2, phase plot with $\varepsilon_1 = -1$, $\varepsilon_2 = 1$ and $\alpha = 0$, $\beta = 1$, $0 \leq t \leq 30$.

tive process that consumes very short computing time when dealing with ordinary differential equations. The number of iterations required for convergent solutions depends on the complexity of solutions and the length of time increments. For average problems, the solutions may converge in about ten to twenty iterations. In the present two examples, the iteration number of differential transform is 30, i.e., the order of Taylor expansion of the approximation solution is n=30. To increase the convergence rate in the present examples, the time domain is divided into several sub-intervals. The values of x(t) and $\dot{x}(t)$ in at the end of each sub-interval are adopted as the initial values of the next domain. The time increment of each interval is chosen to be $\Delta t = 0.05$ or $\Delta t = 0.02$. If the solution goes to infinite as shown in Fig. 6, a smaller time increment is necessary. Otherwise, the time increment doesn't have to be very small to achieve good results. This paper shows that the differential transformation method can solve these nonlinear equations effectively. This method is a useful tool in the investigation of the behavior of strongly nonlinear equations.

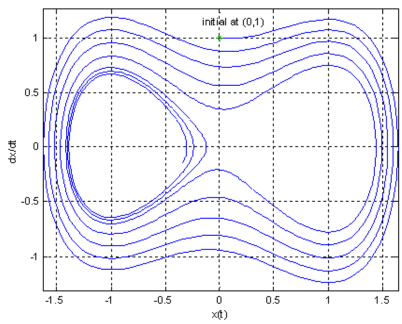

Figure 17: Example 2, phase plot with $\varepsilon_1 = -1$, $\varepsilon_2 = 0.1$ and $\alpha = 0$, $\beta = 1$, $0 \le t \le 60$.

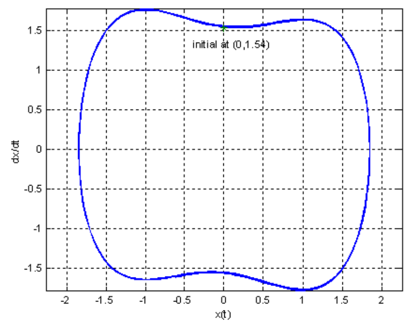

Figure 18: Example 2, phase plot with $\varepsilon_1 = -0.1$, $\varepsilon_2 = 0.1 =$ and $\alpha = 0$, $\beta = 2$, $0 \le t \le 20$.

5 Conclusions

The differential transformation method is used to solve two strongly nonlinear equations with cubic nonlinearities and self-excitation terms. Different coefficients and initial conditions are adopted to verify the capability of this method. It is shown that the solutions of the strongly nonlinear equations are very sensitive to their initial conditions. Also shown is that the differential transformation method can solve these nonlinear equations effectively.

References

Chen, C.L.; Liu, Y.C. (1998): Solution of two-boundary-value problems using the differential transformation method. *J. Optimiz. Theory App.*, vol. 99, pp. 23-35.

Chen, S.S.; Chen, C.K. (2009): Application of the differential transformation method to the free vibrations of strongly non-linear oscillators. *Nonlinear Analysis*, vol. 10 , pp. 881-888.

Ho, S.H.; Chen, C.K. (2006): Free transverse vibration of an axially loaded non-uniform spinning twisted Timoshenko beam using differential transform. *Int. J. Mech. Sci.* vol. 48, pp. 1323-1331.

Jang, M.J.; Wang, J.S; Shie, H.J. (2004): Using two-dimensional differential transformation to solve the conduction type problems. J. Chin. Soc. Mech. Eng. vol. 25(4), pp. 343-351.

Lakrad, F.; **Belhaq, M.** (2000): Periodic Solutions of Strongly Non-linear Oscillators by the Multiple Scales Method. *J. Sound Vib,* vol. 258, pp. 677-700.

Nayfeh, A.H. (2000): *Perturbation Methods*, John Wiley & Son, New York.

Tsai, P.Y.; Chen, C.K. (2009): Application of the hybrid Laplace Adomian decomposition method to the nonlinear oscillatory systems. J. Chin. Soc. Mech. Eng. vol. 30(6), pp. 493-501.