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Meshless Unsteady Thermo-Elastoplastic Analysis by Triple-Reciprocity Boundary Element Method

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Abstract: In general, internal cells are required to solve unsteady thermo-elastoplastic problems using a conventional boundary element method (BEM). However, in this case, the merit of BEM, which is the easy preparation of data, is lost. The conventional multiple-reciprocity boundary element method (MRBEM) cannot be used to solve thermo-elastoplastic problems, because the distribution of initial stress cannot be determined analytically. In this paper, it is shown that two-dimensional unsteady thermo-elastoplastic problems can be solved without the use of internal cells by using the triple-reciprocity BEM and a thin plate spline. The initial stress formulation is adopted and the initial stress distribution is interpolated using boundary integral equations and a thin plate spline. A new computer program was developed and applied to several problems.

Keywords: Boundary Element Method, Elastoplasticity, Thermal Stress, Computational Mechanics, Initial Stress

1 Introduction

Elastoplastic problems can be solved by a conventional boundary element method (BEM) using internal cells for domain integrals [Brebbia (1984), Wrobel (2002)]. In this case, however, the merit of BEM, which is ease of data preparation, is lost. On the other hand, several countermeasures to prevent this lose have been considered. For example, Nowak and Neves (1994) proposed the conventional multiple-reciprocity boundary element method (MRBEM). In the conventional MRBEM, the distribution of initial stress must be given analytically, and fundamental solutions of higher order are used to make solutions converge. Accordingly, this method is not suitable for thermo-elastoplastic analysis. Dual-reciprocity BEM has been proposed to reduce the dimensionality, which is an advantage of BEM [Partridge (1992)]. However, it is difficult to apply the dual-reciprocity BEM to thermo-elastoplastic problems with arbitrary heat generation. Sladek (2007) applied the lo-

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cal integral equation method to elastoplastic problems without internal cells. Ochiai and Kobayashi (1999, 2001a) proposed triple-reciprocity BEM (improved multiple-reciprocity BEM) without using internal cells for elastoplastic problems. By this method, a highly accurate solution can be obtained using only fundamental solutions of low orders and by reducing the need for data preparation. They applied triple-reciprocity BEM without using internal cells to two-dimensional elastoplastic problems using initial strain formulations. Ochiai (2010, 2011) applied triple-reciprocity BEM to two-dimensional thermo-elastoplastic problems with arbitrary heat generation and three-dimensional elastoplastic problems using initial strain formulations. Only the triple-reciprocity BEM and the local integral equation method have been applied to elastoplastic problems without internal cells.

In the previous papers (2010, 2011), steady thermal stress was assumed. In this study, triple-reciprocity BEM is applied to two-dimensional thermo-elastoplastic problems involving unsteady thermal stress with arbitrary heat generation. The initial stress formulations is adopted and the theory is expressed using a small number of fundamental solutions. In this method, boundary elements and arbitrary internal points are used. The arbitrary distributions of initial stress used for elastoplastic analysis are interpolated using boundary integral equations and internal points. This interpolation corresponds to a thin plate spline. In this method, the strong singularities that appear in the calculation of stresses at internal points become weak. A new computer program is developed and applied to several thermo-elastoplastic problems to clearly demonstrate the theory. A time-dependent solution is used to calculate unsteady heat conduction. The temperature distribution is used for the calculation of unsteady thermal stress. In this method, which does not use a time-dependent solution for thermal stress, internal points are necessary; however, they are necessary for elastoplastic analysis. This method is demonstrated to be efficient for calculation.

2 Theory

2.1 Unsteady heat conduction

In unsteady heat conduction problems with heat generation $W_1^S(q,\tau)$, temperature T is obtained by solving

$$\nabla^2 T + \frac{W_1^S}{\lambda} = \kappa^{-1}\frac{\partial T}{\partial t} \tag{1}$$

where κ and t are the thermal diffusivity and time, respectively. Denoting an arbitrary time and the initial temperature by τ and $T_1^{0S}(q)$, respectively, the boundary integral equation for the temperature in the case of unsteady heat conduction is expressed by [Brebbia (1984), Wrobel (2002)]

$$cT(P,t) = -\kappa \int_0^t \int_\Gamma \left[T(Q,\tau)\frac{\partial T_1^*(P,Q,t,\tau)}{\partial n} - \frac{\partial T(Q,\tau)}{\partial n} T_1^*(P,Q,t,\tau) \right] d\Gamma d\tau$$

$$+\kappa\lambda^{-1} \int_0^t \int_\Omega T_1^*(P,q,t,\tau)\, W_1^S(q,\tau)\, d\Omega d\tau + \int_\Omega T_1^*(P,q,t,0)\, T_1^{0S}(q)\, d\Omega, \tag{2}$$

where c=0.5 on the smooth boundary and c=1 in the domain. λ is the heat conduction coefficient. Superscript S denotes a distributed value. Γ and Ω represent the boundary and the domain, respectively, and p and q become P and Q on the boundary. In two-dimensional problems, the time-dependent fundamental solution $T_1^*(q,p,t,\tau)$ in Eq.(2) used for unsteady temperature analysis and its normal derivative are given by

$$T_1^*(p,q,t,\tau) = \frac{1}{4\pi\kappa(t-\tau)} \exp(-a) \tag{3}$$

$$\frac{\partial T_1^*(p,q,t,\tau)}{\partial n} = \frac{-r}{8\pi\kappa^2(t-\tau)^2} \frac{\partial r}{\partial n} \exp(-a), \tag{4}$$

where

$$a = \frac{r^2}{4\kappa(t-\tau)}. \tag{5}$$

Here, r is the distance between the observation point p and the loading point q. As shown in Eq.(2), when there exists an arbitrary distribution of initial temperature $T_1^{0S}(q,0)$ or heat generation $W_1^S(q,\tau)$ in a domain, the domain integral is necessary. Therefore, the triple-reciprocity BEM [Ochiai (2003, 2004)] is used to avoid the need for internal cells.

