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The Superconvergence of Certain Two-Dimensional Hilbert Singular Integrals

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Abstract: The composite rectangle (midpoint) rule for the computation of multi-dimensional singular integrals is discussed and the superconvergence results is obtained. When the local coordinate is coincided with certain priori known coordinates, we get the convergence rate one order higher than the global one. At last, numerical examples are presented to illustrate our theoretical analysis which agree with it very well.

1 Introduction

We consider certain two-dimensional Hilbert kernel integral of the form

$$I(f,t,s) = \int_0^{2\pi}\!\!\!\!\!\!\!-\,\int_0^{2\pi}\!\!\!\!\!\!\!-\,\cot\frac{x-t}{2}\cot\frac{y-s}{2}f(x,y)dxdy = g(t,s), \tag{1}$$

where $\int_0^{2\pi}\!\!\!\!\!\!\!-\,\int_0^{2\pi}\!\!\!\!\!\!\!-\,$ denotes a Cauchy Principle value integral and $(t,s)\in(0,2\pi)\times(0,2\pi)$ the singular point.

In recent years, Cauchy principal value integrals have attracted a lot of attention. The main reason for this interest is probably due to the fact that Cauchy principal value integral equations have shown to be an adequate tool in boundary element methods [Yu (1985); Yu (1993); Yu (2002)] and many engineering problems [Chen and Hong (1999);Zhou, Li, and Yu (2008)] for the modeling of many physical situations[Yu and Huang (2008); Liu and Yu (2008); Young, Chen, Chen and Kao (2007)], such as acoustics, fluid mechanics, elasticity, fracture mechanics and electromagnetic scattering problems and so on. There are numerous works devoted to develop efficient quadrature formulae, such as the Gaussian method [Ioakimidis (1985); Hui and Shia (1999);Monehato (1994)], the transformation method [Elliott

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and Venturino (1997)]; the Newton-Cote methods [Linz (1985); Yu (1992); Wu and Yu (1999); Du (2001);Li Wu and Yu (2009);Li Zhang and Yu (2010);Li and Yu (2011)] and so on.

Certain kind of two-dimensional Cauchy principal value integrals which have not been extensively and references [Criscuolo and G. Mastroianni (1987);Monegato (1984);Li and Yu (2011)] may be the entire literature on the subject. In the reference of [Criscuolo and G. Mastroianni (1987)], numerical approximation of certain two-dimensional Cauchy principal value integrals with respect to generalized smooth Jacobi weight functions was considered by product rules of Gauss type. Then in [Monegato (1984)], generalized quadrature rule for two-dimensional Cauchy principal value integrals which based on the quadrature of one-dimensional Cauchy principal value integral was presented. Recently, Li et al.[Li and Yu (2011)] discussed the superconvergence phenomenon of rectangle rules with the singular point located at the middle area of each subinterval away from the boundary.

In this paper, we pay our attentions to investigate the superconvergence phenomenon of rectangle rule for Hilbert kernel integrals and to derive the error estimates. The superconvergence phenomenon for hypersingular integral is firstly studied by [Wu and Sun (2005); Wu and Sun (2008)]. They proved the superconvergence points occurred at the roots of certain special functions. Then in the year of 2010, the superconvergence phenomenon of the Cauchy principal value integral was studied in [Liu, Wu and Yu (2010)]. In this paper, we examine the convergence property of rectangle rule for certain kind of the two-dimensional Hilbert kernel integrals and generalize the above one-dimensional convergence results to cover this new situation. Moreover, we give an error expansion of the corresponding remainder with the density function $f(x,y) \in C^2$. Based on the error expansion functional, we get the superconvergence phenomenon, i.e.,when the singular point coincides with some a priori known point, the convergence rate is higher than what is globally possible.

The rest of this paper is organized as follows. In Sect.2, after introducing some basic formulae of the rectangle rule and lemmas, we present the main results. In Sect.3, the proof of the main results is finished. Finally, several numerical examples are provided to validate our analysis.

2 Main result

Let $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 2\pi$ and $0 = y_0 < y_1 < \cdots < y_{m-1} < y_m = 2\pi$ be a uniform partition of the area $[0,2\pi] \times [0,2\pi]$ with mesh size $h_x = 2\pi/n$ and $h_y = 2\pi/m$. To simplify our analysis, we set $h = h_x = h_y = 2\pi/n$, it is not difficult to extend our analysis to the qusi-uniform meshes.

Define $f_C(x,y)$ as the piecewise linear interpolation for $f(x,y)$:

$$f_C(x,y) = f(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}),\ \ x_{i+\frac{1}{2}} = x_i + \frac{h}{2},\ \ y_{j+\frac{1}{2}} = y_j + \frac{h}{2} \tag{2}$$

then we get

$$\begin{aligned} I_n(f,t,s) &= \int_0^{2\pi}\!\!\int_0^{2\pi} \cot\frac{x-t}{2}\cot\frac{y-s}{2} f_C(x,y)dxdy \\ &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} f(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}})\omega_{i,j}(t,s) \\ &= I(f,t,s) - E_n(f,t,s) \end{aligned} \tag{3}$$

where $E_n(f,t,s)$ denotes the error functional and

$$\omega_{i,j}(t,s) = 4\log\left|\frac{\sin 0.5(y_{i+1}-t)}{\sin 0.5(t_i - t)}\right| \log\left|\frac{\sin 0.5(x_{i+1}-s)}{\sin 0.5(x_i - s)}\right|. \tag{4}$$

We also define

$$k_t(x) = \begin{cases} (x-t)\cot\dfrac{x-t}{2}, & x \neq t, \\ 2, & x = t \end{cases} \tag{5}$$

and

$$k_s(y) = \begin{cases} (y-s)\cot\dfrac{y-s}{2}, & y \neq s, \\ 2, & y = s. \end{cases} \tag{6}$$

We present the error estimate for the rectangle rule with Certain Two-Dimensional Cauchy principal value integrals in the following theorem.

