Implementation of a Parallel Dual Reciprocity Boundary Element Method for the Solution of Coupled Thermoelasticity and Thermoviscoelasticity Problems

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Abstract: A parallel dual reciprocity boundary element method solution to thermoelasticity and thermoviscoelasticity problems is proposed. The DR-BEM formulation is given in Fourier Transform Space where the Time Space solutions are obtained through inverse Fourier Transform. The parallellization of the code is achieved through solving each frequency at a distinct computational node. The implemented parallel code is tested on 64-core IBM blade servers and it is seen that a linear speed-up is achieved.

Keywords: Thermoelasticity, Thermoviscoelasticity, dual reciprocity, boundary element method, parallel algorithm.

1   Introduction

The boundary element method (BEM) is one of the powerful numerical techniques in the field of solid and fluid mechanics. In a very general sense, when applied to a specific problem, the BEM establishes integral equations derived from the governing differential equations of the problem, transforms the resulting integral equations into boundary integral equations and solves them numerically by discretizing only the boundary of the solution domain. There are several stated advantages and disadvantages of the BEM, most of which are given in comparison to the finite element method (FEM) - the numerical method which is relatively more penetrated into the engineering life in practical sense. The most common known advantage is that, with its boundary-only nature, the BEM reduces the dimension of problem by one, e.g., for a 3D problem, the boundary element (BE) mesh is 2D and for a 3D problem, the BE mesh is 1D.

In view of the power of computers attained presently to handle large linear systems

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of equations, the major portion of the time spent in the analysis of a problem stems from modeling, e.g., meshing of the domain and imposition of the boundary conditions, which require human attention and time. Compared to the FEM, the BEM reduces the time spent in modeling considerably due to its boundary-only nature.

A drawback of the BEM is that it requires knowledge of some pre-determined fundamental solutions (FS). The FS can be thought as the weighting functions used in the weighted residual methods. In literature, the FS can be found for many mathematical models - yet in some problems, as in thermoelasticity (TE) and thermoviscoelasticity (TVE), the FS can be very complex. The integration of those complex FS, considering their complexity in singular integrals too, would be a hard task. In some problems, on the other hand, there exists no such FS in literature. Such problems are mostly the problems defined by inhomogenous differential equation(s) where the inhomogenous term would result in a domain integral associated with the inhomogenous term. The domain integration of these resulting domain integrals is possible, yet this would impair the boundary-only nature of the BEM. To retain the boundary-only property of the BEM, some methods are proposed in literature, carrying the domain integral to the boundary. Among these methods, the most commonly used is the dual reciprocity (DR) method [Partridge, Brebbia, and Wrobel (1992)], which is based on expressing the solution as the sum of homogenous and particular solutions. The DR method (DRM) involves: (i) expansion of internal excitation distribution in terms of pre-assumed base functions, in most cases, the radial basis functions, (ii) determination of the coefficients appearing in the expansion by matching these internal excitation values at the collocation points, (iii) finally, evaluation of approximate particular solution in view of expansion assumed for internal excitation.

In the DRM, only the boundary of the solution domain is discretized, but in the solution domain there is a need of sufficient number of additional collocation points. No mesh is needed inside the domain, which retains the boundary-only nature of the BEM.

In the present study, the DRM is applied to TE and TVE problems. As mentioned above, the FS for these problems are provided in literature [Sladék and Sladék (1983)], yet these functions are rather complicated. In this study, to simplify the complexity of these FS, we will treat the coupling terms in GE of TE and TVE as nonhomogenous terms (internal excitations) and apply DRM which makes it possible to carry out the analysis using the FS of elastodynamics and diffusion equation, which are less complicated than the FS of TE and TVE.

The use of DRM in TE and TVE problems is very limited and recent. Henry and Banerjee was first to implement DRM in steady-state and uncoupled quasi-static TE [Henry and Banerjee (1988)] in 1987. Later, Golberg, Chen and Bowman worked

on the base functions that can be used in TE problems [Golberg, Chen, and Bowman (1999)]. A summary for the use of DRM in uncoupled TE may be found in [Cheng, Chen, Golberg, and Rashed (2001)].The only reference to the research on DRM applied to coupled TE and TVE problems is by Baranoglu and Mengi [Baranoglu and Mengi (2006)].

In this study we propose a parallel solution to the coupled TE and TVE problems. Though the direct solution involves the evaluation of large system of linear equations, the computational time considerably increases when the number of unknowns are increased in the problem. To cope with the increase in computational time, parallel solution is needed. There are some research on parallel application of BEM and DRM in literature [Chen and Shanazari (2010)], but no reference occurs to the parallel solution of TE and TVE problems.