Two different numerical procedures can be employed for the numerical solution of Eq.(2). One method requires internal cells. In this paper, internal points are used. At the end of each time step, the temperature at a sufficient number of internal points must be computed for use as the initial temperature distribution in the next time step. In the proposed method, the temperature distributions in some time steps are assumed to be the initial temperature distribution and are interpolated using integral equations and internal points. The deformation of an imaginary thin plate is utilized to interpolate the distribution of $W^{[1]S}$ in the two-dimensional case. The following equations can be used for the two-dimensional interpolation of the heat generation distribution $W_1^S(p,\tau)$ and initial temperature distribution $T_1^{0S}(p)$:

$$\nabla^2 W_1^S(q,\tau) = -W_2^S(q,\tau), \tag{6}$$

$$\nabla^2 W_2^S(q,\tau) = -\sum_{m=1}^{M} W_3^P(q_m,\tau)\delta(q-q_m), \tag{7}$$

$$\nabla^2 T_1^{0S}(q) = -T_2^{0S}(q), \tag{8}$$

$$\nabla^2 T_2^{0S}(q) = -\sum_{m=1}^{M} T_3^{0P}(q_m)\delta(q-q_m), \tag{9}$$

where M is the number of interpolation points. On the other hand, the polyharmonic function $T_f^*(p,q,t,\tau)$ in the unsteady heat conduction problem is defined by

$$\nabla^2 T_{f+1}^*(p,q,t,\tau) = T_f^*(p,q,t,\tau). \tag{10}$$

Using Green's theorem twice and Eqs. (6)-(9), Eq.(2) becomes [Ochiai (2004, 2009)]

$$\begin{aligned}
cT(P,t) &= -\kappa\int_0^t\int_\Gamma [T(Q,\tau)\frac{\partial T_1^*(P,Q,t,\tau)}{\partial n} - \frac{\partial T(P,Q)}{\partial n}T_1^*(P,Q,t,\tau)]d\Gamma d\tau \\
&+\sum_{f=1}^{2}(-1)^f\int_\Gamma [T_{f+1}^*(P,Q,t,0)\frac{\partial T_f^{0S}(Q)}{\partial n} - \frac{\partial T_{f+1}^*(P,Q,t,0)}{\partial n}T_f^{0S}(Q)]d\Gamma + \\
&+\sum_{m=1}^{M} T_3^{0P}(q_m)T_3^*(P,q_m,t,0) + \sum_{f=1}^{2}(-1)^f\int_0^t\int_\Gamma [T_{f+1}^*(P,Q,t,\tau)\frac{\partial W_f^S(Q,\tau)}{\partial n} - \\
&-\frac{\partial T_{f+1}^*(P,Q,t,\tau)}{\partial n}W_f^S(Q,\tau)]d\Gamma d\tau + \sum_{m=1}^{M}\int_0^t W_3^P(q_m,\tau)T_3^*(P,q_m,t,\tau)d\tau.
\end{aligned} \tag{11}$$

Using Eq.(10), the two-dimensional polyharmonic function $T_f^*(P,q,t,\tau)$ in Eq.(11) is obtained from

$$T_{f+1}^*(p,q,t,\tau) = \int\frac{1}{r}\int rT_f^*(p,q,t,\tau)drdr. \tag{12}$$

in the unsteady state and its normal derivative are concretely given by

$$T_2^*(p,q,t,\tau) = \frac{1}{4\pi}[E_1(a)+\ln(a)+C], \tag{13}$$

$$\frac{\partial T_2^*(p,q,t,\tau)}{\partial n} = \frac{1}{2\pi r}\frac{\partial r}{\partial n}[1-\exp(-a)], \tag{14}$$

$$T_3^*(p,q,t,\tau) = \frac{r^2}{16\pi}[E_1(a)+\ln(a)+C+\frac{1-\exp(-a)}{a}+\frac{E_1(a)+\ln(a)+C}{a}-2], \tag{15}$$

$$\frac{\partial T_3^*(p,q,t,\tau)}{\partial n} = \frac{r}{8\pi}\frac{\partial r}{\partial n}[E(a)+\ln(a)+C-1+\frac{1-\exp(-a)}{a}], \tag{16}$$

where $E_1()$ is the exponential integral function and C is Euler's constant.

Numerical solutions are obtained by using the interpolation functions for time and space. If a constant time interpolation and time step $(t_k - t_{k-1})$ are used, the time integral can be treated analytically. The actual time is T_F, and the time integrals for $T_f^*(P,q,t,\tau)$ are given as follows:

$$\int_{t_f}^{t_F} T_1^*(P,q,t,\tau)d\tau = \frac{1}{4\pi\kappa}E_1(a_f), \tag{17}$$

$$\int_{t_f}^{t_F} T_2^*(P,q,t,\tau)d\tau = \frac{r^2}{16\kappa\pi}\left[E_1(a_f)+\frac{1}{a_f}\left\{E_1(a_f)-\exp(-a_f)+\ln(a_f)+1+C\right\}\right], \tag{18}$$

$$\int_{t_f}^{t_F} T_3^*(P,q,t,\tau)d\tau = \frac{r^4}{256\kappa\pi}\left[E_1(a_f)+\frac{4E_1(a_f)+4\ln(a_f)-\exp(-a_f)-4+4C}{a_f}\right.$$
$$\left.+\frac{2E_1(a_f)+2\ln(a_f)-3\exp(-a_f)+3+2C}{a_f^2}\right], \tag{19}$$

with $a_f = r^2/4\kappa(t_F - t_f)$.

2.2 Thermo-elastoplastic analysis

To analyze thermo-elastoplastic problems using the initial stress formulation, the following boundary integral equation must be solved.

$$c_{ij}\dot{u}_j(P) = \int_\Gamma [u_{ij}^{[1]}(P,Q)\dot{p}_j(Q) - p_{ij}(P,Q)\,\dot{u}_j(Q)]\,d\Gamma + \int_\Omega \varepsilon_{ijk}^{[1]}(P,q)\,\dot{\sigma}_{Ijk}^{[1]}(q)\,d\Omega$$
$$+\int_\Omega T(q)\nabla^2 u_i^{[1]}(P,q)d\Omega \tag{20}$$