Theorem 1 *Assume $f(x,y) \in C^1[0,2\pi]\times[0,2\pi]$. For the rectangle rule $I_n(f,t,s)$ defined as Eq. 3. Assume that $t = x_k + (1+\tau)h/2, s = y_l + (1+\tau)h/2$, there exists a positive constant C, independent of h and s, such that*

$$|E_n(f,t,s)| \leq C\max\{|k_t(x)|, |k_s(y|)\}(|\ln h| + |\ln\gamma(\tau)|)^2 h, \tag{7}$$

where

$$\gamma(\tau) = \min_{0\leq i\leq n-1}\frac{|s-x_i|}{h} = \frac{1-|\tau|}{2}. \tag{8}$$

Compared with Riemann integrals, the global convergence rate of the (composite) rectangle rule for Eq. 3 integrals is one order lower for the Riemann integral.

To define a linear transformation

$$x = \hat{x}_i(\tau) := (\tau+1)(x_{i+1}-x_i)/2 + x_i, \quad \tau \in [-1,1],$$

$$y = \hat{y}_j(\xi) := (\xi+1)(y_{j+1}-y_j)/2 + y_j, \quad \xi \in [-1,1]$$

from the reference element $[-1,1]$ to the subinterval $[x_i, x_{i+1}]$ and $[y_j, y_{j+1}]$.

Setting

$$I_{n;x_i}(t,s) = \begin{cases} \displaystyle\fint_{x_i}^{x_{i+1}}\fint_{y_j}^{y_{j+1}} (x - x_{i+\frac{1}{2}}) \cot\frac{x-t}{2} \cot\frac{y-s}{2} dxdy, & i=k,\ j=l, \\ \displaystyle\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} (x - x_{i+\frac{1}{2}}) \cot\frac{x-t}{2} \cot\frac{y-s}{2} dxdy, & i \neq k,\ j \neq l \end{cases} \tag{9}$$

and

$$I_{n;y_j}(t,s) = \begin{cases} \displaystyle\fint_{x_i}^{x_{i+1}}\fint_{y_j}^{y_{j+1}} (y - y_{j+\frac{1}{2}}) \cot\frac{x-t}{2} \cot\frac{y-s}{2} dxdy, & i=k,\ j=l, \\ \displaystyle\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} (y - y_{j+\frac{1}{2}}) \cot\frac{x-t}{2} \cot\frac{y-s}{2} dxdy, & i \neq k,\ j \neq l. \end{cases} \tag{10}$$

Lemma 1 *Let $K_t(x)$ and $K_s(y)$ defined in Eq. 5 and Eq. 6. For $t \in (x_{i-1}, x_i), s \in (y_{j-1} y_j)$, by linear transformation, we have*

$$K_t(x) = K_{c_i}(\tau), \quad \tau \in (-1,1), \tag{11}$$

$$K_s(y) = K_{d_j}(\xi), \quad \xi \in (-1,1) \tag{12}$$

where

$$K_{c_i}(\tau) = 2 + 2\sum_{l=1}^{\infty} \frac{\tau - c_i}{\tau - c_i - 2ln} + 2\sum_{l=1}^{\infty} \frac{\tau - c_i}{\tau - c_i + 2ln}, \tag{13}$$

$$K_{d_j}(\xi) = 2 + 2\sum_{l=1}^{\infty} \frac{\xi - d_j}{\xi - d_j - 2ln} + 2\sum_{l=1}^{\infty} \frac{\xi - d_j}{\xi - d_j + 2ln} \tag{14}$$

and

$$c_i = 2(t - x_{i-1})/h - 1,$$

$$d_j = 2(s - y_{j-1})/h - 1.$$

Proof: By the identity in [Andrews (2002)]

$$\pi \cot \pi x = \sum_{l=-\infty}^{l=\infty} \frac{1}{(x+l)}, \tag{15}$$

then we get

$$\cot \frac{x-t}{2} = \frac{2}{x-t} + \sum_{l=1}^{\infty} \frac{2}{x-t-2l\pi} + \sum_{l=1}^{\infty} \frac{2}{x-t+2l\pi} \tag{16}$$

and

$$\begin{aligned} K_t(x) &= (x-t)\cot\frac{x-t}{2} = 2+2\sum_{l=1}^{\infty}\frac{\tau-c_i}{(\tau-c_i-4l\pi/h}+2\sum_{l=1}^{\infty}\frac{\tau-c_i}{(\tau-c_i+4l\pi/h} \\ &= 2+2\sum_{l=1}^{\infty}\frac{\tau-c_i}{\tau-c_i-2ln}+2\sum_{l=1}^{\infty}\frac{\tau-c_i}{\tau-c_i+2ln} \\ &= K_{c_i}(\tau). \end{aligned}$$

The proof is completed.

Lemma 2 *Assume* $t = x_k + (\tau+1)h/2, s = y_l + (\xi+1)h/2$ *with* $\xi, \tau \in (-1,1)$. *Let* $I_{n;x_i}(t,s)$ *and* $I_{n;y_j}(t,s)$ *be defined by Eq. 9 and Eq. 10 respectively, then there holds that*

$$\begin{aligned} I_{n;x_i}(t,s) &= \left\{ h\sum_{k=1}^{\infty}\frac{1}{k}\left(\cos k(x_{i+1}-t)+\cos k(x_i-t)\right) \right. \\ &+ \left. \sum_{k=1}^{\infty}\frac{1}{k^2}\left(\sin k(x_{i+1}-t)-\sin k(x_i-t)\right)\right\} \\ &\cdot \left[\sum_{l=1}^{\infty}\frac{1}{l}\left(\cos l(y_{j+1}-s)-\cos l(y_j-s)\right)\right] \end{aligned} \tag{17}$$

and

$$\begin{aligned} I_{n;y_j}(t,s) &= \left\{ h\sum_{l=1}^{\infty}\frac{1}{l}\left(\cos l(y_{j+1}-s)+\cos l(y_j-s)\right) \right. \\ &+ \left. \sum_{l=1}^{\infty}\frac{1}{l^2}\left(\sin l(y_{i+1}-s)-\sin l(y_j-s)\right)\right\} \\ &\cdot \left[\sum_{k=1}^{\infty}\frac{1}{k}\left(\cos k(x_{i+1}-t)-\cos k(x_i-t)\right)\right]. \end{aligned} \tag{18}$$