For the problem under consideration, parallelism can be achieved at different stages of the solution. For example, data-decomposition is always possible for the concurrent matrix operations involved (e.g., matrix multiplication, inversion, Gauss-Seidel iteration), or frequency space can be partioned for the so-called functional-decomposition [Chandlar and Taylor (1992)]. This study takes the latter view to obtain a parallel solution of TE and TVE problems.

2   Boundary Element Formulation and single-processor (sequential) solution algorithm

It may be shown that, in Fourier Transform Space (FTS), the governing equations (GE) of thermoelasticity (and thermoviscoelasticity using correspondance principle and ignoring viscous dissipation term) can be given as

$$L_{ij}(u_j) = \beta \theta_{,i} \; ; \; L(\theta) = \eta u_{k,k} \tag{1}$$

where

$$L_{ij} = \delta_{ij}\mu\nabla^2 + (\lambda+\mu)\partial_{ij} + \rho\omega^2\delta_{ij} \; ; \; L = k\nabla^2 - i\omega C_v \; ; \; \eta = T_0\beta i\omega \tag{2}$$

In the above equations, Einstein's summation convention is in place requiring summation over a repeated index. Also, a comma in the indices imply differentiation, e.g., $[\,]_{,i} = \frac{\partial[\,]}{\partial x_i}$. The field variables are; the components of the displacement vector, u_i and the temperature difference from reference temperature θ (where the reference temperature is denoted by T_0). The material constants: shear modulus, μ, Lame's modulus, λ, thermal expansion coefficient, k, the heat capacity under constant volume, C_v, the density, ρ and $\beta = (3\lambda+2\mu)\alpha$ with α being the thermal expansion coefficient. The Fourier Transform (FT) parameter is denoted with ω, which also can be regarded as the angular frequency. In the equations $i = \sqrt{-1}$

and δ_{ij} is the Kronecker's delta with $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. The left-equation in Eq.1 will be called the mechanical equations (ME) and the right-equation is called the thermal equation (ThE). In correspondance principle we replace the material constants with their complex forms (which are defined using a specific viscoelastic model), e.g., $\mu \to \tilde{\mu}$, $\lambda \to \tilde{\lambda}$ and $\beta \to \tilde{\beta}$ in the ME. The Eq.1 are coupled through the terms $\beta \theta_{,i}$ in the ME (which will be called the thermal coupling) and $\eta u_{k,k}$ in the ThE (which will be called as the mechanical coupling).

In this study, we utilize a direct solution to the given set of differential equations using the DR-BEM. For this, we consider the coupling terms as body force and internal heat generation and write the displacement and temperature solutions as a summation of homogenous and particular solutions as

$$u_i = u_i^h + u_i^p \quad ; \quad \theta = \theta^h + \theta^p \tag{3}$$

where, the associated tractions and flux would be

$$t_i = t_i^h + t_i^p - \beta n_i \theta \quad ; \quad q = q^h + q^p \tag{4}$$

where n_i are the components of the unit normal vector on the boundary. Note here that, the homogenous parts are solutions to the equations

$$L_{ij}(u_j^h) = 0 \quad ; \quad L(\theta^h) = 0 \tag{5}$$

and the particular parts are particular solutions to the equations

$$L_{ij}(u_j^p) = \beta \theta_{,i} \quad ; \quad L(\theta^p) = \eta u_{k,k} \tag{6}$$

It may be shown that, with above definitions, the boundary element equations for ME and ThE are

$$C_{ij} u_j - \int_S G_{ij} t_j \, dA + \int H_{ij} u_j \, dA - \beta \int_S G_{ij} n_j \theta \, dA$$

$$= C_{ij} u_j^p - \int_S G_{ij} t_j^p \, dA + \int H_{ij} u_j^p \, dA \tag{7}$$

$$C\theta - \int_S Gq \, dA + \int_S H\theta \, dA = C\theta^p \int_S Gq^p \, dA + \int_S H\theta^p \, dA \tag{8}$$