Here, $\dot{\sigma}_{Ijk}^{[1]}(q)$ is the initial stress rate, which is treated iteratively, and is the free coefficient. Moreover, and are the jth components of the displacement and surface traction rates, respectively. Γ and Ω are the boundary and domain, respectively. In this formulation, the concept of the thermoelastic displacement potential is used. As shown in Eq. (20), when there is an arbitrary initial strain rate, a domain integral becomes necessary. Denoting the distance between the observation point and loading point as r, the function $u_i^{[1]}(p,q)$ is given by

$$u_i^{[1]}(p,q) = \frac{-m_0 r r_{,i}}{8\pi}[2\ln(r)+1], \tag{21}$$

where α is the linear coefficient of thermal expansion, $m_0 = (1+\nu)\alpha$ for plane stress problems, and $m_0 = (1+\nu)\alpha/(1-\nu)$ for plane strain problems. δ_{ij} is the Kroneker delta function and $\dot{T}$ is the temperature increment. c_{ij} is $0.5\delta_{ij}$ on a smooth boundary, and δ_{ij} at an internal point. Kelvin's solutions, namely, $u_{ij}^{[1]}(p,q)$ and $p_{ij}(p,q)$, are given by

$$u_{ij}^{[1]}(p,q) = \frac{1}{8\pi(1-\bar{\nu})G}[(3-4\bar{\nu})\,\delta_{ij}\ln(\frac{1}{r}) + r_{,i}r_{,j}], \tag{22}$$

$$p_{ij}(p,q) = \frac{-1}{4\pi(1-\bar{\nu})\,r}[\{(1-2\bar{\nu})\delta_{ij} + 2r_{,i}r_{,j}\}\frac{\partial r}{\partial n} - (1-2\bar{\nu})(r_{,i}n_j - r_{,j}n_i)], \tag{23}$$

where ν is Poisson's ratio and G is the shear modulus and

$$\bar{\nu} = \begin{cases} \nu, & \text{plane strain} \\ \nu/(1+\nu), & \text{plane stress} \end{cases}$$

The ith component of a unit normal vector is denoted by . Moreover, we set . The functions $\varepsilon_{ijk}^{[1]}$ in Eq. (20) are given by

$$\varepsilon_{ijk}^{[1]}(p,q) = [(1-2\bar{\nu})(\delta_{ij}r_{,k} + \delta_{ik}r_{,j}) - \delta_{jk}r_{,i} + 2r_{,i}r_{,j}r_{,k}]\frac{-1}{8\pi(1-\bar{\nu})G\,r}. \tag{24}$$

2.3 Interpolation of initial stress rate and temperature distribution

Interpolation using boundary integrals is introduced to avoid the domain integral in Eq. (20). The distribution of the initial stress rate $\dot{\sigma}_{Ijk}^{[1]}(q)$ is interpolated using the integral equation to transform the domain integral into a boundary integral. Then, the approximation of the initial stress $\dot{\sigma}_{Ijk}^{[1]}(q)$ is denoted as $\dot{\sigma}_{Ijk}^{[1]S}(q)$ and the new field $\dot{\sigma}_{Ijk}^{[2]S}(q)$ as well as the interior nodal point unknowns $\dot{\sigma}_{Ijk}^{[3]P}(q_m)$ are introduced as [Ochiai (2000, 2011, 2003)]

$$\nabla^2\dot{\sigma}_{Ijk}^{[1]S}(q) = -\dot{\sigma}_{Ijk}^{[2]S}(q), \tag{25}$$

$$\nabla^2\dot{\sigma}_{Ijk}^{[2]S}(q) = -\sum_{m=1}^{M}\dot{\sigma}_{Ijk}^{[3]P}(q_m)\delta(q-q_m). \tag{26}$$

From Eqs.(25) and (26), the following equation can be obtained.

$$\nabla^4\sigma_{Ijk}^{[1]S}(q) = \sum_{m=1}^{M}\sigma_{Ijk}^{[3]P}\delta(q-q_m) \tag{27}$$

This equation is similar to the equation for the deformation (T_1^{0S}) of a thin plate with an unknown point load (T_3^{0P}). In this paper, the distribution of initial strain or stress is assumed to be a 2.5-dimensional free-form surface. In this method, each component of initial stress rate $\dot{\sigma}_{Ijk}^{[1]}$ $(j,k=x,y)$ is interpolated.

Similarly, the temperature distribution $T_1^{0S}(q)$ is interpolated as $T_1^S(q)$with introducing the new field $T_2^S(q)$ and a new interior nodal point unknowns $T_3^P(q_m)$ which are governed as

$$\nabla^2 T_1^S(q) = -T_2^S(q), \tag{28}$$

$$\nabla^2 T_2^S(q) = -\sum_{m=1}^{M} T_3^P(q_m)\delta(q-q_m). \tag{29}$$

2.4 Representation of initial stress rate and temperature distribution by integral equations

As will be shown in Sect. 2.5, the domain integrals of the initial stress rate and the temperature field in the integral equation (20) can be converted into boundary integrals of certain nodal unknowns and the interior nodal point unknowns introduced in Sect. 2.3. Now, we derive the boundary integral equations which must be solved for computation of the new unknowns. For this purpose, the higher-order fundamental solutions of the Laplacian operator will be useful

$$T^{[f]}(p,q) = \frac{r^{2(f-1)}}{2\pi[(2f-2)!!]^2}F_f(r), \quad F_f(r) = C_f - \ln r, \tag{30}$$

with $C_{f+1} = C_f + sgn(f)\frac{1}{f}$ or $C_{f+1} = C_0 + sgn(f)\sum_{e=1}^{f}\frac{1}{e}$, $(2f)!! = 2f(2f-2)\cdots 4\cdot 2$ and C_0 is an arbitrary constant.

Beside the fundamental solutions also their derivatives will be required. In advance, we present here all the derivatives which will be needed in this paper

$$T_{,i}^{[f+1]}(p,q) = \frac{r^{2(f-1)}r_i}{2\pi[(2f)!!]^2}(2fF_{f+1}(r)-1) \tag{31}$$

$$T_{,ij}^{[f+1]}(p,q) = \frac{r^{2(f-1)}}{2\pi[(2f)!!]^2}\left\{(2fF_f+1)\delta_{ij} + 2[2f(f-1)F_f-1]r_{,i}r_{,j}\right\} \tag{32}$$

$$T_{,ijk}^{[f+1]}(p,q) = \frac{r^{2f-3}}{2\pi[(2f)!!]^2}\left\{2[2f(f-1)F_f-1](\delta_{ik}r_{,j} + \delta_{jk}r_{,i} + \delta_{ij}r_{,k}) + \right.$$