Proof For $i = k, j = l$, by the definition of cauchy principal value integral, we have

$$I_{n;x_i}(t,s) = \int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}}(x-x_k)\cot\frac{x-t}{2}\cot\frac{y-s}{2}dxdy$$
$$= \lim_{\varepsilon_1\to 0}\left(\int_{x_k}^{t-\varepsilon_1}+\int_{t+\varepsilon_1}^{x_{k+1}}\right)(x-x_k)\cot\frac{x-t}{2}dx\lim_{\varepsilon_2\to 0}\left(\int_{y_l}^{s-\varepsilon_2}+\int_{s+\varepsilon_2}^{y_{l+1}}\right)\cot\frac{y-s}{2}dy$$
$$= \left[-h\ln\left(2\sin\frac{x_k-t}{2}\right)-h\ln\left(2\sin\frac{x_{k+1}-t}{2}\right)+2\int_{x_k}^{x_{k+1}}\ln\left(2\sin\frac{x-t}{2}\right)dx\right]$$
$$\cdot\left[\ln\left(2\sin\frac{y_{j+1}-s}{2}\right)-\ln\left(2\sin\frac{y_j-s}{2}\right)\right]. \qquad (19)$$

For $i \neq k, j \neq l$,taking integration by parts on the correspondent Riemann integral, we have

$$I_{n;x_i}(t,s) = \int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}(x-x_i)\cot\frac{x-t}{2}\cot\frac{y-s}{2}dxdy$$
$$= \left[-h\ln\left(2\sin\frac{x_i-t}{2}\right)-h\ln\left(2\sin\frac{x_{i+1}-t}{2}\right)+2\int_{x_i}^{x_{i+1}}\ln\left(2\sin\frac{x-t}{2}\right)dx\right]$$
$$\cdot\left[\ln\left(2\sin\frac{y_{j+1}-s}{2}\right)-\ln\left(2\sin\frac{y_j-s}{2}\right)\right] \qquad (20)$$

where we have used the well-known identity in [Andrews (2002)]

$$\ln\left|2\sin\frac{x}{2}\right| = -\sum_{n=1}^{\infty}\frac{1}{n}\cos nx \qquad (21)$$

and the proof is completed.

Lemma 3 *Under the same assumptions of Lemma 2, there holds that*

$$\sum_{i=1}^{n} I_{n;x_i}(t,s) = -2h\ln 2\cos\frac{\tau\pi}{2}\int_{y_l}^{y_{l+1}}\cot\frac{y-s}{2}dy \qquad (22)$$

and

$$\sum_{j=1}^{n} I_{n;y_j}(t,s) = -2h\ln 2\cos\frac{\xi\pi}{2}\int_{x_k}^{x_{k+1}}\cot\frac{x-t}{2}dx. \qquad (23)$$

Proof By Eq. 17, we have

$$\begin{aligned}
\sum_{i=1}^{n} I_{n;x_i}(t,s) &= h\sum_{i=1}^{n}(\sum_{k=1}^{\infty}\frac{1}{k}\left(\cos k(x_{i+1}-s)+\cos k(x_i-s)\right) \\
&+ \sum_{k=1}^{\infty}\frac{1}{k^2}\left(\sin k(x_{i+1}-s)-\sin k(x_i-s)\right))\,\int_{y_l}^{y_{l+1}}\cot\frac{y-s}{2}dy \\
&= 2h\sum_{k=1}^{\infty}\sum_{i=1}^{n}\frac{1}{k}\cos k(x_i-s)\,\int_{y_l}^{y_{l+1}}\cot\frac{y-s}{2}dy \\
&= 2h\sum_{k=1}^{\infty}\frac{n\cos k(x_i-s)}{k}\,\int_{y_l}^{y_{l+1}}\cot\frac{y-s}{2}dy \\
&= 2h\sum_{k=1}^{\infty}\frac{\cos j(1+\tau)\pi}{j}\,\int_{y_l}^{y_{l+1}}\cot\frac{y-s}{2}dy \\
&= -2h\ln 2\sin\frac{(1+\tau)\pi}{2}\,\int_{y_l}^{y_{l+1}}\cot\frac{y-s}{2}dy \\
&= -2h\ln 2\cos\frac{\tau\pi}{2}\,\int_{y_l}^{y_{l+1}}\cot\frac{y-s}{2}dy
\end{aligned} \tag{24}$$

where we have used

$$\sum_{i=1}^{n}\cos k(x_i-s)=\begin{cases} n\cos k(x_1-s), & k=nj, \\ 0, & k\neq nj. \end{cases} \tag{25}$$

The proof is completed.

Let

$$\phi_x(t,s)=\begin{cases} -\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{K_{c_i}(\tau)K_{d_j}(\xi)\tau}{(\tau-t)(\xi-s)}d\tau d\xi, & |t|<1,|s|<1, \\ -\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{K_{c_i}(\tau)K_{d_j}(\xi)\tau}{(\tau-t)(\xi-s)}d\tau d\xi, & |t|>1,|s|>1 \end{cases} \tag{26}$$

and

$$\phi_y(t,s)=\begin{cases} -\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{K_{c_i}(\tau)K_{d_j}(\xi)\xi}{(\tau-t)(\xi-s)}d\tau d\xi, & |t|<1,|s|<1, \\ -\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{K_{c_i}(\tau)K_{d_j}(\xi)\xi}{(\tau-t)(\xi-s)}d\tau d\xi, & |t|>1,|s|>1. \end{cases} \tag{27}$$

We also set $J := (-\infty, -1)\bigcup(-1,1)\bigcup(1,\infty)$ and the operator $W : C(J) \to C(-1,1)$ be defined by

$$Wf(\tau) := f(\tau) + \sum_{j=1}^{\infty} f(2j+\tau) + \sum_{j=1}^{\infty} f(-2j+\tau). \tag{28}$$

Then we have

$$S_{0x}(\tau) := \phi_x(\tau) + \sum_{j=1}^{\infty} \phi_x(2j+\tau) + \sum_{j=1}^{\infty} \phi_x(-2j+\tau) \tag{29}$$

$$S_{0y}(\xi) := \phi_y(\xi) + \sum_{j=1}^{\infty} \phi_y(2j+\xi) + \sum_{j=1}^{\infty} \phi_y(-2j+\xi). \tag{30}$$

The superconvergence results of constant rectangle rules are given in the following.