To proceed further, we represent the temperature field and the displacement field in truncated series expansion as

$$\theta(\mathbf{P}) = \sum_{n=1}^{N+M} \phi_n(\mathbf{P}) \psi_n(\mathbf{P}, \mathbf{P}_n) \quad ; \quad u_i(\mathbf{P}) = \sum_{n=1}^{N+M} \phi_{in}(\mathbf{P}) \psi_n(\mathbf{P}, \mathbf{P}_n) \tag{9}$$

where ϕ_n and ϕ_{in} are the coefficients of the series expansion and ψ_n are the base functions. In this study, for simplicity, base functions are selected to be $\psi_n(\mathbf{P}, \mathbf{P}_n) = 1 + r(\mathbf{P}, \mathbf{P}_n)$ where $r(\mathbf{P}, \mathbf{P}_n)$ is the distance between points $\mathbf{P}$ and $\mathbf{P}_n$. In the truncated series expansions, the truncation limit is $N + M$ where N is the number of nodes taken on the boundary of the solution domain and M is the additional collocation points selected within the solution domain. Defining particular solutions θ_{in}^p and u_{in}^p as the solutions to the equations

$$L(\theta_{in}^p) = \eta\,\psi_{n,i} \quad ; \quad L_{ij}(u_{jn}^p) = \beta\,\psi_{n,i} \tag{10}$$

with their associated flux and traction, q_{in}^p and t_{in}^p, we obtain, after several mathematical manipulations,

$$u_i^p = \phi_n u_{in}^p \quad ; \quad t_i^p = \phi_n t_{in}^p \quad ; \quad \theta^p = \phi_{in}\theta_{in}^p \quad ; \quad q^p = \phi_{in}q_{in}^p \tag{11}$$

The general solutions to Eq.11 can be found in literature. Inserting Eq.11 into Eq.7 and 8 would yield

$$C_{ij}u_j - \int\limits_S G_{ij}t_j\mathrm{d}A + \int H_{ij}u_j\mathrm{d}A - \tilde{\beta}\int\limits_S G_{ij}n_j\theta\mathrm{d}A$$

$$= \sum_{n=1}^{N+M} \phi_n \left(C_{ij}u_{jn}^p - \int\limits_S G_{ij}t_{jn}^p\mathrm{d}A + \int\limits_S H_{ij}u_{jn}^p\mathrm{d}A \right) \tag{12}$$

$$C\theta - \int\limits_S Gq\mathrm{d}A + \int\limits_S H\theta\mathrm{d}A = \sum_{n=1}^{N+M} \phi_{in}\left(C\theta_{in}^p - \int\limits_S Gq_{in}^p\mathrm{d}A + \int\limits_S H\theta_{in}^p\mathrm{d}A \right) \tag{13}$$

To compute the coefficients of the truncated series expansion, ϕ_n and ϕ_{in}, we use the collocation method: at the collocation points $\mathbf{P}_n$ ($n = 1..N + M$) we match the values of u_i and θ obtained from the series expansion with the given data. When we write these equations at all collocation points, we obtain two systems of linear algebraic equations which can be represented in matrix relations as

$$\underline{u}_i = \underline{\psi}\cdot\underline{\phi}_i \rightarrow \underline{\phi}_i = \underline{\psi}^{-1}\cdot\underline{u}_i \quad ; \quad \underline{\theta} = \underline{\psi}\cdot\underline{\phi} \rightarrow \underline{\phi} = \underline{\psi}^{-1}\cdot\underline{\theta} \tag{14}$$

Inserting Eq.14 into 12 and 13 and writing Eq.12 and 13 at each boundary node, we obtain the matrix relations

$$\mathbf{H}^{(b)}\cdot\mathbf{u}^{(b)} - \mathbf{G}^{(b)}\cdot\mathbf{t} - \tilde{\beta}\mathbf{G}^{(b)}\mathbf{n}\Theta^{(b)} = \left(\mathbf{H}^{(b)}\cdot\mathbf{U} - \mathbf{G}^{(b)}\cdot\mathbf{T}\right)\cdot\mathbf{F}^{-1}\cdot\Theta$$

$$\bar{\mathbf{H}}^{(b)}\cdot\Theta^{(b)} - \bar{\mathbf{G}}^{(b)}\cdot\mathbf{q} = \left(\bar{\mathbf{H}}^{(b)}\cdot\hat{\Theta} - \bar{\mathbf{G}}^{(b)}\cdot\mathbf{Q}\right)\cdot\bar{\mathbf{F}}^{-1}\cdot\mathbf{u} \tag{15}$$

Here, the superscript (b) denotes that these matrices (or column vectors) are evaluated over the boundary nodes. Matrices with no superscript are either not defined outside the boundary (like traction, $\mathbf{t}$, normal flux, $\mathbf{q}$, and unit outward normal at the boundary, $\mathbf{n}$) or contain values from both boundary and internal, e.g.,

$$\Theta = \left\{ \begin{array}{c} \Theta^{(b)} \\ \Theta^{(i)} \end{array} \right\} \; ; \; \mathbf{u} = \left\{ \begin{array}{c} \mathbf{u}^{(b)} \\ \mathbf{u}^{(i)} \end{array} \right\} \tag{16}$$