$$+4[2f(f-1)(f-2)F_f - f^2 + 2]r_{,i}r_{,j}r_{,k}\big\} \tag{33}$$

$$T_{,ijkl}^{[f+2]}(p,q) = \frac{r^{2(f-1)}}{2\pi[(2f+2)!!]^2}\big\{4[2f(f^2-1)F_{f+1} - (f+1)^2 + 2](\delta_{ik}r_{,j}r_{,l} + \delta_{jk}r_{,i}r_{,l} +$$
$$+\delta_{ij}r_{,k}r_{,l} + \delta_{il}r_{,j}r_{,k} + \delta_{jl}r_{,i}r_{,k} + \delta_{kl}r_{,i}r_{,j}) +$$
$$+8(f-1)[2f(f^2-1)F_{f+1} - 2f^2 - 3f + 1]\, r_{,i}r_{,j}r_{,k}r_{,l} +$$
$$+2[2f(f+1)F_{f+1} - 1](\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il} + \delta_{ij}\delta_{kl})\big\} \tag{34}$$

$$T_{,ijklm}^{[f+2]}(p,q) = \frac{r^{2f-3}}{2\pi[(2f+2)!!]^2}\big\{8(f-1)[2f(f^2-1)F_{f+1} - 2f^2 - 3f + 1][(\delta_{ik}r_{,j}r_{,l} +$$
$$+\delta_{jk}r_{,i}r_{,l} + \delta_{ij}r_{,k}r_{,l} + \delta_{il}r_{,j}r_{,k} + \delta_{jl}r_{,i}r_{,k} + \delta_{kl}r_{,i}r_{,j})r_{,m} + (\delta_{im}r_{,j}r_{,k} +$$
$$+\delta_{jm}r_{,i}r_{,k} + \delta_{km}r_{,i}r_{,j})r_{,l} + \delta_{lm}r_{,i}r_{,j}r_{,k}] +$$
$$+4[2f(f^2-1)F_{f+1} - (f+1)^2 + 2](\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il} + \delta_{ij}\delta_{kl})r_{,m} +$$
$$+\delta_{ik}(\delta_{jm}r_{,l} + \delta_{lm}r_{,j} - 2r_{,j}r_{,l}r_{,m}) + \delta_{jk}(\delta_{im}r_{,l} + \delta_{lm}r_{,i} - 2r_{,i}r_{,l}r_{,m}) +$$
$$+\delta_{ij}(\delta_{km}r_{,l} + \delta_{lm}r_{,k} - 2r_{,k}r_{,l}r_{,m}) + \delta_{il}(\delta_{jm}r_{,k} + \delta_{km}r_{,j} - 2r_{,j}r_{,k}r_{,m}) +$$
$$+\delta_{jl}(\delta_{im}r_{,k} + \delta_{km}r_{,i} - 2r_{,i}r_{,k}r_{,m}) + \delta_{kl}(\delta_{im}r_{,j} + \delta_{jm}r_{,i} - 2r_{,i}r_{,j}r_{,m})] +$$
$$+16(f-1)[2f(f-3)(f^2-1)F_{f+1} - 3f^3 + 3f^2 + 11f - 3]\, r_{,i}r_{,j}r_{,k}r_{,l}r_{,m}\big\}. \tag{35}$$

Then, it can be seen that

$$\nabla^2 T^{[f+1]} = T^{[f]}, \quad \nabla^2 T^{[1]} = -\delta(p-q). \tag{36}$$

Making use the kernel functions, one can derive from Eqs. (25) and (26) two boundary integral equations [8-13]

$$c\dot{\sigma}_{Ijk}^{[1]S}(P) = -\sum_{f=1}^{2}(-1)^f \int_\Gamma [T^{[f]}(P,Q)\frac{\dot{\sigma}_{Ijk}^{[f]S}(Q)}{\partial n} - \frac{\partial T^{[f]}(P,Q)}{\partial n}\dot{\sigma}_{Ijk}^{[f]S}(Q)]d\Gamma - .$$
$$-\sum_{m=1}^{M} T^{[2]}(P,q_m)\dot{\sigma}_{Ijk}^{[3]P}(q_m). \tag{37}$$

$$c\dot{\sigma}_{Ijk}^{[2]S}(P) = \int_{\Gamma} [T^{[1]}(P,Q)\frac{\partial \dot{\sigma}_{Ijk}^{[2]S}(Q)}{\partial n} - \frac{\partial T^{[1]}(P,Q)}{\partial n}\dot{\sigma}_{Ijk}^{[2]S}(Q)]d\Gamma + $$

$$+\sum_{m=1}^{M} T^{[1]}(P,q_m)\dot{\sigma}_{Ijk}^{[3]P}(q_m). \tag{38}$$

Supplementing the integral equations (37) and (38) with the integral representation of initial stress rates $\dot{\sigma}_{Ijk}^{[1]S}$calculated at interior nodes $q_m$iteratively, one can solve them for unknown $\dot{\sigma}_{Ijk}^{[2]S}(Q)$, $\partial\dot{\sigma}_{Ijk}^{[2]S}(Q)/\partial n(Q)$ and $\dot{\sigma}_{Ijk}^{[3]P}(q_m)$. Similarly, from Eqs. (28) and (29), one can derive the integral equations

$$cT_1^S(P) = -\sum_{f=1}^{2}(-1)^f\int_{\Gamma}[T^{[f]}(P,Q)\frac{T_f^S(Q)}{\partial n} - \frac{\partial T^{[f]}(P,Q)}{\partial n}T_f^S(Q)]d\Gamma -$$

$$-\sum_{m=1}^{M} T^{[2]}(P,q_m)T_3^P(q_m), \tag{39}$$

$$cT_2^S(P) = \int_{\Gamma}[T^{[1]}(P,Q)\frac{\partial T_2^S(Q)}{\partial n} - \frac{\partial T^{[1]}(P,Q)}{\partial n}T_2^S(Q)]d\Gamma + \sum_{m=1}^{M} T^{[1]}(P,q_m)T_3^P(q_m) \tag{40}$$

and supplement them by the integral representation of T_1^Sat interior nodes q_m.

Bearing in mind the expressions of the kernels $T^{[f]}$ and their derivatives given by Eqs. (30) and (31), one can see that the highest order of the singularity in the boundary integral equations (37)-(40) is the strong singularity r^{-1}which should be considered in the CPV sense and can be treated by using various regularization techniques [Sladek and Sladek (1998)].