Theorem 2 *Let $S_{0x}(\tau), S_{0y}(\tau)$ be defined by Eq. 29,Eq. 30 respectively. Assume that $t \neq x_i, s \neq y_j$ for any $i, j = 0,1,2,\cdots,n$. For the constant rectangle rule Eq. 3, there exists a positive constant C, independent of h and s, such that*

$$E_{kn}(f,t,s) = f_y(t,s)S_{0x}(\tau)h + f_x(t,s)S_{0y}(\tau)h + \mathscr{R}_n(t,s), \tag{31}$$

where $t = x_k + \frac{1+\tau}{2}h, s = y_l + \frac{1+\tau}{2}h \;\; k,l = 1,2,\cdots,n, \;\; \tau \in (-1,1)$.

$$|\mathscr{R}_n(t,s)| \leq C\max\{|k_t(x)|, |k_s(y|)\}[(|\ln h| + |\ln\gamma(\tau)|)^2]h^2. \tag{32}$$

and $\gamma(\tau)$ defined as

$$\gamma(\tau) = \min_{0\leq i\leq n-1}\left\{\frac{t-x_i}{h}\right\} = \frac{1-|\tau|}{2}. \tag{33}$$

From Theorems 2, when the special function $S_{0x}(\tau) = S_{0y}(\tau) = 0$, one can see that the superconvergence rate of the (composite) constant rectangle at certain points is one order higher than their global convergence rate which the same as the Riemann integral .

3 Proof of Theorem 2

In the following analysis, C will denote a generic positive constant which is independent of h and t,s.

Lemma 4 *Let $f(x,y) \in C^2[0,2\pi] \times [0,2\pi]$ and $f_C(x,y)$ be defined as Eq. 2, there holds*

$$f(x,y) - f_C(x,y) = f_x(x,y)(x - x_{i+0.5}) + f_y(x,y)(y - y_{j+0.5}) + \mathscr{R}_{ij}(\alpha_i, \beta_j), \tag{34}$$

where $\alpha_i \in (x_i, x_{i+1}), \beta_j \in (y_j, y_{j+1})$

$$\begin{aligned} \mathscr{R}_{ij}(\alpha_i, \beta_j) &= \frac{1}{2} f_{xx}(x,y)(x - x_{i+0.5})^2 + f_{xy}(x,y)(x - x_{i+0.5})(y - y_{j+0.5}) \\ &+ \frac{1}{2} f_{yy}(y - y_{j+0.5})^2 \end{aligned} \tag{35}$$

and

$$|\mathscr{R}_{ij}(\alpha_i, \beta_j)| \le Ch^2. \tag{36}$$

Lemma 5 *Assume $(t,s) \in (x_{k-1}, x_k) \times (y_{l-1}, y_l)$ and let $a_i = 2(t - x_{i-1})/h - 1, b_j = 2(s - x_{j-1})/h - 1, 0 \le i, j \le n-1$. Then, we have*

$$\phi_x(a_i, b_j) = \begin{cases} -\dfrac{1}{h} \displaystyle\fint_{x_i}^{x_{i+1}} \fint_{y_j}^{y_{j+1}} (x - x_{i+0.5}) \cot \frac{x-t}{2} \cot \frac{y-s}{2} dxdy, & i = k, j = l, \\ -\dfrac{1}{h} \displaystyle\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} (x - x_{i+0.5}) \cot \frac{x-t}{2} \cot \frac{y-s}{2} dxdy, & i \ne k, j \ne l \end{cases} \tag{37}$$

and

$$\phi_y(a_i, b_j) = \begin{cases} -\dfrac{1}{h} \displaystyle\fint_{x_i}^{x_{i+1}} \fint_{y_j}^{y_{j+1}} (y - y_{j+0.5}) \cot \frac{x-t}{2} \cot \frac{y-s}{2} dxdy, & i = k, j = l, \\ -\dfrac{1}{h} \displaystyle\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} (y - y_{j+0.5}) \cot \frac{x-t}{2} \cot \frac{y-s}{2} dxdy, & i \ne k, j \ne l. \end{cases} \tag{38}$$

Proof: By the definition of Eq. 1 and $i=k, j=l$, we have:

$$
\begin{aligned}
&\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}(x-x_{i+0.5})\cot\frac{x-t}{2}\cot\frac{y-s}{2}dxdy\\
&=\int_{x_i}^{x_{i+1}}(x-x_{i+0.5})\cot\frac{x-t}{2}\int_{y_j}^{y_{j+1}}\cot\frac{y-s}{2}dy\\
&=\lim_{\varepsilon\to 0}\left\{\left(\int_{x_i}^{t-\varepsilon}+\int_{t+\varepsilon}^{x_{i+1}}\right)(x-x_{i+0.5})\cot\frac{x-t}{2}dx\right\}\\
&\lim_{\varepsilon\to 0}\left\{\left(\int_{y_j}^{s-\varepsilon}+\int_{s+\varepsilon}^{y_{j+1}}\right)\cot\frac{y-s}{2}dy\right\}\\
&=\frac{h}{2}\lim_{\varepsilon\to 0}\left\{\left(\int_{-1}^{a_i-\frac{2\varepsilon}{h}}+\int_{a_i+\frac{2\varepsilon}{h}}^{1}\right)\frac{K_{c_i}(\tau)\tau}{\tau-a_i}d\tau\right\}\\
&\lim_{\varepsilon\to 0}\left\{\left(\int_{-1}^{b_l-\frac{2\varepsilon}{h}}+\int_{b_j+\frac{2\varepsilon}{h}}^{1}\right)\frac{K_{d_j}(\xi)}{\xi-b_j}d\xi\right\}\\
&=\frac{h}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{K_{c_i}(\tau)K_{d_j}(\xi)\tau}{(\tau-a_i)(\xi-b_j)}d\tau d\xi\\
&=-h\phi_x(a_i,b_j).
\end{aligned}
$$

The first identity in Eq. 37 is then verified. The second identity can be obtained by applying the approach to the correspondent Riemann integral.
Define

$$
\mathscr{H}^{kl}(x,y)=f(x,y)-f_C(x,y)-f_x(t,s)(x-x_{k+0.5})-f_y(t,s)(y-y_{l+0.5}). \tag{39}
$$

Lemma 6 *Under the assumption of Theorem 2, there holds that*

$$
\begin{aligned}
&\left|\int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}}\cot\frac{x-t}{2}\cot\frac{y-s}{2}\mathscr{H}^{kl}(x,y)dxdy\right|\\
&\le C\max\{|k_t(x)|,|k_s(y|)\}\,|\ln\gamma(\tau)|^2h^2.
\end{aligned} \tag{40}
$$