Note that, in Eq.15, the matrix $\mathbf{F}$ contains the values of the base functions evaluated at the collocation points. The matrix $\bar{\mathbf{F}}^{-1}$ on the other hand, obtained from $\mathbf{F}^{-1}$ using appropriate row and column changes. Defining

$$\mathbf{S} = \left(\mathbf{H}^{(b)} \cdot \mathbf{U} - \mathbf{G}^{(b)} \cdot \mathbf{T} \right) \cdot \mathbf{F}^{-1} \; ; \; \bar{\mathbf{S}} = \left(\bar{\mathbf{H}}^{(b)} \cdot \hat{\Theta} - \bar{\mathbf{G}}^{(b)} \cdot \mathbf{Q} \right) \cdot \bar{\mathbf{F}}^{-1} \tag{17}$$

and defining new sub matrices such that

$$\mathbf{S} \cdot \Theta \rightarrow \left[\begin{array}{c} \mathbf{S}^{(b)} \\ \mathbf{S}^{(i)} \end{array} \right] \cdot \left\{ \begin{array}{c} \Theta^{(b)} \\ \Theta^{(i)} \end{array} \right\} \; ; \; \bar{\mathbf{S}} \cdot \mathbf{u} \rightarrow \left[\begin{array}{c} \bar{\mathbf{S}}^{(b)} \\ \bar{\mathbf{S}}^{(i)} \end{array} \right] \cdot \left\{ \begin{array}{c} \mathbf{u}^{(b)} \\ \mathbf{u}^{(i)} \end{array} \right\} \tag{18}$$

Eq.15 can be re-written as

$$\begin{aligned} \mathbf{H}^{(b)} \cdot \mathbf{u}^{(b)} - \mathbf{G}^{(b)} \cdot \mathbf{t} &= \left[\begin{array}{c} \mathbf{S}^{(b)} \\ \mathbf{S}^{(i)} \end{array} \right] \cdot \left\{ \begin{array}{c} \Theta^{(b)} \\ \Theta^{(i)} \end{array} \right\} \\ \bar{\mathbf{H}}^{(b)} \cdot \Theta^{(b)} - \bar{\mathbf{G}}^{(b)} \cdot \mathbf{q} &= \left[\begin{array}{c} \bar{\mathbf{S}}^{(b)} \\ \bar{\mathbf{S}}^{(i)} \end{array} \right] \cdot \left\{ \begin{array}{c} \mathbf{u}^{(b)} \\ \mathbf{u}^{(i)} \end{array} \right\} \end{aligned} \tag{19}$$

The integral equations in Eq.12 and 13, when written for the internal DR points, lead to

$$\begin{aligned} \mathbf{u}^{(i)} &= \mathbf{G}^{(i)} \cdot \mathbf{t} - \mathbf{H}^{(i)} \cdot \mathbf{u}^{(b)} + \tilde{\beta} \mathbf{G}^{(i)} \mathbf{n} \Theta^{(i)} + \left(\mathbf{U}^{(i)} + \mathbf{H}^{(i)} \cdot \mathbf{U} - \mathbf{G}^{(i)} \cdot \mathbf{T} \right) \cdot \mathbf{F}^{-1} \cdot \Theta \\ \Theta^{(i)} &= \bar{\mathbf{G}}^{(i)} \cdot \mathbf{q} - \bar{\mathbf{H}}^{(i)} \cdot \Theta^{(b)} + \left(\hat{\Theta}^{(i)} + \bar{\mathbf{H}}^{(i)} \cdot \hat{\Theta} - \bar{\mathbf{G}}^{(b)} \cdot \mathbf{Q} \right) \cdot \bar{\mathbf{F}}^{-1} \cdot \mathbf{u} \end{aligned} \tag{20}$$

Through the steps defined above, these matrix equations can be re-written as

$$\begin{aligned} \mathbf{u}^{(i)} &= \mathbf{G}^{(i)} \cdot \mathbf{t} - \mathbf{H}^{(i)} \cdot \mathbf{u}^{(b)} + \left[\begin{array}{c} \mathbf{Y}^{(b)} \\ \mathbf{Y}^{(i)} \end{array} \right] \cdot \left\{ \begin{array}{c} \Theta^{(b)} \\ \Theta^{(i)} \end{array} \right\} \\ \Theta^{(i)} &= \bar{\mathbf{G}}^{(i)} \cdot \mathbf{q} - \bar{\mathbf{H}}^{(i)} \cdot \Theta^{(b)} + \left[\begin{array}{c} \bar{\mathbf{Y}}^{(b)} \\ \bar{\mathbf{Y}}^{(i)} \end{array} \right] \cdot \left\{ \begin{array}{c} \mathbf{u}^{(b)} \\ \mathbf{u}^{(i)} \end{array} \right\} \end{aligned} \tag{21}$$