2.5 Triple-reciprocity boundary element method

In this section, we derive the thermo-elastoplastic boundary integral equation without domain integrals. Making use the triple-reciprocity interpolations for initial stress rates as well as for temperature field, the domain integrals in (20) can be converted into boundary integrals and certain interior points terms. In view of Eqs. (21), (30) and (31), one can easily verify the following relationships [Ochiai (1995)]

$$u_i^{[1]}(p,q) =$$

$$-\frac{m_o r_i}{8\pi}(1+2\ln r) = \frac{m_o r_i}{8\pi}[2(C_o+1)-2\ln r-1]\Big|_{C_o=-1} = m_o T_{,i}^{[2]}(p,q)\Big|_{C_o=-1} \tag{41}$$

Similarly, from Eqs. (22), (30) and (32), one obtains

$$u_{ij}^{[1]}(p,q) = \left.\left[-\frac{1}{2G(1-\bar{\nu})}T_{,ij}^{[2]}(p,q) + \frac{1}{G}\delta_{ij}T^{[1]}(p,q)\right]\right|_{C_o=C_o^e}, \quad C_o^e = -\frac{1}{2(4\bar{\nu}-3)} \tag{42}$$

Since

$$\varepsilon_{ijk}^{[1]}(p,q) = \frac{1}{2}\left[u_{ij,k}^{[1]}(p,q) + u_{ik,j}^{[1]}(p,q)\right] \tag{43}$$

we may write

$$\varepsilon_{ijk}^{[1]}(p,q) = \left.\left\{-\frac{1}{2G(1-\bar{\nu})}T_{,ijk}^{[2]}(p,q) + \frac{1}{G}\left[\delta_{ij}T_{,k}^{[1]}(p,q) + \delta_{ik}T_{,j}^{[1]}(p,q)\right]\right\}\right|_{C_o=C_o^e} \tag{44}$$

Now, generalizing Eqs. (41)-(43), one can define the higher-order kernel functions as

$$u_i^{[f]}(p,q) = m_o T_{,i}^{[f+1]}(p,q)\Big|_{C_o=-1} \tag{45}$$

$$u_{ij}^{[f]}(p,q) = \left.\left[-\frac{1}{2G(1-\bar{\nu})}T_{,ij}^{[f+1]}(p,q) + \frac{1}{G}\delta_{ij}T^{[f]}(p,q)\right]\right|_{C_o=C_o^e} \tag{46}$$

$$\varepsilon_{ijk}^{[f]}(p,q) = \left.\left\{-\frac{1}{2G(1-\bar{\nu})}T_{,ijk}^{[f+1]}(p,q) + \frac{1}{G}\left[\delta_{ij}T_{,k}^{[f]}(p,q) + \delta_{ik}T_{,j}^{[f]}(p,q)\right]\right\}\right|_{C_o=C_o^e}. \tag{47}$$

Hence and from (36), we have immediately

$$\nabla^2 u_i^{[f+1]} = u_i^{[f]}, \quad \nabla^2 u_{ij}^{[f+1]} = u_{ij}^{[f]}, \quad \nabla^2 \varepsilon_{ijk}^{[f+1]} = \varepsilon_{ijk}^{[f]} \text{ for } f \geq 1 \tag{48}$$

Let us assume the triple-reciprocity approximations (25), (26) and (28), (29) for the initial stress rates and the temperature, respectively, in Eq. (20) with denoting these fields as $\dot{\sigma}_{Ijk}^{[1]S}(q)$ and $T_1^S(q)$. Then, in view of Eqs. (40), (25), (26), (28), (29), the boundary integral equation (20) can be rearranged into the form without domain integrals

$$c_{ij}\dot{u}_j(P) = \int_\Gamma [u_{ij}^{[1]}(P,Q)\dot{p}_j(Q) - p_{ij}(P,Q)\dot{u}_j(Q)]\, d\Gamma +$$

$$+\sum_{f=1}^{2}(-1)^{f+1}\int_{\Gamma}\{T_f^S(Q)\frac{\partial u_i^{[f]}(P,Q)}{\partial n}-\frac{\partial T_f^S(Q)}{\partial n}u_i^{[f]}(P,Q)\}d\Gamma(Q)+$$

$$+\sum_{m=1}^{M}u_i^{[2]}(P,q_m)T_3^P(q_m)+$$

$$+\sum_{f=1}^{2}(-1)^{f+1}\int_{\Gamma}\{\frac{\partial\varepsilon_{ijk}^{[f+1]}(P,Q)}{\partial n}\dot{\sigma}_{Ijk}^{[f]S}(Q)-\varepsilon_{ijk}^{[f+1]}(P,Q)\frac{\partial\dot{\sigma}_{Ijk}^{[f]S}(Q)}{\partial n}\}\,d\Gamma+$$

$$+\sum_{m=1}^{M}\varepsilon_{ijk}^{[3]}(P,q_m)\dot{\sigma}_{Ijk}^{[3]P}(q_m). \tag{49}$$

in which the normal derivatives $\partial u_i^{[f]}(p,Q)/\partial n(Q)$ and $\partial\varepsilon_{ijk}^{[f+1]}(p,Q)/\partial n(Q)$ are given by

$$\frac{\partial u_i^{[f]}(p,Q)}{\partial n(Q)}=m_o n_k(Q)T_{,ik}^{[f+1]}(p,Q)\Big|_{C_o=-1} \tag{50}$$

$$\frac{\partial\varepsilon_{ijk}^{[f+1]}(p,Q)}{\partial n(Q)}=\Big\{-\frac{n_m(Q)}{2G(1-\bar{\nu})}T_{,ijkm}^{[f+2]}(p,Q)+$$

$$+\frac{n_m(Q)}{G}\left[\delta_{ij}T_{,km}^{[f+1]}(p,Q)+\delta_{ik}T_{,jm}^{[f+1]}(p,Q)\right]\Big\}\Big|_{C_o=C_o^e}. \tag{51}$$

In view of the expressions for the integral kernels involved in the BIE (49), it can be seen that the highest order singularity is the weak logarithmic singularity integrable in the ordinary sense.