Proof: By the definition of Cauchy principal integral, we have

$$\begin{aligned}
&\int_a^b\int_c^d \cot\frac{x-t}{2}\cot\frac{y-s}{2}f(x,y)dxdy\\
&=f(t,s)\int_a^b\int_c^d \cot\frac{x-t}{2}\cot\frac{y-s}{2}dxdy\\
&+f_x(t,s)\int_a^b\int_c^d (x-t)\cot\frac{x-t}{2}\cot\frac{y-s}{2}dxdy\\
&+f_y(t,s)\int_a^b\int_c^d (y-s)\cot\frac{x-t}{2}\cot\frac{y-s}{2}dxdy\\
&+\int_a^b\int_c^d R(x,y)\cot\frac{x-t}{2}\cot\frac{y-s}{2}dxdy,
\end{aligned}\tag{41}$$

then following Eq. 39, we have

$$\begin{aligned}
&\int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}}\cot\frac{x-t}{2}\cot\frac{y-s}{2}\mathscr{H}^{kl}(x,y)dxdy\\
&=\mathscr{H}^{kl}(t,s)\int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}}\cot\frac{x-t}{2}\cot\frac{y-s}{2}dxdy\\
&+\mathscr{H}_x^{kl}(t,s)\int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}}\cot\frac{x-t}{2}\cot\frac{y-s}{2}(x-t)dxdy\\
&+\mathscr{H}_y^{kl}(t,s)\int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}}\cot\frac{x-t}{2}\cot\frac{y-s}{2}(y-s)dxdy\\
&+\int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}}\cot\frac{x-t}{2}\cot\frac{y-s}{2}R(x,y)dxdy.
\end{aligned}\tag{42}$$

Now let we estimate Eq. 42 term by term. For the first term of Eq. 42, we have

$$\begin{aligned}
&\left|\mathscr{H}^{kl}(t,s)\int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}}\cot\frac{x-t}{2}\cot\frac{y-s}{2}dxdy\right|\\
&=\left|\mathscr{H}^{kl}(t,s)\int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}}\frac{K_t(x)K_s(y)}{(x-t)(y-s)}dxdy\right|\\
&\le C\max\{|k_t(x)|,|k_s(y|)\}\int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}}\frac{1}{|(x-t)(y-s)|}dxdy\\
&\le C\max\{|k_t(x)|,|k_s(y|)\}(\ln|\gamma(\tau)|)^2h^2.
\end{aligned}\tag{43}$$

For the second term, we have

$$\begin{aligned}
&\left|\mathscr{H}_x^{kl}(t,s)\,\fint_{x_k}^{x_{k+1}}\fint_{y_l}^{y_{l+1}}\cot\frac{x-t}{2}\cot\frac{y-s}{2}(x-t)dxdy\right| \\
&=\left|\mathscr{H}_x^{kl}(t,s)\,\fint_{x_k}^{x_{k+1}}\fint_{y_l}^{y_{l+1}}\frac{K_t(x)K_s(y)}{y-s}dxdy\right| \\
&\leq C\max\{|k_t(x)|,|k_s(y)|\}\fint_{x_k}^{x_{k+1}}\fint_{y_l}^{y_{l+1}}\frac{1}{|y-s|}dxdy \\
&\leq C\max\{|k_t(x)|,|k_s(y)|\}\ln|\gamma(\tau)|h^2
\end{aligned} \tag{44}$$

and

$$\begin{aligned}
&\left|\mathscr{H}_y^{kl}(t,s)\,\fint_{x_k}^{x_{k+1}}\fint_{y_l}^{y_{l+1}}\cot\frac{x-t}{2}\cot\frac{y-s}{2}(y-s)dxdy\right| \\
&=\left|\mathscr{H}_y^{kl}(t,s)\,\fint_{x_k}^{x_{k+1}}\fint_{y_l}^{y_{l+1}}\frac{K_t(x)K_s(y)}{x-t}dxdy\right| \\
&\leq C\max\{|k_t(x)|,|k_s(y)|\}\fint_{x_k}^{x_{k+1}}\fint_{y_l}^{y_{l+1}}\frac{1}{|x-t|}dxdy \\
&\leq C\max\{|k_t(x)|,|k_s(y)|\}\ln|\gamma(\tau)|h^2
\end{aligned} \tag{45}$$

As for the last term of Eq. 42, we have

$$\begin{aligned}
&\left|\int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}}\cot\frac{x-t}{2}\cot\frac{y-s}{2}R(x,y)dxdy\right| \\
&=\left|\int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}}\frac{K_t(x)K_s(y)R(x,y)}{(x-t)(y-s)}dxdy\right| \\
&\leq C\max\{|k_t(x)|,|k_s(y)|\}\int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}}dxdy \\
&\leq C\max\{|k_t(x)|,|k_s(y)|\}h^2.
\end{aligned} \tag{46}$$

Combining Eq. 43, Eq. 44, Eq. 45 and Eq. 46 leads to Eq. 40 and the proof is completed.

Lemma 7 *Under the assumption of Theorem 2,we have*

$$\begin{aligned}
&\left|\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\cot\frac{x-t}{2}\cot\frac{y-s}{2}\mathscr{R}_{ij}(x,y)dxdy\right| \\
&\leq C\max\{|k_t(x)|,|k_s(y)|\}(|\ln h|+|\ln\gamma(\tau)|)^2h^2.
\end{aligned} \tag{47}$$

Proof: By the definition of Eq. 35, we have

$$\begin{aligned}
&\left|\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\cot\frac{x-t}{2}\cot\frac{y-s}{2}\mathscr{R}_{ij}(x,y)dxdy\right|\\
&\leq C\max\{|k_t(x)|,|k_s(y|)\}h^2\left|\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\frac{1}{|(x-t)(y-s)|}dxdy\right|\\
&\leq C\max\{|k_t(x)|,|k_s(y|)\}h^2\sum_{i=0,i\neq k}^{n-1}\int_{x_i}^{x_{i+1}}\frac{1}{|x-t|}dx\sum_{j=0,j\neq l}^{n-1}\int_{y_j}^{y_{j+1}}\frac{1}{|y-s|}dy\\
&\leq C\max\{|k_t(x)|,|k_s(y|)\}(|\ln h|+|\ln\gamma(\tau)|)^2h^2.
\end{aligned} \tag{48}$$