At this point the matrix equations in Eq.19 and 21 can be combined to obtain

$$
\begin{bmatrix}
\mathbf{H}^{(b)} & -\left(\tilde{\beta}\mathbf{G}^{(b)}\mathbf{n}\Theta^{(b)}+\mathbf{S}^{(b)}\right) & \mathbf{0} & -\mathbf{S}^{(i)} \\
-\bar{\mathbf{S}}^{(b)} & \bar{\mathbf{H}}^{(b)} & -\bar{\mathbf{S}}^{(i)} & \mathbf{0} \\
\mathbf{H}^{(i)} & -\left(\tilde{\beta}\mathbf{G}^{(i)}\mathbf{n}\Theta^{(i)}+\mathbf{Y}^{(b)}\right) & \mathbf{I} & -\mathbf{Y}^{(i)} \\
-\bar{\mathbf{Y}}^{(b)} & \bar{\mathbf{H}}^{(i)} & -\bar{\mathbf{Y}}^{(i)} & \mathbf{I}
\end{bmatrix}
\begin{Bmatrix}
\mathbf{u}^{(b)} \\
\Theta^{(b)} \\
\mathbf{u}^{(i)} \\
\Theta^{(i)}
\end{Bmatrix}
\tag{22}
$$

$$
=
\begin{bmatrix}
\mathbf{G}^{(b)} & \mathbf{0} \\
\mathbf{0} & \mathbf{G}^{(i)} \\
\bar{\mathbf{G}}^{(b)} & \mathbf{0} \\
\mathbf{0} & \bar{\mathbf{G}}^{(i)}
\end{bmatrix}
\begin{Bmatrix}
\mathbf{t} \\
\mathbf{q}
\end{Bmatrix}
$$

Here, $\mathbf{0}$ represents the zero matrix and $\mathbf{I}$ represents the identity matrix of matching size. After the imposition of the boundary conditions, the resulting system of equations can be represented by

$$
\bar{\mathbf{K}} \cdot \mathbf{x} = \bar{\mathbf{b}}
\tag{23}
$$

which would be solved for the unknown quantities $\mathbf{x}$. When the sub-matrices in Eq.22 are examined, it may be seen that

- Matrix $\mathbf{F}$ is frequency independent, and $\mathbf{F}^{-1}$ is used successively at each frequency to obtain $\mathbf{S}$, $\bar{\mathbf{S}}$, $\mathbf{Y}$ and $\bar{\mathbf{Y}}$. Therefore, it would be profitable to evaluate it in a separate subroutine (evaluate_Finv) together with its inverse, at the beginning of the code, and make it available to all frequencies.

- sub-matrices $\mathbf{G}^{(b)}$, $\mathbf{H}^{(b)}$, $\bar{\mathbf{G}}^{(b)}$ and $\bar{\mathbf{H}}^{(b)}$ involve integrations over the boundary elements when the fixed point $\mathbf{A}$ is on the boundary. To evaluate the off-diagonal elements of these matrices GQ with 10 points are employed, whereas for the diagonal elements, since the resulting integrals become singular, semi-analytical solutions are provided. The evaluation of these matrices involve common computations, therefore implemented in a single subroutine (coupled_2d)

- sub-matrices $\mathbf{G}^{(i)}$, $\mathbf{H}^{(i)}$, $\bar{\mathbf{G}}^{(i)}$ and $\bar{\mathbf{H}}^{(i)}$ involve integrations over the boundary elements when the fixed point $\mathbf{A}$ is within the solution domain - the point $\mathbf{A}$ coincides with a collocation point within the domain. In this case, no singularities would occur, therefore all integrals are evaluated using 10 point GQ. Since the evaluations of these matrices involve common computations, they are implemented in a single subroutine (coupled_2d_drm)

- The evaluation of **S**, **S̄**, **Y** and **Ȳ** involve the evaluation of point-to-point distance functions preceded with matrix multiplications. These matrices are evaluated in a single subroutine (Obtain_DR_matrices) which should be called after the previously mentioned subroutines.