2.6 Internal stress

In order to complete the set of boundary integral equations (37) and (38) for computation of unknowns $\dot{\sigma}_{Iij}^{[2]S}(Q)$, $\partial\dot{\sigma}_{Iij}^{[2]S}(Q)/\partial n(Q)$, $\dot{\sigma}_{Iij}^{[3]P}(q_m)$, we derive the integral representation of the stress rate at interior nodes q_m. The stress rates can be obtained from the displacement rates by

$$\dot{\sigma}_{ij}=\frac{2\bar{\nu}G}{(1-2\bar{\nu})}\delta_{ij}\dot{u}_{k,k}+G(\dot{u}_{i,j}+\dot{u}_{j,i})=c_{ijml}\dot{u}_{m,l}, \tag{52}$$

where

$$c_{ijml}=\frac{2\bar{\nu}G}{(1-2\bar{\nu})}\delta_{ij}\delta_{ml}+G(\delta_{im}\delta_{jl}+\delta_{il}\delta_{jm}). \tag{53}$$

Thus, in view of Eqs.(20) and (51), the integral representation of the stress rates at interior points becomes

$$\dot{\sigma}_{ij}(p) = \int_{\Gamma} [-\sigma^{[1]}_{kij}(p,Q)\dot{p}_k(Q) - S_{kij}(p,Q)\dot{u}_k(Q)]\, d\Gamma + \int_{\Omega} T(q)\nabla^2 \sigma^{T[1]}_{ij}(p,q)d\Omega +$$

$$+ \int_{\Omega} \varepsilon^{[1]}_{ijks}(p,q)\dot{\sigma}^{[1]}_{Iks}(q)d\Omega - \dot{\sigma}^{[1]}_{Iij}(p) \tag{54}$$

where the kernels, $\sigma^{[1]}_{ijk}(p,Q)$, $S_{kij}(p,Q)$, $\varepsilon^{[1]}_{ijks}(p,q)$ and $\sigma^{T[1]}_{ij}(p,q)$ in Eq.(54) are given by

$$\sigma^{[1]}_{kij}(p,Q) = c_{ijml}u^{[1]}_{mk,l}(p,Q) = \frac{-1}{4\pi(1-\bar{\nu})\, r}[(1-2\bar{\nu})(\delta_{jk}r_{,i} + \delta_{ik}r_{,j} - \delta_{ij}r_{,k}) + 2r_{,i}r_{,j}r_{,k}], \tag{55}$$

$$S_{kij}(p,Q) = c_{ijml}c_{ktsv}n_t(Q)u^{[1]}_{sm,vl}(p,Q) = \frac{2G}{4\pi(1-\bar{\nu})r^2}\left\{2\frac{\partial r}{\partial n}[(1-2\bar{\nu})\delta_{ij}r_{,k} + \right.$$

$$+\ \bar{\nu}(\delta_{ik}r_{,j} + \delta_{jk}r_{,i}) - 4r_{,i}r_{,j}r_{,k}\big] + 2\bar{\nu}(n_i r_{,j}r_{,k} + n_i r_{,i}r_{,k}) +$$

$$\left. +(1-2\bar{\nu})(2n_k r_{,i}r_{,j} + n_j\delta_{ik} + n_i\delta_{jk}) - (1-4\bar{\nu})n_k\delta_{ij}\right\}, \tag{56}$$

$$\varepsilon^{[1]}_{ijks}(p,q) = -c_{ijml}\varepsilon^{[1]}_{mks,l}(p,q) = -c_{ijml}\frac{1}{2}\left[u^{[1]}_{mk,sl}(p,q) + u^{[1]}_{ms,kl}(p,q)\right] =$$

$$= \frac{1}{4\pi(1-\bar{\nu})r^2}\left[(1-2\bar{\nu})(\delta_{ik}\delta_{js} + \delta_{jk}\delta_{is} - \delta_{ij}\delta_{ks} + 2\delta_{ij}r_{,k}r_{,s}) + \right.$$

$$\left. +2\bar{\nu}(\delta_{si}r_{,k}r_{,j} + \delta_{jk}r_{,i}r_{,s} + \delta_{ik}r_{,s}r_{,j} + \delta_{js}r_{,k}r_{,i}) - 8r_{,i}r_{,j}r_{,k}r_{,s} + 2\delta_{kl}r_{,i}r_{,j}\right]. \tag{57}$$

$$\sigma^{T[1]}_{ij}(p,q) = -c_{ijml}u^{[1]}_{m,l}(p,q). \tag{58}$$

Introducing the kernels

$$\sigma^{T[f]}_{ij}(p,q) = -c_{ijml}u^{[f]}_{m,l}(p,q), \quad \varepsilon^{[f]}_{ijks}(p,q) = -c_{ijml}\varepsilon^{[f]}_{mks,l}(p,q), \tag{59}$$

one can rewrite them, in view of Eqs. (45) and (47), as

$$\sigma^{T[f]}_{ij}(p,q) = -2Gm_o\left\{T^{[f+1]}_{,ij}(p,q) + \frac{\bar{\nu}}{1-2\bar{\nu}}\delta_{ij}T^{[f]}(p,q)\right\} \tag{60}$$

with $C_o = -1$.

$$\varepsilon_{ijks}^{[f]}(p,q) = \frac{1}{1-\bar{\nu}} T_{,iksj}^{[f+1]}(p,q) + \frac{\bar{\nu}(4\bar{\nu}-3)}{(1-\bar{\nu})(1-2\bar{\nu})} \delta_{ij} T_{,ks}^{[f]}(p,q) - \delta_{is} T_{,kj}^{[f]}(p,q) -$$

$$-\delta_{jk} T_{,si}^{[f]}(p,q) - \delta_{js} T_{,ki}^{[f]}(p,q) \tag{61}$$

with $C_o = C_o^e$.

In view of Eqs. (32), (60) and (61), one can see that

$$\nabla^2 \sigma_{ij}^{T[f+1]} = \sigma_{ij}^{T[f]}, \quad \nabla^2 \varepsilon_{ijks}^{[f+1]} = \varepsilon_{ijks}^{[f]} \tag{62}$$

for $f \geq 1$.

According to the assumption of the triple-reciprocity approximation for the initial stress rates and the temperature, we denote these fields in Eq. (54) as $\dot{\sigma}_{Iij}^{[1]S}(q)$ and $T_1^S(q)$. Then, bearing in mind Eqs. (25), (26), (28), (29) and (62), the integral representation (54) can be transformed into the form without domain integrals

$$\dot{\sigma}_{ij}(p) = \int_\Gamma [-\sigma_{kij}^{[1]}(p,Q)\dot{p}_k(Q) - S_{kij}\dot{u}_k(Q)]\, d\Gamma +$$

$$+ \sum_{f=1}^{2} (-1)^{f+1} \int_\Gamma \{T_f^S(Q) \frac{\partial \sigma_{ij}^{T[f]}(p,Q)}{\partial n(Q)} - \frac{\partial T_f^S(Q)}{\partial n(Q)} \sigma_{ij}^{T[f]}(p,Q)\} d\Gamma(Q) +$$