3.1 Proof of Theorem 2

Proof: By Lemma 4, we have

$$\begin{aligned}
&\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\cot\frac{x-t}{2}\cot\frac{y-s}{2}\left[f(x,y)-f_C(x,y)\right]dxdy\\
&=\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\cot\frac{x-t}{2}\cot\frac{y-s}{2}f_x(x,y)(x-x_{i+0.5})dxdy\\
&+\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\cot\frac{x-t}{2}\cot\frac{y-s}{2}f_y(x,y)(y-y_{j+0.5})dxdy\\
&+\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\cot\frac{x-t}{2}\cot\frac{y-s}{2}\mathscr{R}_{ij}(\alpha_i,\beta_j)dxdy.
\end{aligned} \tag{49}$$

Following the equation of $\mathscr{H}^{kl}(x,y)$, we have

$$\begin{aligned}
&⨍_{x_k}^{x_{k+1}}⨍_{y_l}^{y_{l+1}}\cot\frac{x-t}{2}\cot\frac{y-s}{2}\left[f(x,y)-f_C(x,y)\right]dxdy\\
&=f_x(t,s)⨍_{x_k}^{x_{k+1}}⨍_{y_l}^{y_{l+1}}\cot\frac{x-t}{2}\cot\frac{y-s}{2}(x-x_{i+0.5})dxdy\\
&+f_y(t,s)⨍_{x_k}^{x_{k+1}}⨍_{y_l}^{y_{l+1}}\cot\frac{x-t}{2}\cot\frac{y-s}{2}(y-y_{j+0.5})dxdy\\
&+⨍_{x_k}^{x_{k+1}}⨍_{y_l}^{y_{l+1}}\cot\frac{x-t}{2}\cot\frac{y-s}{2}\mathscr{H}^{kl}(x,y)dxdy,
\end{aligned} \tag{50}$$

and by Taylor expansion of $f_x(x,y)$ and $f_y(x,y)$, we have

$$f_x(x,y)=f_x(t,s)+f_{xx}(t,s)(x-t)+f_{xy}(t,s)(y-s)$$

$$f_y(x,y) = f_y(t,s) + f_{yx}(t,s)(x-t) + f_{yy}(t,s)(y-s).$$

Combining Eq. 49, Eq. 50 together, we have

$$\begin{aligned}
&\int_0^{2\pi}\!\!\int_0^{2\pi} \cot\frac{x-t}{2}\cot\frac{y-s}{2}\left[f(x,y)-f_C(x,y)\right]dxdy \\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} \cot\frac{x-t}{2}\cot\frac{y-s}{2}\left[f(x,y)-f_C(x,y)\right]dxdy \\
&= f_y(t,s)S_{0x}(\tau)h + f_x(t,s)S_{0y}(\tau)h + \mathscr{R}_n(t,s),
\end{aligned} \tag{51}$$

where

$$\mathscr{R}_n(t,s) = \mathscr{R}_n^{(1)}(t,s) + \mathscr{R}_n^{(2)}(t,s) + \mathscr{R}_n^{(3)}(t,s) + \mathscr{R}_n^{(4)}(t,s)$$

and

$$\begin{aligned}
&\mathscr{R}_n^{(1)}(t,s) \\
&= \sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} f_{xy}(t,s)\cot\frac{x-t}{2}\cot\frac{y-s}{2}(x-x_{i+0.5})(x-t)dxdy \\
&\quad + \sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} f_{xx}(t,s)\cot\frac{x-t}{2}\cot\frac{y-s}{2}(x-x_{i+0.5})(y-s)dxdy,
\end{aligned}$$

$$\begin{aligned}
&\mathscr{R}_n^{(2)}(t,s) \\
&= \sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} f_{yx}(t,s)\cot\frac{x-t}{2}\cot\frac{y-s}{2}(y-y_{j+0.5})(x-t)dxdy \\
&\quad + \sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} f_{yy}(t,s)\cot\frac{x-t}{2}\cot\frac{y-s}{2}(y-y_{j+0.5})(y-s)dxdy,
\end{aligned}$$

$$\mathscr{R}_n^{(3)}(t,s) = \sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} \cot\frac{x-t}{2}\cot\frac{y-s}{2}\mathscr{R}_{ij}(\alpha_i,\beta_j)dxdy,$$

$$\mathscr{R}_n^{(4)}(t,s) = \int_{x_k}^{x_{k+1}}\!\!\int_{y_l}^{y_{l+1}} \cot\frac{x-t}{2}\cot\frac{y-s}{2}\mathscr{H}^{ij}(x,y)dxdy.$$

For the first term of $\mathscr{R}_n^1(t,s)$, we have

$$\begin{aligned}
&\left|\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} f_{xy}(t,s)\cot\frac{x-t}{2}\cot\frac{y-s}{2}(x-x_{i+0.5})(x-t)dxdy\right| \\
&=\left|\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} \frac{k_t(x)k_s(y)f_{xy}(t,s)(x-x_{i+0.5})}{y-s}dxdy\right| \\
&\leq C\max\{|k_t(x)|,|k_s(y|\}\sum_{i=0,i\neq k}^{n-1}\int_{x_i}^{x_{i+1}}|x-x_{i+0.5}|dx\sum_{j=0,j\neq l}^{n-1}\int_{y_j}^{y_{j+1}}\frac{1}{|y-s|}dy \\
&\leq C\max\{|k_t(x)|,|k_s(y|\}(|\ln h|+|\ln\gamma(\tau)|)h^2.
\end{aligned}$$

For the second term of $\mathscr{R}_n^1(t,s)$, we have

$$\begin{aligned}
&\left|\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} f_{xx}(t,s)\cot\frac{x-t}{2}\cot\frac{y-s}{2}(x-x_{i+0.5})(y-ss)dxdy\right| \\
&=\left|\sum_{i=0,i\neq k}^{n-1}\sum_{j=0,j\neq l}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} \frac{k_t(x)k_s(y)f_{xy}(t,s)(x-x_{i+0.5})}{x-t}dxdy\right| \\
&\leq C\max\{|k_t(x)|,|k_s(y|\}\sum_{i=0,i\neq k}^{n-1}\int_{x_i}^{x_{i+1}}|x-x_{i+0.5}|dx\sum_{j=0,j\neq l}^{n-1}\int_{y_j}^{y_{j+1}}\frac{1}{|x-t|}dy \\
&\leq C\max\{|k_t(x)|,|k_s(y|\}(|\ln h|+|\ln\gamma(\tau)|)h^2.
\end{aligned}$$

which means

$$|\mathscr{R}_n^1(t,s)|\leq C\max\{|k_t(x)|,|k_s(y|\}(|\ln h|+|\ln\gamma(\tau)|)h^2.$$

Similarly, we also have

$$|\mathscr{R}_n^2(t,s)|\leq C\max\{|k_t(x)|,|k_s(y|\}(|\ln h|+|\ln\gamma(\tau)|)h^2.$$

By Lemma Eq. 6 and Lemma Eq. 7, we have

$$|\mathscr{R}_n(t,s)|\leq C\max\{|k_t(x)|,|k_s(y|\}[(|\ln h|+|\ln\gamma(\tau)|)^2]h^2.$$

The proof is completed.