The algorithm for the sequential program can be found in Fig. 1
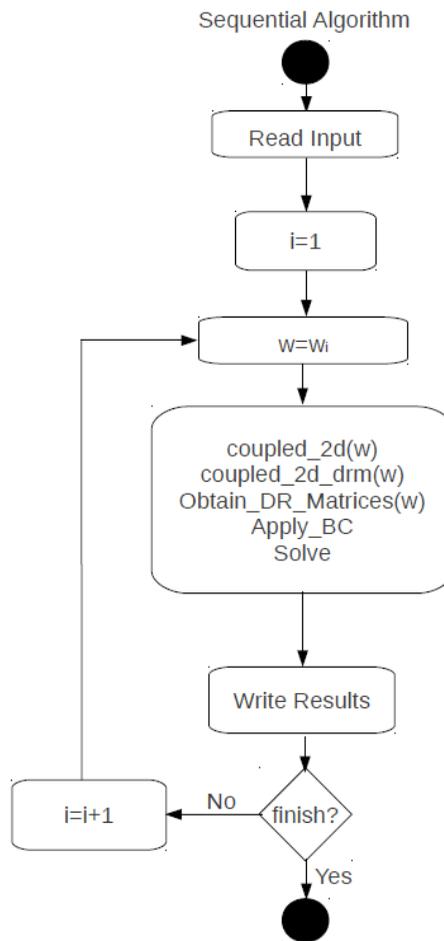
Figure 1: Sequential Algorithm

3 Frequency parallellization

One of the advantages of working in FTS is that the solution at each frequency is independent of one another. In TS, however, the solution at a specific time step depends on all previous time solutions (if time integration is employed), or the previous time step and the value of the time increment selected (if time stepping algorithm is used as proposed by Brebbia). The fact that each frequency can be solved individually and independently gives the opportunity of parallellization of the solution algorithm in such a way that each frequency may be solved at a free and distinct computation node (CN).

The algorithm for the parallel program can be found in Fig. 2
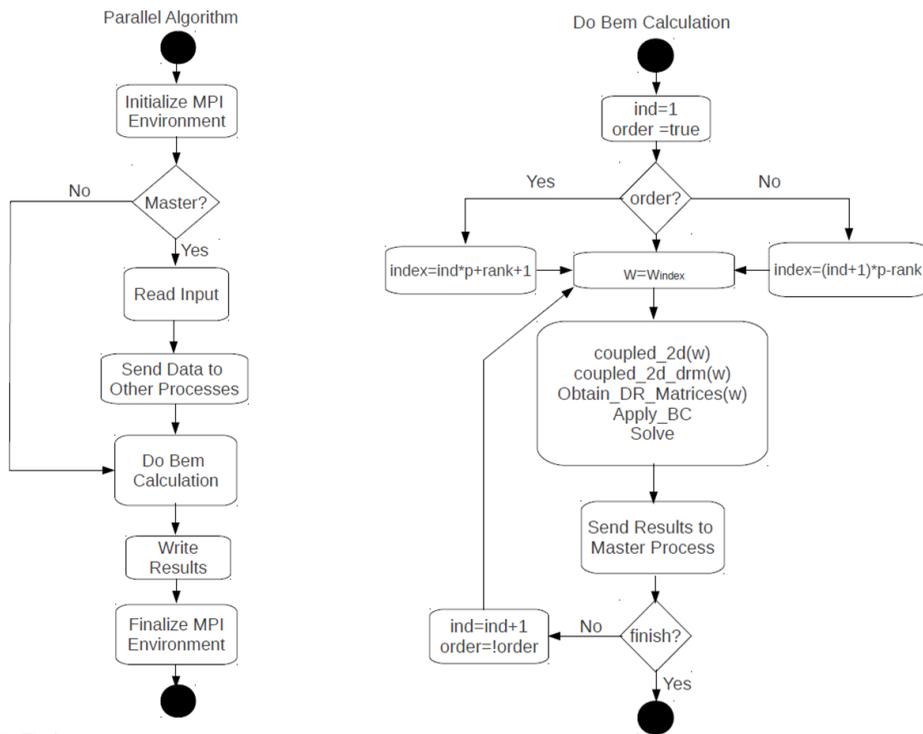
Figure 2: The parallel algorithm

4 Assessment of code and results

The algorithm described above is implemented in a FORTRAN code. The code is written in Intel-FORTRAN on IBM blade servers - total of 112 cores, 64 of which

are reserved for this study. Also, for comparison, a sequential code is prepared using the same functions and subroutines with the parallel code.