$$+ \sum_{m=1}^{M} \sigma_{ij}^{T[2]}(P,q_m) T_3^P(q_m) +$$

$$+ \sum_{f=1}^{2} (-1)^{f+1} \int_\Gamma [\frac{\partial \varepsilon_{ijks}^{[f+1]}(p,Q)}{\partial n(Q)} \dot{\sigma}_{Iks}^{[f]S}(Q) - \varepsilon_{ijks}^{[f+1]}(p,Q) \frac{\partial \dot{\sigma}_{Iks}^{[f]S}(Q)}{\partial n(Q)}]\, d\Gamma +$$

$$+ \sum_{m=1}^{M} \varepsilon_{ijks}^{[3]}(p,q_m) \dot{\sigma}_{Iks}^{[3]P}(q_m) - \dot{\sigma}_{Iij}^{[1]}(p), \tag{63}$$

where the normal derivative of the kernels $\sigma_{ij}^{T[f]}(p,Q)$ and $\varepsilon_{ijks}^{[f+1]}(p,Q)$ are given as

$$\frac{\partial \sigma_{ij}^{T[f]}(p,Q)}{\partial n(Q)} = -2Gm_o n_k(Q) \left\{ T_{,ijk}^{[f+1]}(p,Q) + \frac{\bar{\nu}}{1-2\bar{\nu}} \delta_{ij} T_{,k}^{[f]}(p,Q) \right\} \tag{64}$$

with $C_o = -1$.

$$\frac{\partial \varepsilon_{ijks}^{[f+1]}(p,Q)}{\partial n(Q)} = n_l(Q) \left\{ \frac{1}{1-\bar{\nu}} T_{,iksjl}^{[f+2]}(p,Q) + \frac{\bar{\nu}(4\bar{\nu}-3)}{(1-\bar{\nu})(1-2\bar{\nu})} \delta_{ij} T_{,ksl}^{[f+1]}(p,Q) - \right.$$

$$-\delta_{is}T^{[f+1]}_{,kjl}(p,Q)-\delta_{jk}T^{[f+1]}_{,sil}(p,Q)-\delta_{js}T^{[f+1]}_{,kil}(p,Q)\Big\} \quad (65)$$

with $C_o = C_o^e$.

Since the integral representation (63) is considered only at interior points, there are no singular integral. Nevertheless, there are involved nearly singular integrals with kernels proportional to r^{-1}. If the distance of the interior points in comparable with the size of boundary elements, such boundary integrals can be computed sufficiently accurately without using any special techniques [Sladek and Sladek (1998)].

An iterative process is used for thermo-elastoplastic analysis. The temperature load at the first yield T_S is obtained. Denoting the final temperature load as T_O and the number of iterations as N, the incremental load $(T_O - T_S)/N$ is added in each iteration.

3 Numerical examples

To verify the accuracy of the present method, the unsteady thermal stress in a circular region, made of an elastoplastic material, that is subjected to thermal loading is obtained. The temperature is 500 $^\circ C$ at the outer surface. Thermal diffusivity of $\kappa = 16mm^2/s$ and relative heat conduction of $h = 0.2\text{mm}^{-1}$ are assumed. The outer diameter is 20 mm. The von Mises yield criterion is used. Young's modulus of E =210 GPa, Poisson's ratio of $\nu = 0.3$, thermal expansion of $\alpha = 0.000011$, yield stress of $\sigma_0 = 100MPa$ and strain hardening of $H = 0.1E$ are assumed. The numbers of discretized boundary elements and internal points are 144 and 193, respectively, as shown in Fig. 1. Internal points are used to interpolate the distribution of initial stress. Constant boundary elements are used. Figure 2 shows the temperature distributions at times $t = 0.3s$ and $0.5s$ along with the exact solution. A plane stress state is assumed. Figure 3 shows the circumferential and radial stress distributions at time $t = 0.3s$. Figure 4 show the equivalent strain distributions at time $t = 0.3s$. BEM results are shown with FEM solutions in Figs. 3 and 4. The stress and strain distributions agree well with the FEM solutions [Nakasone (2007)].

The next numerical example is the unsteady thermal stress in a circular region with a circular hole made of an elastoplastic material which is subjected to thermal loading. A temperatures T =500 $^\circ C$ at the inner surface and relative heat conduction of $h = 0.1\text{mm}^{-1}$ at the inner surface are assumed. The temperature at the outer surface is 0 $^\circ C$. The inner diameter is 20 mm and the outer diameter is 60 mm. The von Mises yield criterion is used. Young's modulus of E =210 GPa, Poisson's ratio of $\nu = 0.3$, thermal expansion of $\alpha = 0.000011$ and yield stress of $\sigma_0 = 100MPa$ is assumed. The numbers of discretized boundary elements and internal points are 216 and 324, respectively, as shown in Fig. 5. Internal points are used to inter-

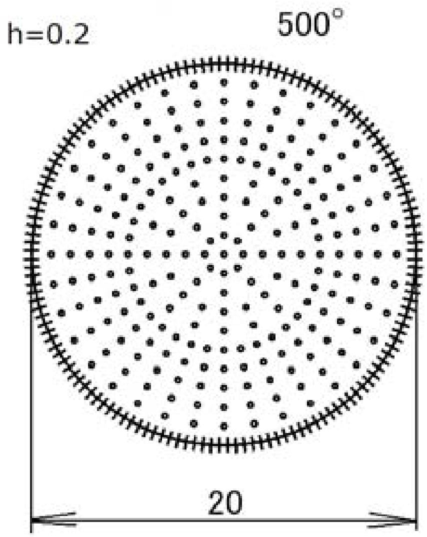

Figure 1: Circular region

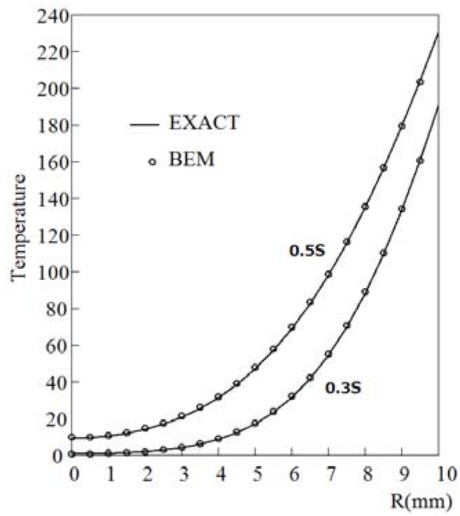

Figure 2: Temperature distribution in a circular region

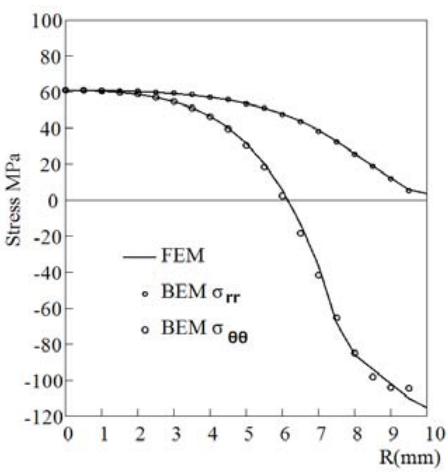

Figure 3: Thermal stress distribution (plane stress)