For the one-dimensional Hilbert integral, we can easily get the superconvergence point is $\pm\frac{2}{3}$. As we have proved above, the superconvergence point of certain two-dimensional Hilbert integrals are $(\pm\frac{2}{3},\pm\frac{2}{3})$.

4 Numerical analysis

In this part, we present several examples to illustrate our theorems.

Example 1 *We consider the one dimensional Hilbert integral*

$$\int_0^{2\pi} \cot\frac{x-t}{2}\sin x dx = g(t) \tag{52}$$

and the analysis solution is

$$g(t) = -2\pi\cos t.$$

We adopt the uniform meshes to examine the convergence rate of the rectangle rule with the dynamic point with $t = y_{n/3} + (\tau+1)h/2$ *and* $t = a + (\tau+1)h/2$.

Table 1: Error estimate of rectangle rule for $t = a + x_{[n/3]} + (\tau+1)h/2$

	$\tau = 0$	$\tau = \frac{2}{3}$	$\tau = -\frac{2}{3}$	$\tau = \frac{1}{2}$
36	1.4331e-1	-2.8376e-3	1.2500e-2	7.1381e-2
72	6.6068e-2	-9.2338e-4	3.1271e-3	3.2859e-2
144	3.1638e-2	-2.5728e-4	7.8189e-4	1.5762e-2
288	1.5470e-2	-6.7600e-5	1.9547e-4	7.7192e-3
576	7.6481e-3	-1.7308e-5	4.8868e-5	3.8199e-3
1162	3.8023e-3	-4.3781e-6	1.2217e-5	1.9001e-3

Table 2: Error estimate of rectangle rule for $t = a + (\tau+1)h/2$

	$\tau = 0$	$\tau = \frac{2}{3}$	$\tau = -\frac{2}{3}$	$\tau = \frac{1}{2}$
36	-2.4891e-1	-9.2064e-3	-7.5746e-3	-1.2888e-1
72	-1.2285e-1	-2.1535e-3	-1.9441e-3	-6.2478e-2
144	-6.0971e-2	-5.1876e-4	-4.9225e-4	-3.0742e-2
288	-3.0367e-2	-1.2717e-4	-1.2384e-4	-1.5247e-2
576	-1.5153e-2	-3.1474e-5	-3.1055e-5	-7.5922e-3
1152	-7.5688e-3	-7.8283e-6	-7.7759e-6	-3.7883e-3

From the table 1 and table 2, we know that when the local coordinate of singular point is $\pm\frac{2}{3}$, the quadrature reach the convergence rate of $O(h^2)$ as for the non-supersingular point the convergence rate is $O(h)$ which agree with our theorematically analysis.

Example 2 *Consider the supersingular integral*

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \cot\frac{x-t}{2}\cot\frac{y-s}{2} f(x,y)dxdy, \tag{53}$$

with

$$f(x,y) = \sin(x)\sin(y)$$

and the exact solution is

$$4\pi^2\cos(t)\cos(s).$$

Table 3: Error estimate of rectangle rule with $t = x_{n/3} + (\tau+1)h/2$ and $s = y_{n/3} + (\tau+1)h/2$

	$(0,0)$	$(2/3,2/3)$	$(-2/3,-2/3)$	$(2/3,-2/3)$	$(0/3,2/3)$
24	4.6827e+0	5.8360e-2	7.9151e-1	5.9389e-1	2.9177e+0
48	1.7189e+0	-3.7697e-2	1.8886e-1	1.0437e-1	9.6767e-1
96	7.1047e-1	-1.3659e-2	4.5806e-2	2.0156e-2	3.7754e-1
192	3.1951e-1	-3.7717e-3	1.1258e-2	4.2832e-3	1.6474e-1
384	1.5105e-1	-9.7623e-4	2.7893e-3	9.7588e-4	7.6702e-2

Table 4: Error estimate of rectangle rule with $t = a + (\tau+1)h/2$ and $s = y_{n/3} + (\tau+1)h/2$

	$(0,0)$	$(2/3,2/3)$	$(-2/3,-2/3)$	$(2/3,-2/3)$	$(0/3,2/3)$
24	-6.3967e+0	-5.4351e-1	-9.1008e-1	-9.8242e-1	-3.9857e+0
48	-2.7995e+0	-6.5981e-2	-2.3183e-1	-2.4573e-1	-1.5760e+0
96	-1.2761e+0	-7.1651e-3	-5.8228e-2	-6.0246e-2	-6.7810e-1
192	-6.0473e-1	-6.3922e-4	-1.4573e-2	-1.4842e-2	-3.1181e-1
384	-2.9378e-1	-1.8277e-5	-3.6441e-3	-3.6787e-3	-1.4918e-1

We adopt the uniform meshes to examine the convergence rate of the rectangle rule with the dynamic point with $t = x_{[n/3]} + (1+\tau)h/2, s = y_{n/3} + (\tau+1)h/2$, $t = x_{n/3} + (\tau+1)h/2, s = a + (\tau+1)h/2$ and $t = a + (\tau+1)h/2, s = a + (\tau+1)h/2$.