For the application of FFT algorithm, 2^n frequency points should be specified at which the solution will be obtained. In the code, for simplicity but without loss of generality, two dimensional formulation is implemented using constant elements. The formulation, however, is applicable to three dimensional problems and higher order elements. Also, for DR formulation, in the truncated series expansions, the simplest radial basis function ($\psi = 1 + r$) is employed.

The developed code determines the response in FTS, from which the TS response can be obtained through inversion using FFT algorithm. For FFT is not in the scope of this study, the Fouriér transform and inverse transform is made using a separate small program.

To assess the code, we consider the dynamic response of an elastic/viscoelastic tube to a harmonic thermal input. The object of this example is to examine the performance of the developed code using 2, 4, 8, 16, 32 and 64 CN. In the example, we consider an infinitely long circular elastic tube whose inner and outer boundaries are free of forces. The thermal input is specified at the inner boundary as a uniform temperature rise of ($\theta = \theta^*$). The outer boundary remains at the same temperature ($\theta = 0$). Since the problem is symmetric in two cartesian coordinate axes, only $\frac{1}{4}$ of it is enough to be modeled (see Fig. 3 where the modeling and boundary conditions are presented together with a boundary element mesh). The exact solution to the problem in case of uncoupled thermoelasticity is known, also it is shown that the results obtained are promising [Baranoglu and Mengi (2006)].

The main focus of this present example will be to test the performance of the proposed parallel solution in comparison with the sequential code, therefore no attention will be given to the change in accuracy while changing the size of the problem (which may be a topic for a later study). The problem is modeled using 4 different meshes as given in Tab.1. For obtaining the frequency response of the problem, 256 discrete frequency points are employed with $\Delta f = 0.1$. In the study no major effect of increasing the number of frequency points is obtained (a linear increase in time occurs with respect to increase in the number of frequency points), so the response of the program to 256 frequency points is considered to be characteristic.

It is seen in the study that, the difference in the solution times of the sequential program and the single-core parallel program is too small, when the total computation time is considered (See Tab.2)

Instead of giving the change of solution time with the number of cores, we present the relative change with respect to the sequential program, $\frac{t_{sequential}}{t_{core}}$. This will show the speed-up with respect to the expected speedup (which is the number of
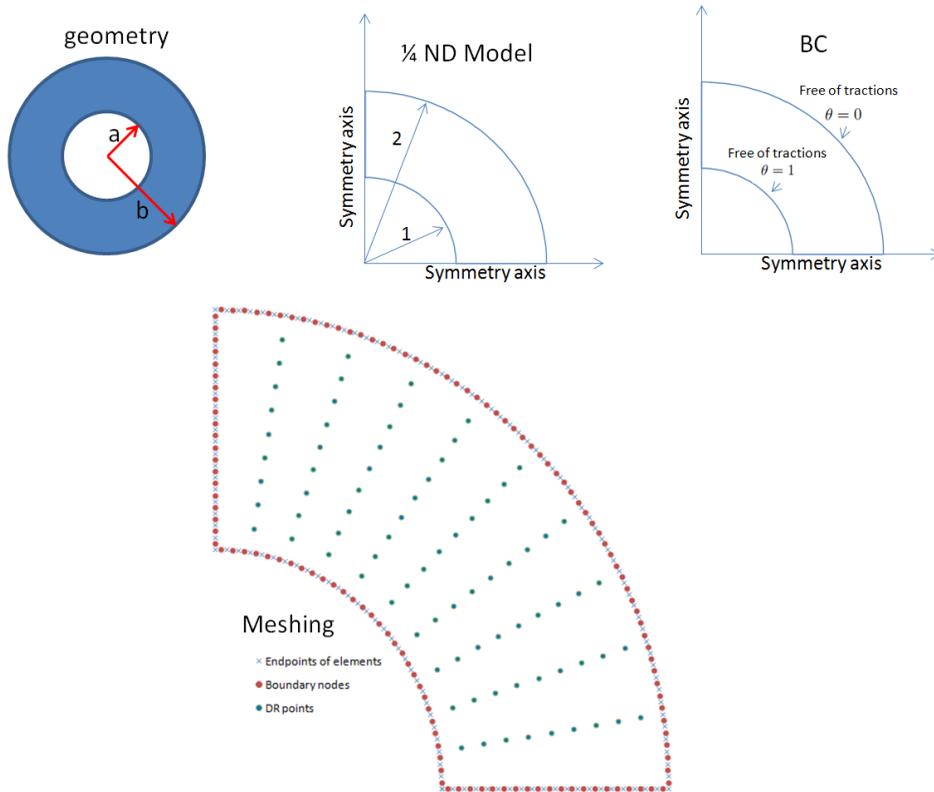
Figure 3: Geometry, boundary conditions and meshing of the problem