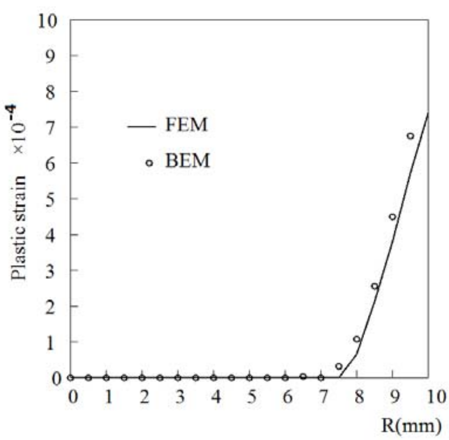

Figure 4: Equivalent plastic strain

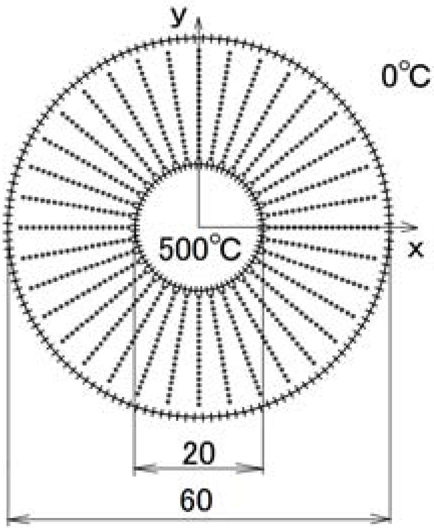

Figure 5: Circular cylinder

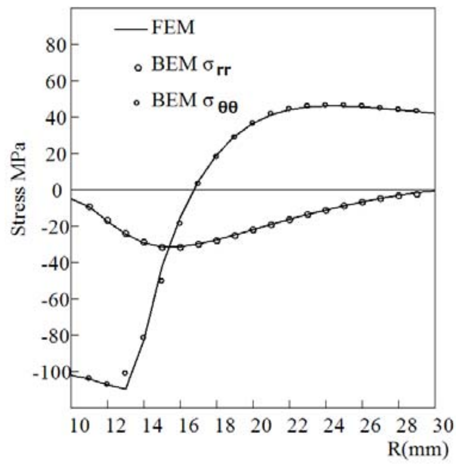

Figure 6: Thermal stress distribution (plane stress)

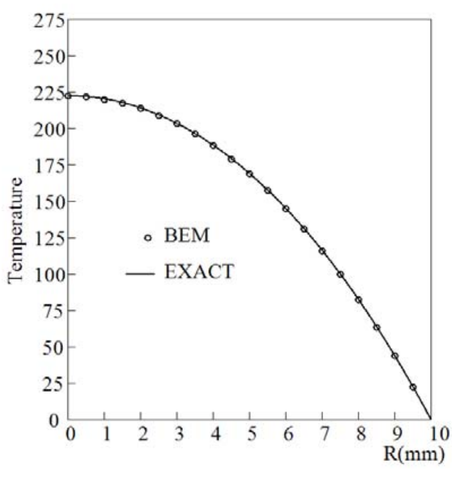

Figure 7: Temperature distribution in a circular region

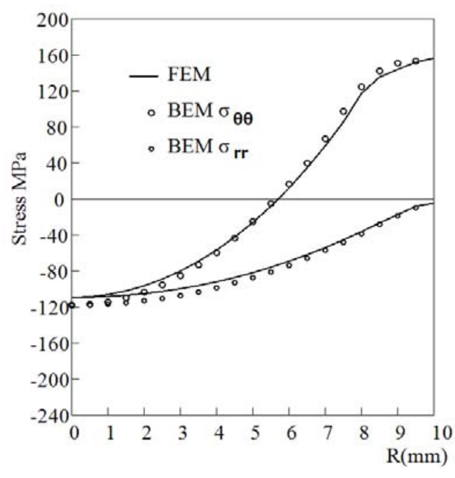

Figure 8: Thermal stress distribution with heat generation (plane stress)

polate the distribution of initial stress. Constant boundary elements are used. A plane stress state is assumed. Figure 6 shows the circumferential and radial stress distributions at time $t = 1.5s$. BEM results are shown with FEM solutions in Fig. 6. The stress distributions agree well with the FEM solutions.

The next example is the unsteady thermal stress in the circular region shown in Fig. 1 with heat generation of $W/\lambda = 10°C\text{mm}^{-2}$. The temperatures is 0 $°C$ at the outer surface and the outer diameter is 20 mm. The von Mises yield criterion is used. The thermal diffusivity of $\kappa = 16mm^2/s$, Young's modulus of E =210 GPa, Poisson's ratio of $\nu = 0.3$, thermal expansion of $\alpha = 0.000011$, yield stress of $\sigma_0 = 150MPa$ and strain hardening of $H = 0.1E$ are assumed. The numbers of discretized boundary elements and internal points are 144 and 193, respectively, as in Fig. 1. Figure 7 shows the temperature distribution at time $t = 2.5s$ along with the exact solution. A plane stress state is assumed. Figure 8 shows the circumferential and radial stress distributions. BEM results are shown with FEM solutions in Fig. 8. The stress distributions agree well with the FEM solutions.

4 Conclusion

It was shown that two-dimensional thermo-elastoplastic analysis can be carried out in the case of unsteady thermal stress without the use of internal cells, using the triple-reciprocity BEM. The integral kernels involved in the initial stress formulations have been expressed in terms of the higher-order fundamental solutions $T^{[f]}$ and their derivatives. In this method, the highest order of singularities does not exceed that in the conventional BEM. In this formulation, a time-dependent solution is not used to obtain the unsteady thermal stress. Using numerical examples, the effectiveness and accuracy of this method were demonstrated. In this method, the merit of BEM, which is the ease of data preparation, is not lost because internal cells are not necessary.

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