Table 5: Error estimate of rectangle rule with $t = a + (\tau+1)h/2$ and $s = b + (\tau+1)h/2$

	(0,0)	(2/3,2/3)	(−2/3,−2/3)	(2/3,−2/3)	(0/3,2/3)
24	8.7381e+0	1.1226e+0	7.5598e-1	9.6190e-1	4.9035e+0
48	4.5594e+0	2.6862e-1	2.0791e-1	2.3904e-1	2.4433e+0
96	2.2920e+0	6.2626e-2	5.4225e-2	5.8451e-2	1.1893e+0
192	1.1446e+0	1.4924e-2	1.3828e-2	1.4377e-2	5.8331e-1
384	5.7137e-1	3.6300e-3	3.4903e-3	3.5602e-3	2.8846e-1

From the table 3, we know that when the local coordinate of singular point is $(\pm\frac{2}{3}, \pm\frac{2}{3})$, the quadrature reach the convergence rate of $O(h^2)$ as for the non-supersingular point the convergence rate is $O(h)$ which agree with our theorematically analysis. For the case of $t = x_{n/3} + (\tau+1)h/2, s = a + (\tau+1)h/2$ and $t = a + (\tau+1)h/2, s = a + (\tau+1)h/2$, table 4 and table 5 show that the convergence phenomenon for the mid-rectangle rue can also reach $O(h^2)$ for superconvergence point, which is different with the case of Cauchy Principal integral on interval,as there are no the influence of the boundary condition.

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References

Andrews, L. C. (2002): *Special Functions of Mathematics for Engineers*. McGraw-Hill, Inc.

Chen, J.T.; Hong,H.K.(1999): Review of dual boundary element methods with emphasis on hypersingular integrals and divergent series. *Appl. Mech.Rev.* , vol 52, no.1, pp. 17-33.

Choi, U.J.; Kim, S.W.; Yun, B.I.(2004): Improvement of the asymptotic behaviour of the Euler-Maclaurin formula for Cauchy principal value and Hadamard finite-part integrals. *Int. J. Numer. Methods.*, vol.61, pp. 496-513.

Criscuolo,G. Mastroianni,G.(1987): Convergence of Gauss type product fomulas for the evaluation of two dimensional cauchy pricipal value integral. *BIT* vol.27,pp. 72-84

Du, Q.K. (2001): Evaluations of certain hypersingular integrals on interval. *Int. J. Numer. Methods.*, vol. 512, pp. 1195-1210.

Elliott, D.; Venturino, E. (1997): Sigmoidal transformations and the Euler-Maclaurin expansion for evaluating certain Hadamard finite-part integrals. *Numer. Math.*, vol. 77, pp. 453-465.

Hui, C.Y.; Shia, D.(1999): Evaluations of hypersingular integrals using Gaussian quadrature. *Int. J. Numer. Methods.*, vol. 44, pp. 205-214.

Ioakimidis, N.I.(1985): On the uniform convergence of Gaussian quadrature rules for Cauchy principal value integrals and their derivatives. *Math. Comp.*, vol 44, pp. 191-198.

Linz, P. (1985): On the approximate computation of certain strongly singular integrals. *Computing.*, vol. 352, pp. 345–353.

Li.J; Wu,J.M; Yu, D. H.(2009): Generalized extrapolation for computation of hypersingular integrals in boundary element methods. *CMES:Computer Modeling in Engineering & Sciences,* vol.42,no.2, pp.151-175.

Li.J; Yu, D. H.(2011): Superconvergence of certain two-dimensional Cauchy Principal Value integrals. *CMES:Computer Modeling in Engineering & Sciences,* vol.71,no.4, pp.331-346.

Li.J; Zhang X.P.; Yu, D. H.(2010): Superconvergence and ultraconvergence of Newton-Cotes rules for supersingular integrals. *J. Comput. Appl. Math.,* vol.233,no.11, pp.2841-2854.

Liu, D.J; Wu,J.M; Yu, D. H.(2008): The coupling method of natural boundary element and mixed finite element for stationary Navier-Stokes equation in unbounded domains. *CMES:Computer Modeling in Engineering & Sciences,* vol. 37, no. 3, pp. 305-330.

Liu, D.J;Wu,J.M, Yu, D. H.(2010): The superconvergence of the Newton-Cotes rule for Cauchy principal value integrals. *J. Comput. Appl. Math.*, vol. 235, no. 3, pp. 696-707.

Monegato,G.(1984): Convergence of Product Formulas for the Numerical Evaluation of Certain Two-Dimensional Cauchy Principal Value Integrals. *Numer. Math.* vol.43,pp. 161-173.

Monegato, G.(1994): Numerical evaluation of hypersingular integrals. *J. Comput. Appl. Math.*, vol.50, pp. 9-31.

Wu, J.M.; Sun, W.W. (2008): The superconvergence of Newton-Cotes rules for the Hadamard finite-part integral on an interval. *Numer. Math.*, vol. 109, pp. 143–165.

Wu, J.M.; Sun, W.W. (2005): The superconvergence of the composite trapezoidal rule for Hadamard finite part integrals. *Numer. Math.*, vol. 102, pp. 343–363.

Wu, J.M.; Yu,D.H.(1999): The approximate computation of hypersingular integrals on interval. *Chinese J. Numer. Math. Appl.*, vol.21 pp. 25-33.

Yu, D. H.; Huang,H.Y.(2008): The artificial boundary method for a nonlinear interface problem on unbounded domain. *CMES:Computer Modeling in Engineering & Sciences,* vol.35, no.3, pp. 227-252.

Yu, D. H.(2002): *Natural Boundary Integrals Method and its Applications*. Kluwer Academic Publishers.

Yu, D. H.(1993): *Mathematical Theory of Natural Boundary Element Method*. Science Press, Beijing.

Yu,D.H.(1992): The approximate computation of hypersingular integrals on interval. *Numer. Math. J. Chinese Univ.*,vol.1, pp. 114-127.

Yu,D.H.(1985): Approximation of boundary conditions at infinity for a harmonic equation. *J. Comp. Math.*,vol. 3, no. 3, pp. 219-227.

Young, D.L.;Chen, K.H.; Chen,J.T.;Kao, J.H. (2007): A modified method of fundamental solutions with source on the boundary for solving laplace equations with circular and arbitrary domains. *CMES:Computer Modeling in Engineering & Sciences,* vol.19, no.3, pp.197-221.

Zhou, Y.T.; Li,X.; Yu, D. H.(2008): Integral method for contact problem of bonded plane material with arbitrary cracks. *CMES:Computer Modeling in Engineering & Sciences,* vol. 36, no.2, pp. 147-172.