Table 1: Number of elements and internal collocation points (DR point) used in 4 different meshes

Mesh #	# of nodes	# of DR points
1	126	130
2	260	260
3	520	600
4	1040	1200

CN used), as given in Fig. 4

Note that, the performance of the algorithm can be considered to be good in Mesh 1, where the system of equations is smaller, and as the size of the system is increased, the performance drops. This is mainly due to the fact that communication time increase between the cores as the size of the matrices that are delivered in and out

Table 2: Solution times (s) on IBM Blade server

# of Processors	Mesh 1	Mesh 2	Mesh 3	Mesh 4
Sequential	479	2818	21943	153607
1	491	2823	21948	153612
2	246	1411	10984	76981
4	124	763	6666	47576
8	66	503	4497	36635
16	34	278	2563	18906
32	18	145	1286	9615
64	9	73	651	4895
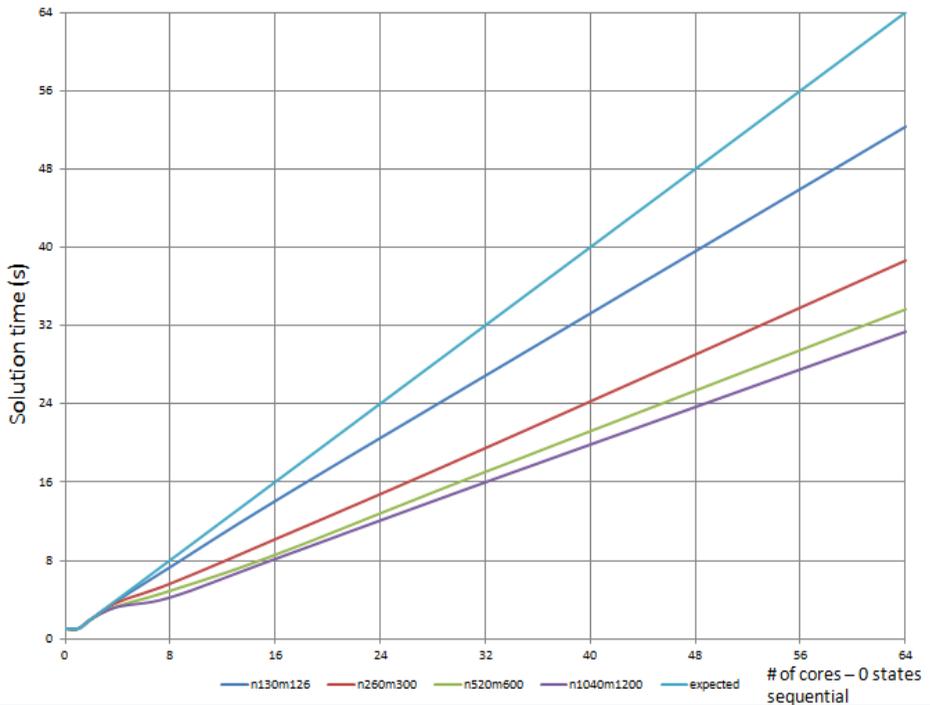
Figure 4: Speed-up ratios for given meshes

is increased.

5   Conclusion

In this study, a parallel code is implemented for the solution of TE and TVE problems using DR-BEM. The formulation is done in FTS and the parallellization is made by solving each frequency at different CN. In the analysis, without losing generality, 256 frequency points are considered.

From the obtained results, it can be concluded that, the functional-decomposition of the frequency space provides linear speed-ups for all meshes; yet as the total number of computation points (nodes on the boundary and DR points inside the solution domain) increases, the speed-up ratio decreases. This is mainly due to the increase in the communication times of broadcasting large amount of matrix data over the computational nodes. The following approaches may alliviate this problem and will be considered in the future:

- Parallellization of the formulation by parallel execution of the subroutines called in the program, instead. This would decrease the communication time, yet there is a possibility of cores which finish the execution waiting for the still executing cores, if a good load balancing is not done.

- Parallellization of the evaluation of the integrals (Gaussian Quadratures), instead. This would decrease the data transferred in communication drastically, but the number of communications increase. The efficiency of the algorithm would depend on which one is more dominant.

- A hybrid approach, where frequency parallellization is coupled with parallellizations described above. In such a program, each frequency will be solved in parallel and by more than one CN. This would be a very good alternative, yet it is much more complicated to implement and also the load balancing would be very important.

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